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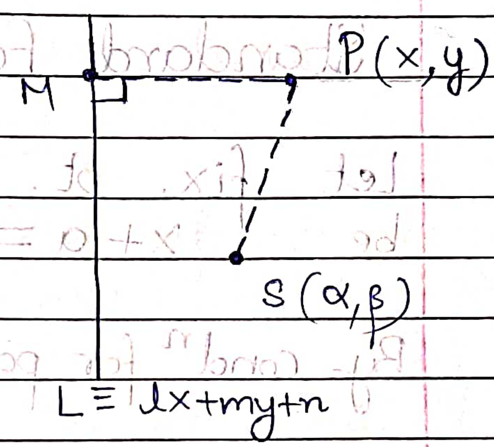
Parabola

Def^n: Locus of $\cdot$ in plane whose dist.^{n} from a fix. pt. & fix. line is same.

Eccentricity of Parabola is 1



As per given cond^n,

$$PS = PM$$

$$\Rightarrow (x-\alpha)^2+(y-\beta)^2 = \frac{(lx+my+n)^2}{(l^2+m^2+n^2)}$$

$$\Rightarrow \boxed{(lx-my)^2 + 2gx + 2fy + d = 0}$$ for some g, f, d

This is General Eq^n of Parabola

It is clear that second degree terms in this eq^n always form a perfect sq.

The converse is also true, i.e. if in an eq^n of second degree, second degree terms form perfect sq., then eq^n represents parabola.

unless it represents a pair of st. lines.

★ For general parabola, $\Delta \neq 0$ & $h^2 = ab$ in general 2nd degree eq^n.

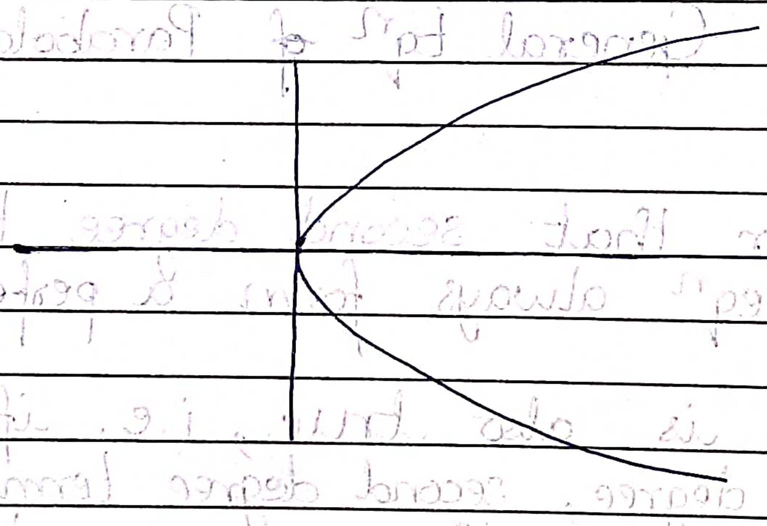
Standard Form

Let fix. pt. be $(a, 0)$ & fix line be $x + a = 0$, where $a > 0$

By cond^n for parabola, $\sqrt{(x-a)^2 + y^2} = (x+a)$

$$\Rightarrow (x-a)^2 + y^2 = (x+a)^2$$

$$\Rightarrow \boxed{y^2 = 4ax}$$



✓ Focus – $(a, 0)$ ✓ Directrix – $x + a = 0$

✓ Axis – $y = 0$ (Strt. Line thru Focus ⊥ to Directrix)

✓ Vertex – $(0,0)$ (Pt. of ∩ of conic with Axis)

✓ Chord – Line joining any 2 pts. of curve.

✓ Focal Chord – Chord thru focus

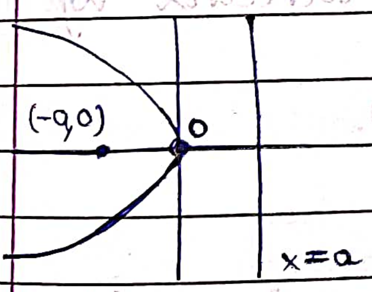
✓ Latus Rectum (L.R.) – $x = a$ Length $= 4a$
(focal chord ⊥ to Axis)

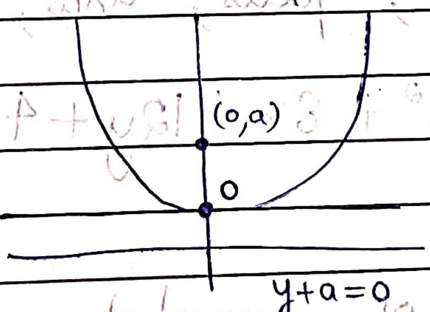
✓ Focal dist. – Dist., from focus, of any pt. on curve.

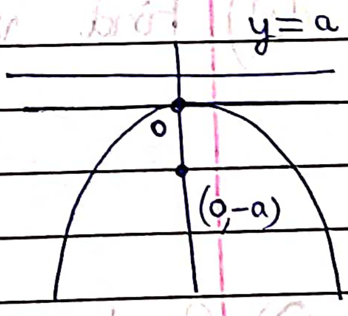
✓ Double Ordinate – Chord ⊥ to Axis.

✓ Tangent at Vertex – Line ⊥ to Axis, thru vertex.

Other Standard forms –







$y^2 = (-4)ax$ $x^2 = 4ay$ $x^2 = (-4)ay$

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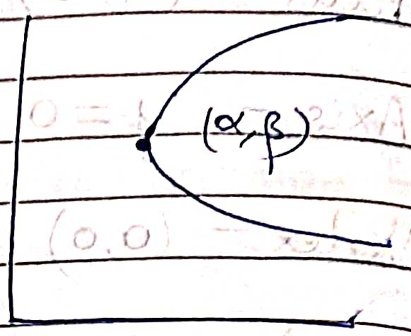
Shifting of Vertex

$$(y-\beta)^2 = 4a(x-\alpha)$$

$$Y^2 = 4aX$$



Shifting origin to (α, β) & then solve.

Q) Focal dist. of pt. on $y^2 = 8x$ is 8. Find the pt.

Q) Find length of side of equi Δ inscribed in $y^2 = 4ax$ s.t. 1 vertex of Δ is at vertex of C.

Q) Find vertex, focus, axis, L.R. & directrix of

$$x^2 + 8x + 12y + 4 = 0.$$

Q) Find eqn of parabola with focus $(4,-3)$ and vertex $(4,-1)$.

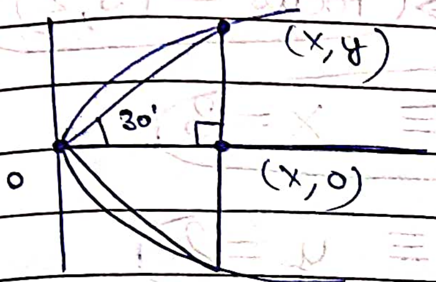
Q) Find eq^n of parabolas with L.R. joining pts. $(3,6)$ & $(3,-2)$.

A) $S(2,0)$; Pt $(2t^2, 4t)$

$8 = (2t^2 + 2) \Rightarrow t = \pm\sqrt{3} \Rightarrow \boxed{\begin{matrix}(6, 4\sqrt{3}) \\ (6, -4\sqrt{3})\end{matrix}}$

(Dist from focus = Dist from directrix)

A)



$x = y\sqrt{3}$

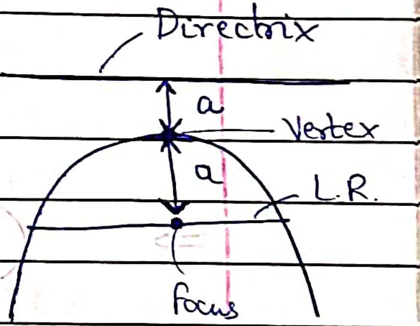
$\Rightarrow y^2 = 4a(y\sqrt{3}) \Rightarrow \begin{matrix} y = 4a\sqrt{3} \\ x = 12a \end{matrix}$

$\Rightarrow \boxed{\text{Side} = 8\sqrt{3}\,a}$

A) $(x^2 + 8x + 16) + (12y - 12) = 0$

$\Rightarrow (x+4)^2 = (-4)(3)(y-1)$

$\Rightarrow X^2 = (-4)\,a\,Y$

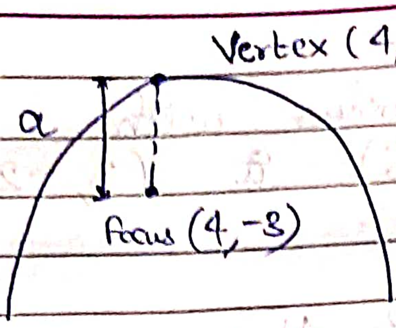


Vertex : $(-4, 1)$ Focus : $(-7, 1)$ $(-4, -2)$

Axis : ~~Y axis~~ $x+4=0$ L.R. : $y = (-2)$

Directrix : $y = 4$

A)

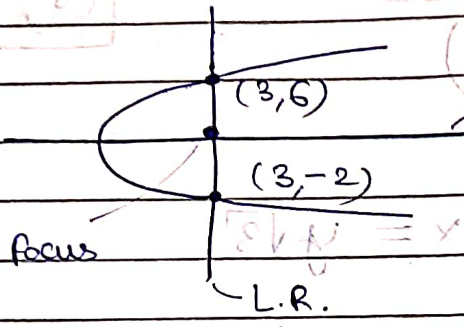


$a = 2$

$(x-4)^2 = (-4)(2)(y+1)$

$$\boxed{(x-4)^2 + 8(y+1) = 0}$$

A)



$\text{Focus} = \frac{(3,6)+(3,-2)}{2}$

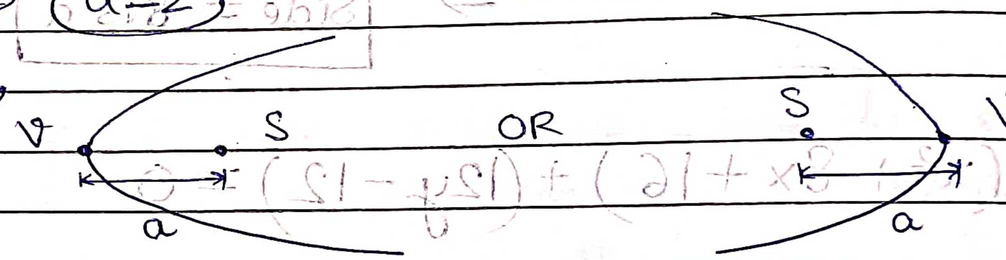
$\Rightarrow$ Focus $= (3,2)$

L.R. $\equiv$ $x = 3$

$|L.R.| = 4a = 8$ $\Rightarrow$ Axis $\equiv$ $y = 2$

$\Rightarrow$ $a = 2$

Now,

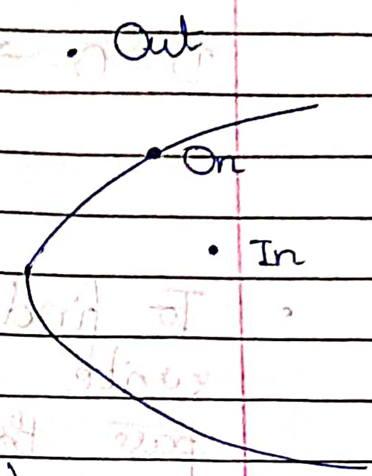


$\Rightarrow$ $V = (1,2)$ $\qquad$ $V = (5,2)$

Eqn: $\boxed{(y-2)^2 = 8(x-1)}$ $\qquad$ Eqn: $\boxed{(y-2)^2 = (-8)(x-5)}$

Posn. of Pt. wrt Parabola

Let $S \equiv y^2 - 4ax$

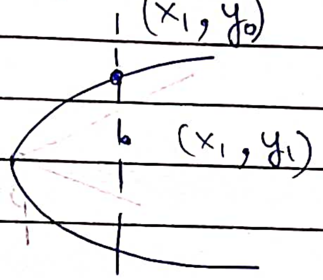


Pt. Inside $\Rightarrow$ $S(x_1, y_1) < 0$

Pt. On $\Rightarrow$ $S(x_1, y_1) = 0$

Pt. Outside $\Rightarrow$ $S(x_1, y_1) > 0$

Proof :



for pt. in. Let (x_1, y_1).

We have $y_0 > y_1$

and $y_0^2 = 4ax_1$ $\Rightarrow$ $(y_1^2 - 4ax_1) < 0$

Similarly for other.

Intersection of Line with Parabola

Let $L: y = mx + c$ & $S: y^2 = 4ax$.

- for Tangent, $\boxed{c = a/m}$

with pts of contact $(a/m^2, 2a/m)$.

Proof: $L \cap S$ has only 1 sol^n in 'm'.

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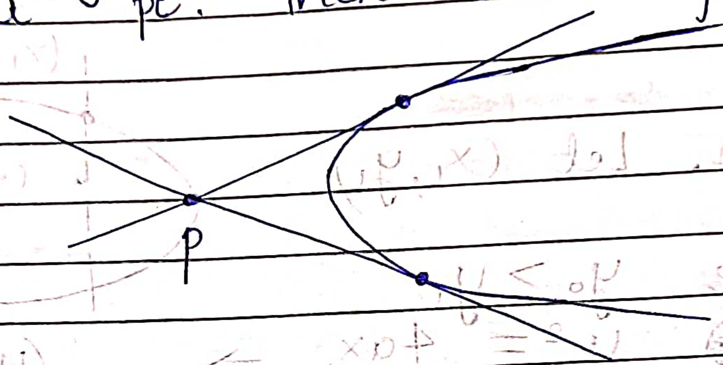
$(mx+c)^2 = 4ax \Rightarrow m^2x^2 + (2mc-4a)x + c^2 = 0$

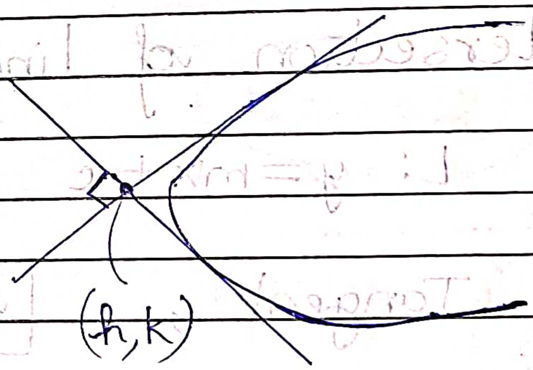
$D=0 \Rightarrow (mc-2a)^2 = (mc)^2 \Rightarrow c = a/m$.

- To find tangents from an ext. pt., write tangent in general form & pass thru pt. Then solve for 'm'.



1) Director $\odot$: For any parabola, directrix is the director $\odot$.

Proof: $y = mx + a/m$ is tangent.



$\Rightarrow m^2h - mk + a = 0$

for $\perp$ lines, $m_1m_2 = (-1)$

$\Rightarrow a/h = (-1) \Rightarrow h = (-a)$

$\Rightarrow$ Locus : $\boxed{x = (-a)}$

☆ Pair of Tangents drawn from any pt. ~~of~~ on directrix are mutually perpendicular.

Imp. Results

1) (Tangent at a pt. on Parabola) $\equiv$ $\boxed{T = 0}$

2) (Joint Eq^n of Pair of Tangents from an ext. pt.) $\equiv$ $\boxed{SS_1 = T^2}$

3) (Chord of Contact from ext. pt.) $\equiv$ $\boxed{T = 0}$

4) (Chord with given midpt.) $\equiv$ $\boxed{T = S_1}$

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Parametric Coordinates

Parabola	't' form	'm' form
$y^2 = 4ax$	$(at^2, 2at)$	$(am^2, -2am)$
$y^2 = (-4)ax$	$(-at^2, 2at)$	$(-am^2, 2am)$
$x^2 = 4ay$	$(2at, at^2)$	$(-2a/m, a/m^2)$
$x^2 = (-4)ay$	$(2at, -at^2)$	$(2a/m, -a/m^2)$

where 'm' is slope of normal at pt. $P(t)$

1) $\begin{pmatrix}\text{Tangent at } P(t) \\ \text{on } y^2 = 4ax\end{pmatrix} \equiv \boxed{ty = x + at^2}$

Slope $= (1/t)$

2) $\begin{pmatrix}\text{Normal at } P(t) \\ \text{on } y^2 = 4ax\end{pmatrix} \equiv \boxed{y = (-t)x + 2at + at^3}$

Slope $= (-t)$

3) $\begin{pmatrix}\text{Normal at } (am^2, -2am) \\ \text{on } y^2 = 4ax\end{pmatrix} \equiv \boxed{y = mx - 2am - am^3}$

Slope $= m$

Imp. Prop^{t}s

1) Slope of line joining $P(t_1)$ & $Q(t_2)$ on $y^2 = 4ax$ is $\boxed{\frac{2}{t_1+t_2}}$

Proof: $m_{PQ} = \left(\frac{2at_2 - 2at_1}{at_2^2 - at_1^2}\right) = \left(\frac{2}{t_1+t_2}\right)$

2) Pt. of $\cap$ of tangents at $P(t_1)$ & $Q(t_2)$ on $y^2 = 4ax$ is $\boxed{(at_1t_2, a(t_1+t_2))}$
$\| $
$\boxed{\begin{pmatrix} \text{G.M.} & \text{A.M.} \\ \text{of } X_{coor}, & \text{of } Y_{coor} \end{pmatrix}}$

Proof: $L_1 : t_1 y = x + at_1^2$
$L_2 : t_2 y = x + at_2^2$

3) Pt. of $\cap$ of normals at $P(t_1)$ & $Q(t_2)$ on $y^2 = 4ax$ is $\boxed{\left(a(t_1^2+t_2^2+t_1t_2+2), -at_1t_2(t_1+t_2)\right)}$

Proof: $y = (-t_1)x + 2at_1 + at_1^3$.
$y = (-t_2)x + 2at_2 + at_2^3$.

4) Endpts. of focal chord are $P(t)$ & $Q(-1/t)$
OR
If $P(t_1)$ & $Q(t_2)$ s.t. PQ thru focus $\Rightarrow$ $\boxed{t_1t_2 = (-1)}$

Proof : PQ : $(y - 2at_1) = \left(\frac{2}{t_1 + t_2}\right)(x - at_1^2)$

Thru focus $(a, 0)$ $\Rightarrow$ $2at_1 = \left(\frac{2a}{t_1 + t_2}\right)(t_1^2 - 1)$

$\Rightarrow t_1^2 + t_1 t_2 = t_1^2 - 1 \Rightarrow t_1 t_2 = (-1)$

5) If normal at $P(t_1)$ $\cap$ parabola at $Q(t_2)$, then $\boxed{t_2 = (-t_1) + \left(\frac{-2}{t_1}\right)}$

Proof : Normal : $y = (-t_1)x + 2at_1 + at_1^3$.

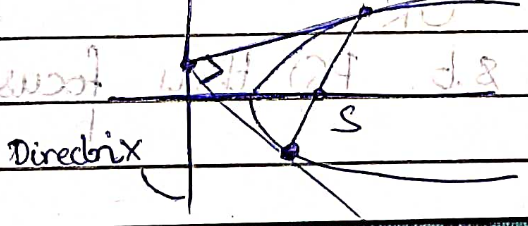
$\left(\text{Slope of PQ}\right) = \left(\frac{2}{t_1 + t_2}\right) = (-t_1) \Rightarrow t_2 = (-t_1) + \left(\frac{-2}{t_1}\right)$

6) If normals at $P(t_1)$ & $Q(t_2)$ meet on parabola, then $\boxed{t_1 t_2 = 2}$

Proof : Let meet at $R(t_3)$. Now,

$t_3 = -t - 2/t \Rightarrow t^2 + t_3 t + 2 = 0$ $\begin{matrix} t_1 \\ t_2 \end{matrix}$ $\rightarrow t_1 t_2 = 2$

7) Chord joining pts. of contact of $\perp$ pair of tangents, passes thru focus.



Proof: Let pts. of contact $\equiv P(t_1)$ & $Q(t_2)$

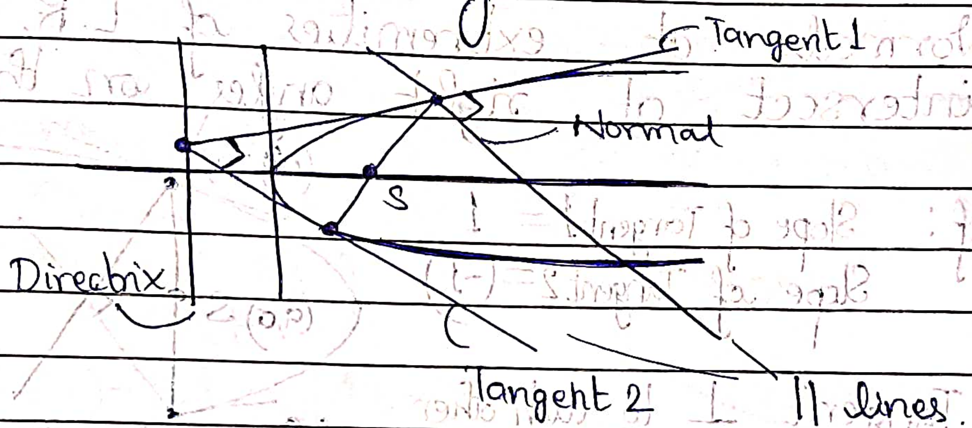
$\left(\cap \text{ of Tangents}\right) \equiv \left(at_1t_2, a(t_1+t_2)\right)$

lies on directrix $\Rightarrow at_1t_2 = (-a) \Rightarrow t_1t_2 = (-1)$

$\Rightarrow$ PQ thru focus.

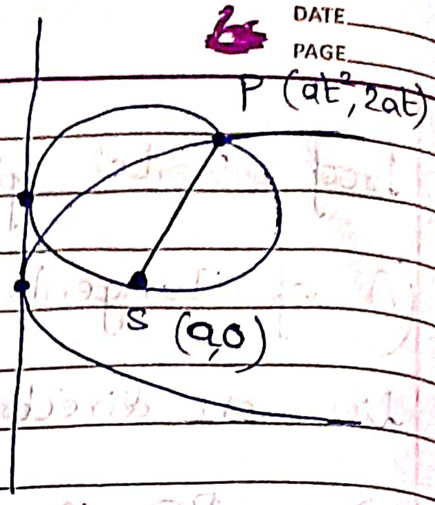
8) Tangent at one extremity of focal chord of parabola, is $\parallel$ to normal at other extremity.

Proof:



9) Circle described on any focal chord of parabola as diameter touches the directrix. Proof: Trivial.

10) Circle described on any focal dist. as diameter touches tangent at vertex.



Proof: ⊙ with diameter SP,

$(x-a)(x-at^2)+y(y-2at)=0$

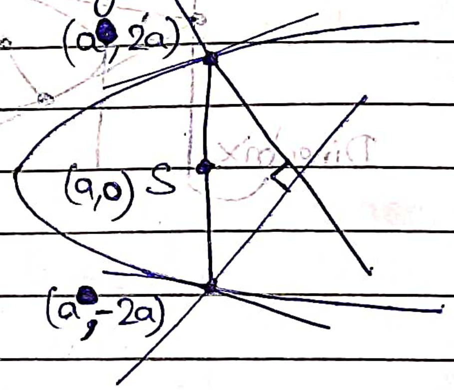
$\Rightarrow x^2-(a+at^2)x+a^2t^2 + y^2-2at=0$

$\Rightarrow \left(x-a\left(\frac{1+t^2}{2}\right)\right)^2+(y-at)^2 = \left(a\left(\frac{1+t^2}{2}\right)\right)^2$

$\Rightarrow$ Tangent to y axis.

11) Normals at extremities of L.R. intersect at right angles on the axis.

Proof: Slope of Tangent 1 = 1
Slope of Tangent 2 = (−1)



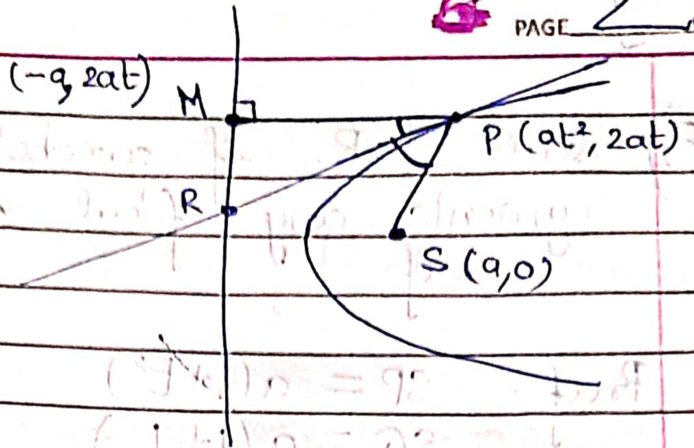
$\Rightarrow$ Tangent ⊥ to each other
$\Rightarrow$ Normals ⊥ to each other

12) Tangent at any pt. P bisects the angle b/w the focal chord thru P and ⊥ from P to directrix.

Proof: $m_{PM} = 0$

$m_{PR} = 1/t$

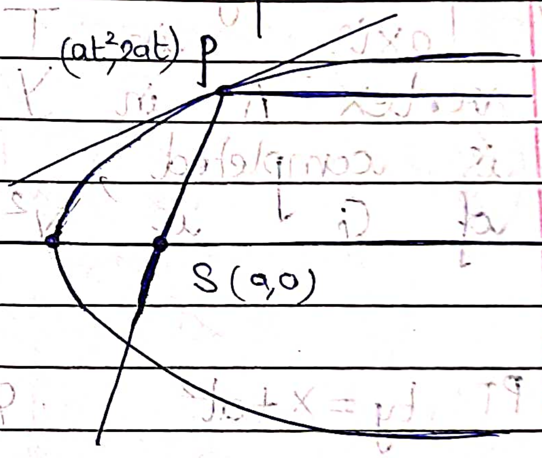
$m_{PS} = \left(\dfrac{2t}{t^2-1}\right)$



$\Rightarrow \angle MPR = \angle SPR$

$\Rightarrow$ Tangent bisects $\angle SPM$

13) A line travelling $\parallel$ to axis of parabola, after getting reflected passes or appears to pass thru focus.



Proof: $m_{ray} = 0$

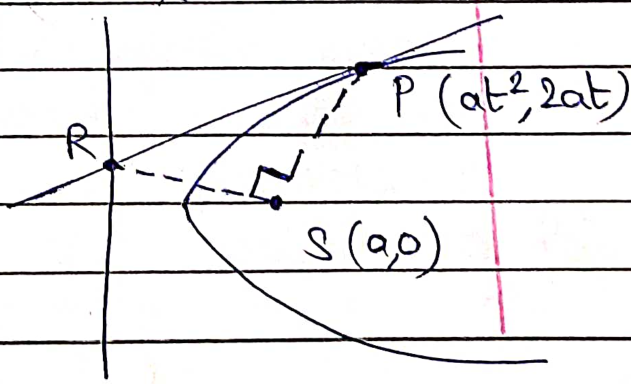
$m_{tangent} = 1/t$

$m_{SP} = \left(\dfrac{2t}{t^2-1}\right)$

$$\Rightarrow \left|\frac{\left(2t/t^2-1\right) - 1/t}{1 + \left(2/t^2-1\right)}\right| = \left|\frac{1/t - 0}{1+0}\right|$$

$\Rightarrow$ Tangent acts like mirror.

14) Portion of Tangent to Parabola ~~cut~~ cut off b/w curve and directrix subtends 90° at focus.



Proof: PR: $ty = x + at^2$

$\Rightarrow R \equiv (-a,\ a(t - 1/t))$

$\Rightarrow RS \perp PS$

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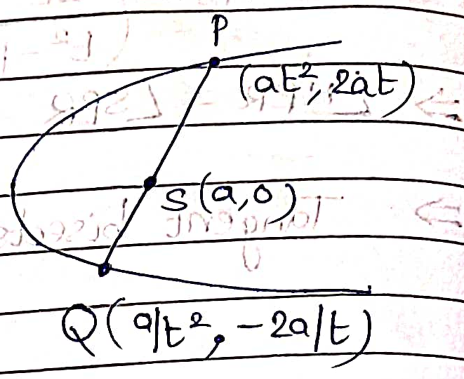
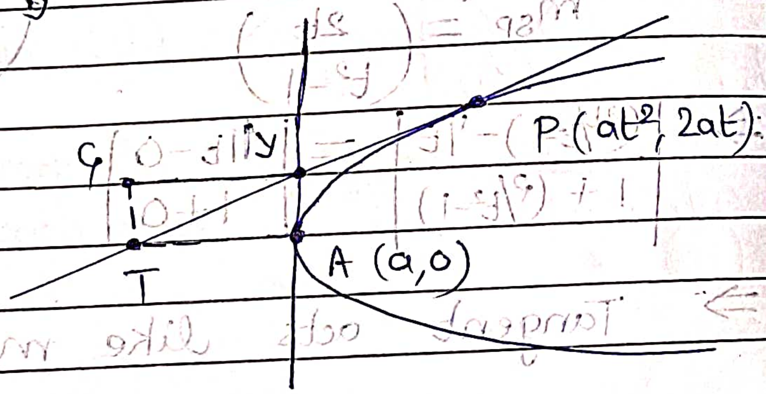
15) Semi L.R. of parabola is H.M. b/w segments of any focal chord of parabola



Proof : $SP = a(1+t^2)$
$SQ = a\left(1+\frac{1}{t^2}\right)$

$\Rightarrow \frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a}$

Q) If tangent to $y^2 = 4ax$ meets axis in T and tangent at vertex A in Y and rect. TAYG is completed, show that locus of G is $y^2 + ax = 0$.



A) PT: $ty = x + at^2$

$\Rightarrow T = (-at^2, 0)$

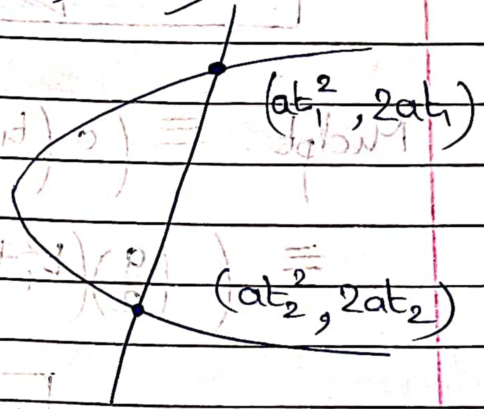
$\Rightarrow Y = (0, at)$

$\Rightarrow G = (-at^2, at) \Rightarrow$ Locus : $\boxed{y^2 + ax = 0}$

Q) Show that locus of pt. that divies a chord of slope 2 of $y^2 = 4ax$ internally in 1:2 is a parabola.

A) ~~$y = 2x + c \Rightarrow 4x^2 + 4cx + c^2 = 4ax$~~

$$\frac{(2a)(t_2 - t_1)}{(a)(t_2^2 - t_1^2)} = 2$$



$$\Rightarrow t_1 + t_2 = 1$$

Locus pt. $\equiv \left(\frac{at_1^2 + 2at_2^2}{3}, \frac{2at_1 + 4at_2}{3}\right)$

$$\equiv \left(\left(\frac{a}{3}\right)\left(2t_2^2 + (1-t_2)^2\right), \left(\frac{2a}{3}\right)\left(2t_2 + (1-t_2)\right)\right)$$

$$\equiv \left(\left(\frac{a}{3}\right)\left(3t_2^2 - 2t_2 + 1\right), \left(\frac{2a}{3}\right)\left(t_2 + 1\right)\right)$$

$\Rightarrow$ Locus : $\left(\frac{3x}{a}\right) = 3\left(\frac{3y}{2a} - 1\right)^2 - 2\left(\frac{3y}{2a} - 1\right)^2 + 1$

$\Rightarrow$ $\boxed{\text{Parabola}}$

Q) Thru vertex O of $y^2 = 4ax$, chords OP and OQ are drawn at right angles to one another. Show that $\forall$ P on parabola, PQ cuts axis of parabola at fix pt. Also find locus of midpt. of PQ.

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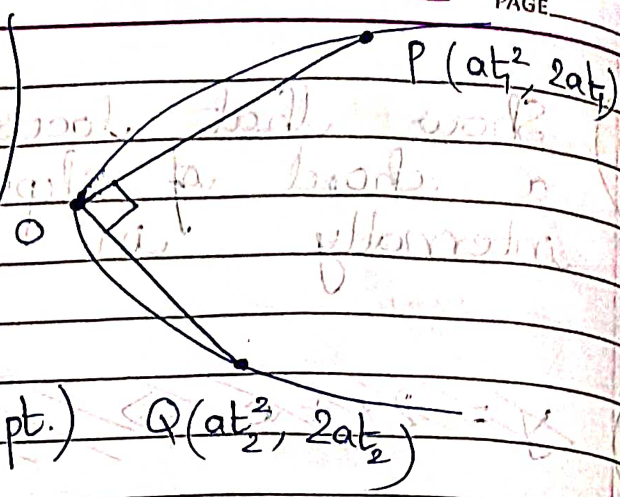
PQ : $(t_1+t_2)(y-2at_1)=2(x-at_1^2)$

$\Rightarrow \boxed{\text{fix pt } (4a,0)}$



A) $\left(\frac{2at_1}{at_1^2}\right)\left(\frac{2at_2}{at_2^2}\right)=(-1)$

$\Rightarrow t_1t_2=(-4)$

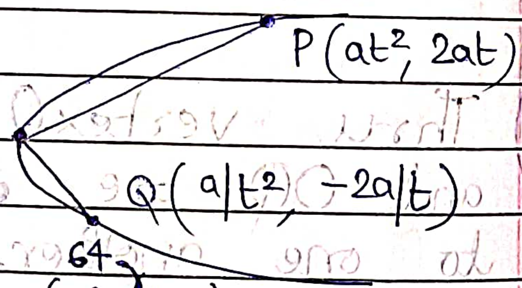
$\Rightarrow$ $\boxed{\text{PQ thru } \cancel{\text{focus}}}$ (fix pt.)

Midpt. $\equiv \left(a\left(\frac{t_1^2+t_2^2}{2}\right),\ a(t_1+t_2)\right)$

$\equiv \left(\left(\frac{a}{2}\right)\left(t_1^2+\frac{16}{t_1^2}\right),\ a\left(t_1-\frac{4}{t_1}\right)\right)$

$\Rightarrow$ Locus : $\boxed{\left(\frac{2x}{a}\right)-8=\left(\frac{y}{a}\right)^2}$

Q) If a chord PQ of $y^2=4ax$ subtends right angle at vertex, show that locus of pt. of $\cap$ of normals at P & Q is $y^2=16a(x-6a)$



A) $\text{Normal}_P : y=(-t)x+2at+at^3$

$\text{Normal}_Q : y=(4/t)x-2a(4/t)-64a/t^3$

$\Rightarrow 0=-(t+4/t)x+2a(t+4/t)+a(t^3+64/t^3)$

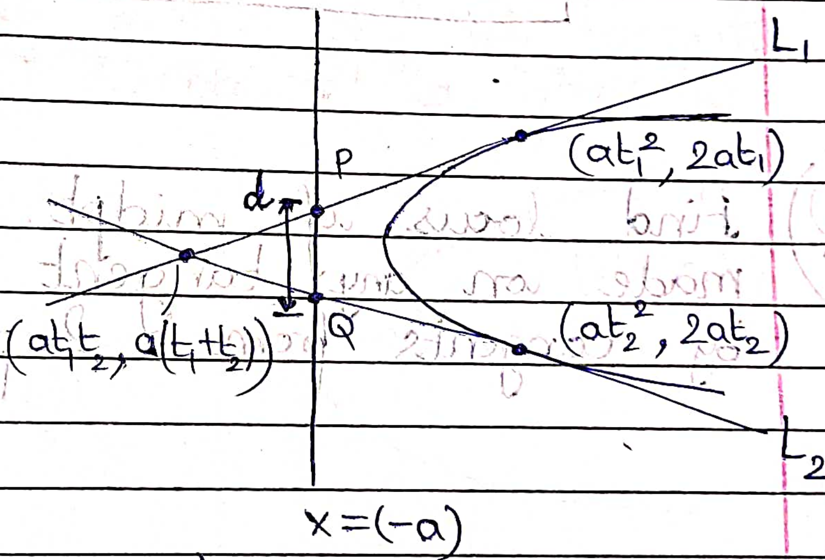
$\Rightarrow x=\left(\frac{at}{t^2+4}\right)\left[2(t+4/t)+(t^3+64/t^3)\right]$

$$\Rightarrow y = \left(\frac{a}{t^2+4}\right)\left[2\left(t+\frac{4}{t}\right) + \left(t^3+\frac{64}{t^3}\right)\right] \quad \frac{8a}{t} \quad \frac{a\cdot 64}{t^3}$$

$$\Rightarrow \cancel{y = \left(\frac{a}{t^2+1}\right)} \quad x = a\left(t^2+\frac{16}{t^2}-2\right) \text{ \& } y = 4a\left(t-\frac{64}{t}\right)$$

$\Rightarrow$ Locus: $\boxed{(x/a) - 6 = (y/a)^2}$ $(4a)$

Q) Show that locus of $\wedge$ of tangents to $y^2 = 4ax$ which intercept a const. length 'd' on directrix is $(y^2-4ax)(x+a)^2 = d^2x^2$



A) $L_1: t_1 y = x + at_1^2$
$L_2: t_2 y = x + at_2^2$

$\Rightarrow P = \left(-a, a\left(t_1 - 1/t_1\right)\right)$

$\Rightarrow Q = \left(-a, a\left(t_2 - 1/t_2\right)\right)$

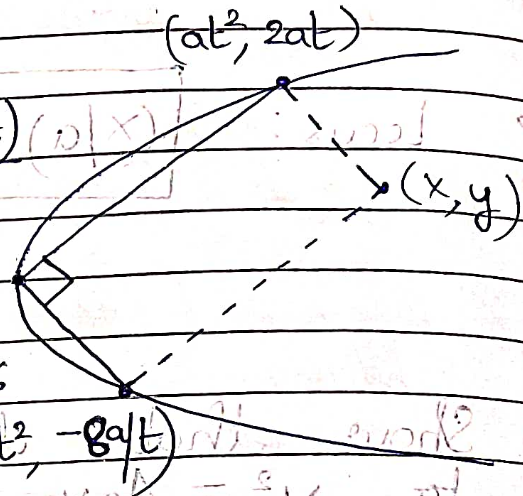
$$d = a\left((t_1-t_2) + \frac{(t_1-t_2)}{t_1t_2}\right)$$

$$\Rightarrow d^2 = a^2(t_1-t_2)^2\left(1+\frac{1}{t_1t_2}\right)^2 \Rightarrow d^2(at_1t_2)^2 = \left[a^2(t_1+t_2)^2 - 4a(at_1t_2)\right]\left[(at_1t_2) + a\right]^2$$

$$\Rightarrow \boxed{x^2d^2 = (y^2-4ax)(x+a)^2}$$

Q) If from vertex of $y^2 = 4ax$, 2 chords are drawn $\perp$ to each other and rectangle with those chords as sides is completed, find locus of vertex farthest from $(0,0)$.

A) $\dfrac{(x,y)+(0,0)}{2} = \dfrac{(at^2, 2at) + \left(\frac{16a}{t^2}, -\frac{8a}{t}\right)}{2}$



$\Rightarrow (x,y) = \left(a\left(t^2+\frac{16}{t^2}\right), 2a\left(t-\frac{4}{t}\right)\right)$

$\Rightarrow$ Locus : $\boxed{\left(\frac{x}{a}\right) - 8 = \left(\frac{y}{2a}\right)^2}$

Q) Find locus of midpt. of intercept made on any tangent to $y^2 = 4ax$ by tangents from 2 pts ($\underbrace{P \;\&\; Q}_{\text{fix}}$)

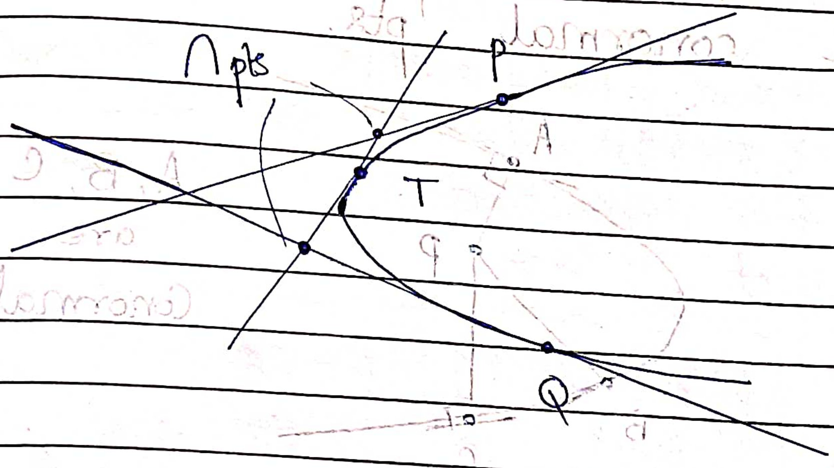
A) Let $P \equiv (ap^2, 2ap)$ & $Q \equiv (aq^2, 2aq)$

Let tangent be at $T \equiv (at^2, 2at)$

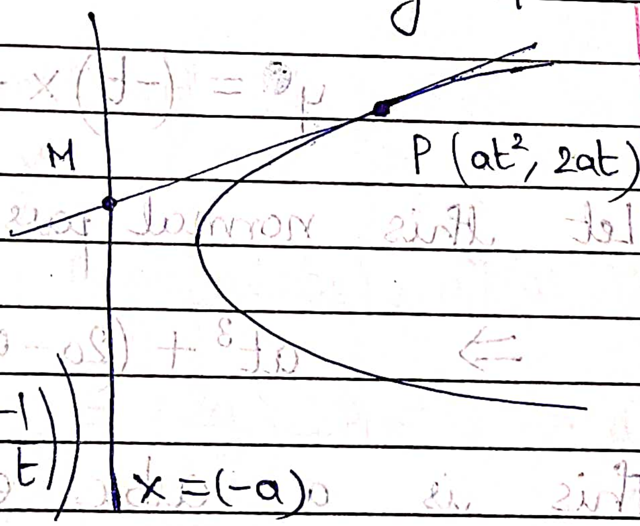
$\Rightarrow \cap$ pts $\equiv (atp, a(t+p))$ & $(atq, a(t+q))$

$\Rightarrow$ Midpt. $\equiv \left(a\left(\frac{p+q}{2}\right)t,\ at + a\left(\frac{p+q}{2}\right)\right)$

$\Rightarrow$ Locus : $\boxed{y = a\left(\frac{p+q}{2}\right) + \left(\frac{2x}{p+q}\right)}$



Q) Find locus of midpt. of all tangents from pts. on directrix to $y^2 = 4ax$.



A) PM: $ty = x + at^2$

$\Rightarrow M = \left(-a,\ a\left(t - \frac{1}{t}\right)\right)$

$\Rightarrow$ Midpt $= \left(a\left(\frac{t^2-1}{2}\right), \left(\frac{a}{2}\right)\left(\frac{3t-1}{t}\right)\right)$

$\Rightarrow$ Locus : $\boxed{\left(\frac{2y}{a}\right) = 3\sqrt{\left(\frac{2x}{a}\right)+1} - \frac{1}{\sqrt{\left(\frac{2x}{a}\right)+1}}}$

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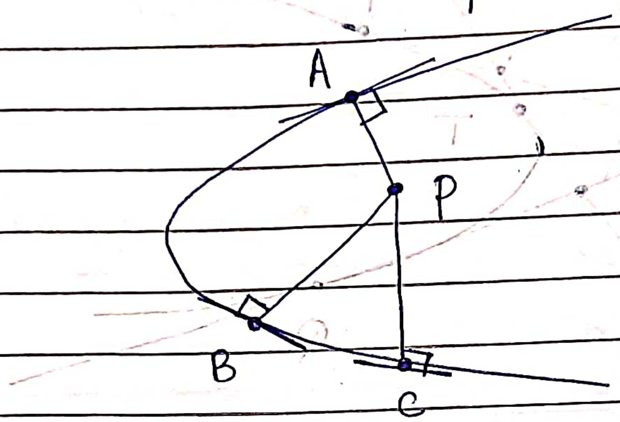
Normal:

Conormal Pts: In general, 3 normals can be drawn from a pt. to parabola. Their feet, pts where they meet the parabola, are called conormal pts.



A, B, C are Conormal.

Consider $y^2 = 4ax$. Let $T(at^2, 2at)$ be a pt. on it. Normal thru T is

$$y = (-t)x + 2at + at^3$$

Let this normal pass thru a fix pt. (α, β)

$$\Rightarrow \quad at^3 + (2a-\alpha)t - \beta = 0.$$

This is a cubic eq^n in 't'.

Clearly, it has either 3 real roots or exactly 1 real roots.

If it has 3 real roots t_1, t_2, t_3, then

$$\sum t_1 = 0, \quad \sum t_1 t_2 = \left(\frac{2a-\alpha}{a}\right), \quad \prod t_1 = \left(\frac{\beta}{a}\right)$$

Since $m = (-t)$, where 'm' is slope of normal.

$$\sum(\text{Slopes}) = 0, \quad \sum\left(\begin{matrix}\text{Slopes taken}\\ \text{2 at a time}\end{matrix}\right) = \left(\frac{2a-\alpha}{a}\right), \quad \prod(\text{Slopes}) = \left(\frac{-\beta}{a}\right)$$

Conclusions –

1) From any pt., at least 1 normal ~~normal~~ can be drawn to parabola.

2) The algebraic sum of slopes of 3 concurrent normals is zero.

3) In general, centroid of Δ formed by conormal pts lies on axis of parabola.

Proof: $A(at_1^2, 2at_1)$; $B(at_2^2, 2at_2)$; $C(at_3^2, 2at_3)$

$$G \equiv \left(\frac{A+B+C}{3}\right) \Rightarrow y_g = \left(\frac{2a}{3}\right)\sum t_1 = 0 \Rightarrow G \text{ on axis of Parabola}$$

4) If 3 normals can be drawn from a pt. (α, β) to parabola $y^2 = 4ax$, then $\boxed{\alpha > 2a}$.

Proof: Cubic eq^n in 't' needs to have 3 real roots.

$at^3 + (2a-\alpha)t - \beta^3 = 0$ has 3 real roots.

$\Rightarrow \quad 3at^2 + (2a-\alpha) = 0$ has 2 real roots

$\Rightarrow \quad \boxed{\alpha > 2a}$ (Necessary but NOT Sufficient)

5) If 3 normals drawn to parabola $y^2 = 4ax$ from (α, β) are real & distinct, then

$$\boxed{27a\beta^2 < (4)(\alpha - 2a)^3}$$

Prof: $\quad at^3 + (2a-\alpha)t - \beta = 0$ has 3 real roots

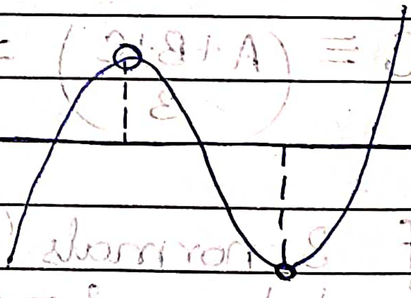
$\Rightarrow \quad 3at^2 + (2a-\alpha) = 0$ has 2 real roots.

$\Rightarrow \quad t = \sqrt{\frac{\alpha-2a}{3a}}, -\sqrt{\frac{\alpha-2a}{3}}$

At these values $f x^n$ has opp. signs.

$$\left[a\left(\frac{\alpha-2a}{3a}\right)^{3/2} + (2a-\alpha)\left(\frac{\alpha-2a}{3a}\right)^{1/2} - \beta\right]$$

$$\times\left[-a\left(\frac{\alpha-2a}{3a}\right)^{3/2} - (2a-\alpha)\left(\frac{\alpha-2a}{3a}\right)^{1/2} - \beta\right] < 0$$



$$\Rightarrow \left[a\left(\frac{\alpha-2a}{3a}\right)^{3/2} + (2a-\alpha)\left(\frac{\alpha-2a}{3a}\right)^{1/2} - \beta\right]\left[a\left(\frac{\alpha-2a}{3a}\right)^{3/2} + (2a-\alpha)\left(\frac{\alpha-2a}{3a}\right)^{1/2} + \beta\right] > 0$$

$$\Rightarrow \left[a\left(\frac{\alpha-2a}{3a}\right) + (2a-\alpha)\right]^2 \left(\frac{\alpha-2a}{3a}\right) > \beta^2 \quad (i)$$

$$\Rightarrow \left[\left(\frac{2}{3}\right)\left(2a-\alpha\right)\right]^2 \left(\frac{\alpha-2a}{3a}\right) > \beta^2$$

$$\Rightarrow \boxed{27a\beta^2 < (4)(\alpha-2a)^3}$$

Q) Show that locus of pts s.t. 2 of the 3 normals drawn from them to $y^2=4ax$ concide is $27ay^2=4(x-2a)^2$.

A) Let roots of $at^3+(2a-\alpha)t-\beta=0$ be t_1, t_1, t_2 where (α,β) is locus pt.

$$\Rightarrow 2t_1+t_2=0, \quad t_1^2+2t_1t_2=(2a-\alpha), \quad t_1^2t_2=\beta$$

$$\Rightarrow t_2=(-2t_1) \Rightarrow (-3t_1^2)=\frac{(2a-\alpha)}{a}, \quad (-2t_1^3)=\frac{\beta}{a}$$

$$\Rightarrow t_1^2=\left(\frac{\alpha-2a}{3a}\right), \quad t_1^3=\left(\frac{-\beta}{2a}\right)$$

$$\Rightarrow t_1^6=\left(\frac{\alpha-2a}{3a}\right)^3=\left(\frac{-\beta}{2a}\right)^2$$

$$\Rightarrow \text{Locus}: \boxed{27ay^2=4(x-2a)^2}$$

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Q) Normals are drawn from pt. P, with slopes m_1, m_2, m_3, to $y^2 = 4ax$. If locus of P with $m_1 m_2 = \alpha$ is a part of parabola itself, then find α.

A) $\cancel{\text{Let } at^3 + (2a - x_0)t - y_0 = 0 \; (t_1, t_2, t_3)}$

$\cancel{t_1 + t_2 + t_3 = 0, \quad t_1t_2 + t_3(t_1+t_2) = \frac{(2a - x_0)}{a}, \quad t_1t_2t_3 = \frac{y_0}{a}}$

$\cancel{(t_1+t_2) = (-t_3) \Rightarrow t_1t_2 - (t_1+t_2)^2 = (2a - x_0)/a}$

$\cancel{-t_1t_2(t_1+t_2) = y_0/a}$

$\cancel{\Rightarrow t_1t_2 - \frac{y_0^2}{a^2(t_1t_2)^2} = \frac{(2a - x_0)}{a}}$

Let $at^3 + (2a - x_0)t - y_0 = 0 \begin{cases} t_1 \\ t_2 \\ t_3 \end{cases}$ s.t.

$t_1t_2 = \alpha$, $t_1+t_2+t_3 = 0$, $t_1t_2 + t_2t_3 + t_3t_1 = \left(\frac{2a - x_0}{a}\right)$, $t_1t_2t_3 = ($

$\Rightarrow (t_1 + t_2) = (-t_3)$, $t_3 = \left(\frac{y_0}{a\alpha}\right)$

Now, $\sum t_1t_2 = t_1t_2 + (t_1+t_2)t_3$

$= t_1t_2 - t_3^2$

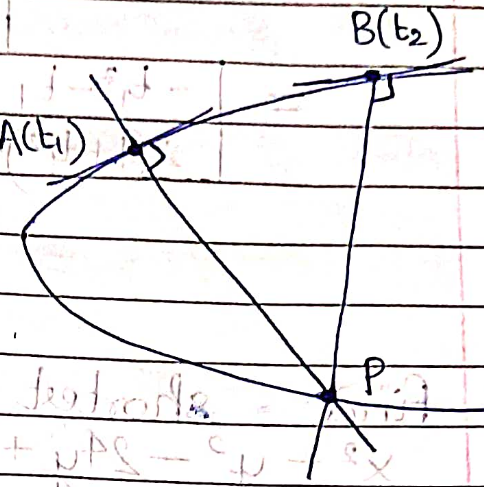
$= \alpha - \left(\frac{y_0}{a\alpha}\right)^2 = \left(\frac{2a - x_0}{a}\right)$

$\Rightarrow y_0^2 = (a\alpha^2)x_0 + (a^2\alpha^3 - 2a^2\alpha^2)$

This should coincide with $y^2 = 4ax$.

$\Rightarrow \boxed{a = 2}$

<u>Unique Observation :</u>



Let A & B be pts. s.t. $\cap$ of their normals P lies on parabola.

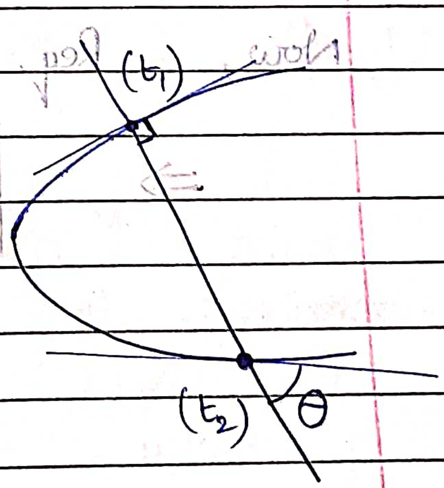
$\forall$ A, B P on parabola
& PA & PB are 2 normals to parabola from P

$\Rightarrow$ P is pt. given in Q !

$\Rightarrow$ $t_1 t_2 = 2 \Rightarrow \boxed{a = 2}$

Q) P.t. normal at $(at^2, 2at)$ on $y^2 = 4ax$ intersects the parabola again at an angle $\tan^{-1}\left|\frac{t}{2}\right|$

A) $t_2 = -t_1 - 2/t_1$



$\text{Slope}_{\text{normal}} = (-t_1)$

$\text{Slope}_{\text{tangent at } \cap} = 1/t_2 = \left(\frac{-t_1}{t_1^2 + 2}\right)$

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Now, $\tan(\theta) = \left| \dfrac{-t_1 - \left(\frac{-t_1}{t_1^2+2}\right)}{1 + (-t_1)\left(\frac{-t_1}{t_1^2+2}\right)} \right| = \left| \dfrac{-t_1^3 - 2t_1 + t_1}{t_1^2 + 2 + t_1^2} \right|$

$= \left| \dfrac{-t_1^3 - t_1}{2(1+t_1^2)} \right| \Rightarrow \theta = \tan^{-1}\left|\dfrac{t}{2}\right|$

Q) Find shortest dist. b/w $y^2 = 4x$ & $x^2 + y^2 - 24y + 128 = 0$

A) $\boxed{\text{Shortest dist. is along Common Normal.}}$

Normal : $y = (-t)x + 2t + t^3$

Thru $(0, 12) \Rightarrow 12 = t^3 + 2t$

$\Rightarrow (t-2)(t^2+2t+6) = 0 \Rightarrow t=2$

$\Rightarrow P(4,4)$'s normal to $y^2 = 4x$.

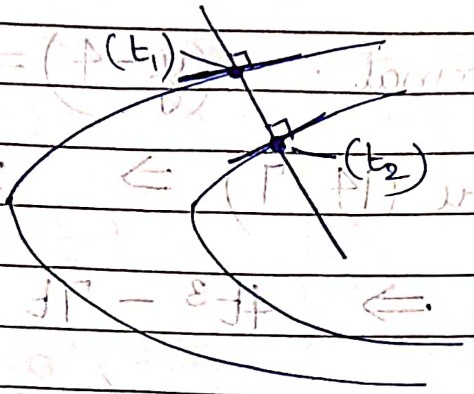
Now, Req. $= \sqrt{4^2 + 8^2} - \sqrt{12^2 - 128}$

$\Rightarrow \boxed{\text{Req.} = 4\sqrt{5} - 4}$

Q) Find condⁿ s.t. $y^2 = 4ax$ & $y^2 = 4b(x-c)$ have a common normal other than the X axis, given $a, b > 0$.

A) $y = (-t_1)x + 2at_1 + at_1^3$

$y = (-t_2)(x-c) + 2bt_2 + bt_2^3$



Both represent same normal.

$\Rightarrow \quad t_1 = t_2 = t$ (say)

& $\quad 2at_1 + at_1^3 = 2bt_2 + bt_2^3 + ct_2$

$\Rightarrow \quad (a-b)t^3 + (2a-2b-c)t = 0$

$\Rightarrow \quad (t)\left[(a-b)t^2 + (2a-2b-c)\right] = 0$

$\Rightarrow \quad \left(\dfrac{2b+c-2a}{a-b}\right) > 0$

$\Rightarrow \quad \boxed{\left(\dfrac{c}{a-b}\right) > 2}$

Q) 3 normals drawn from (14, 7) to $y^2 - 16x - 8y = 0$. Find feet of normals.

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A) $(y-4)^2 = 16(x+1)$

Let normal be at ~~$P(4t^2, 8t)$~~ $P(t)$

Normal : $(y-4) = (-t)(x+1) + 8t + 4t^3$

Thru $(14, 7)$ $\Rightarrow$ $3 = (-15t) + 8t + 4t^3$

$\Rightarrow 4t^3 - 7t - 3 = 0$

$\Rightarrow (t+1)(4t^2 - 4t - 3) = 0$

$\Rightarrow (t+1)(2t+1)(2t-3) = 0$

$\Rightarrow t = (-1), (-1/2), (3/2)$

Now, $P \equiv (4t^2 - 1, 8t + 4)$

$\Rightarrow \boxed{(3, -4) ; (0, 0) ; (8, 16)}$

Q) P.t. 2 strt lines, one tangent to parabola $y^2 = 4a'(x+a')$ and other to $y^2 = 4a(x+a)$, which are at right angles to each other meet on $x + a + a' = 0$. Show also that this line is common chord of both parabolae. Given $aa' < 0$.

A) $T_1 \equiv t_1 y = (x+a') + a' t_1^2$

$T_2 \equiv t_2 y = (x+a) + a t_2^2$

$T_1 \perp T_2 \Rightarrow t_1 t_2 = (-1) \Rightarrow t_1 = t, \; t_2 = (-1/t)$

$\Rightarrow T_1 \equiv ty = x + a'(1+t^2)$

$T_2 \equiv -ty = t^2 x + a(t^2 + 1)$

$\Rightarrow \quad x = \dfrac{-(t^2+1)(a+a')}{(t^2+1)} \Rightarrow \boxed{x + a + a' = 0}$

Locus of $\cap$ pt.

For common chord, $y^2 = 4a(x+a) = 4a'(x+a')$

$\Rightarrow$ Common Chord : $\boxed{x + a + a' = 0}$

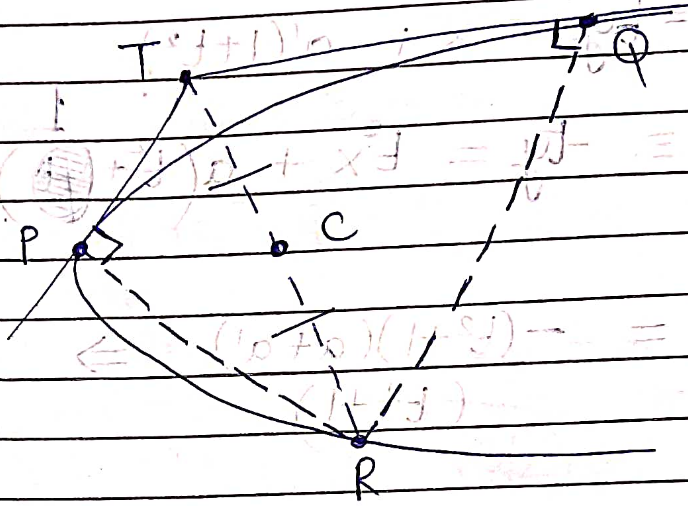
Q) TP & TQ are tangents to $y^2 = 4ax$. Normals at P & Q $\cap$ at R on the curve. P.t. centre of $\odot$ circumscribing ΔTPQ lies on $2y^2 = a(x-a)$.

A) $P \equiv (at_1^2, 2at_1)$; $Q \equiv (at_2^2, 2at_2)$

with $t_1 t_2 = 2$, and $T \equiv (at_1t_2, a(t_1+t_2))$

Observe $\angle TPR = \angle TQR = 90^\circ$.

$\Rightarrow$ TPQR is cyclic quadrilateral.



Let $R \equiv (at_3^2, 2at_3)$ where

$$t_3 = \left(\frac{-2}{t_1} - t_1\right) = \left(\frac{-2}{t_2} - t_2\right) = -(t_1+t_2)$$

Now, Centre $\equiv C \equiv \left(\frac{R+T}{2}\right)$

$$\Rightarrow C \equiv \left(\frac{at_1t_2 + at_3^2}{2}, at_3 + a\left(\frac{t_1+t_2}{2}\right)\right)$$

$$\Rightarrow C \equiv \left(a + \left(\frac{a}{2}\right)t_3^2, \left(\frac{a}{2}\right)t_3\right)$$

$\Rightarrow$ Locus of $C \equiv \left(\frac{2}{a}\right)(x-a) = \left(\frac{2y}{a}\right)^2$

$$\Rightarrow \boxed{2y^2 = a(x-a)}$$

Q) 3 normals drawn from $(3,0)$ to $y^2 = 4x$ meet parabola at P, Q, R.

1) Area of ΔPQR 2) Radius of circum⊙ of ΔPQR

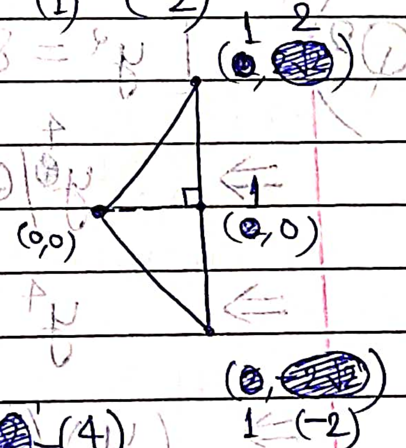
3) Centroid of ΔPQR 4) Circumcentre of ΔPQR.

A) Normal : $y = (-t)x + 2t + t^3$

Thru $(3,0)$ $\Rightarrow$ $t^3 - [illegible]t = 0 \Rightarrow t = 0, 1, -1$

$\Rightarrow P(0,0)$; $Q(1, 2)$; $R(1, -2)$

1) Area $= \frac{1}{2} \cdot 2 \cdot 4 = 2$



3) $\boxed{G \equiv \left(\frac{2}{3}, 0\right)}$

2,4) Let $O \equiv (R, 0) \Rightarrow R = \sqrt{(R-1)^2 + 4}$

$\Rightarrow \boxed{R = \frac{5}{2}}$; $\boxed{O \equiv \left(\frac{5}{2}, 0\right)}$

Q) Consider $x = t^2 - 2t + 2$ & $y = t^2 + 2t + 2$. Which of the following are correct.

1) Tangent at vertex is $x + y = 4$

2) Vertex is $(2, 2)$

3) Directrix is $x + y = 6$.

4) Focus is $(3, 3)$

A) $$\left.\begin{aligned}(x-1) &= (t-1)^2\\ (y-1) &= (t+1)^2\end{aligned}\right\} \Rightarrow t = \left(\frac{y-x}{4}\right)$$

$$\Rightarrow \left(\left(\frac{y-x}{4}\right)+1\right)^2 = (y-1) \Rightarrow (y-x+4)^2 = 16(y-1)$$

$$\Rightarrow x^2+y^2-2xy-8x-8y+32=0$$

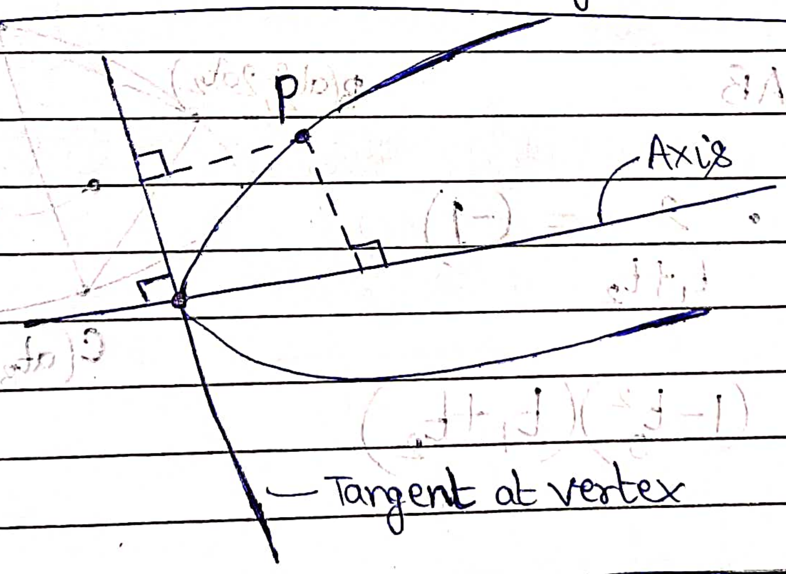
$$\Rightarrow \boxed{\left(\frac{x-y}{\sqrt{2}}\right)^2 = \cancel{8}\,(4\sqrt{2})\left(\frac{x+y-4}{\sqrt{2}}\right)}$$

(Dist. from axis) ~~Tangent at vertex~~ (4a) ~~Axis~~ (Dist. from tangent at vertex)

Now, solve as usual.

☆ In general parabola,

$$\boxed{\left(\begin{matrix}\text{Dist. from}\\ \text{Axis}\end{matrix}\right)^2 = (4a)\left(\begin{matrix}\text{Dist. from}\\ \text{Tangent at Vertex}\end{matrix}\right)}$$



Q) Let $A(t_1)$, $B(t_2)$, $C(t_3)$ be 3 pts. on $y^2 = 4ax$ s.t. orthocentre of ΔABC is focus of parabola, then

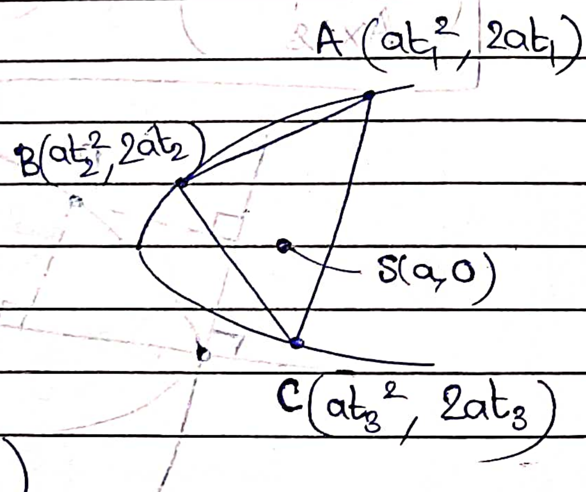
1) $\sum(t_1t_2) = (-5)$

2) $\sum(1/t_1t_2) = (-1)$

3) if $t_1 = 0$, then $t_2 + t_3 = 0$

4) $\prod(1+t_1) = (-4)$

Q) Let PQ be chord of $y^2 = 4ax$. A ⊙ is drawn with PQ as diameter. It passes thru vertex V. If area of $\Delta PVQ = 20$, then find coordinates of P.



A) SC ⊥ AB

$$\Rightarrow \frac{2at_3}{a(t_3^2-1)} \cdot \frac{2}{t_1+t_2} = (-1)$$

$$\Rightarrow 4t_3 = (1-t_3^2)(t_1+t_2)$$

$$\Rightarrow \quad t_3^2(t_1+t_2) + 4t_3 - (t_1+t_2) = 0 \quad \text{— (1)}$$

Similarly, $$t_2^2(t_3+t_1) + 4t_2 - (t_3+t_1) = 0 \quad \text{— (2)}$$

$$t_1^2(t_2+t_3) + 4t_1 - (t_2+t_3) = 0 \quad \text{— (3)}$$

~~Adding,~~ $$\cancel{\sum\left[t_3^2(t_1+t_2+t_3-t_3)\right] + 2\sum t_1 = 0}$$

$$\cancel{\Rightarrow \left(\sum t_1\right)\left(\sum t_1^2\right) - \left(\sum t_1^3\right) + 2\left(\sum t_1\right) = 0}$$

~~(1) − (2)~~
(2) − (3) $$\Rightarrow (t_1t_2)(t_2-t_1) + (t_3)(t_2^2-t_1^2) + 4(t_2-t_1) + (t_2-t_1) = 0$$

$$\Rightarrow (t_2-t_1)\left(t_1t_2 + t_3(t_1+t_2) + 5\right) = 0$$

$$\Rightarrow \boxed{\sum(t_1t_2) = (-5)} \quad \text{— (4)}$$

Using in (1); $$t_3^2(t_1+t_2) + 4t_3 = (t_1+t_2)$$

$$\Rightarrow (t_3)(-5-t_1t_2) + 4t_3 = (t_1+t_2)$$

$$\Rightarrow 0 \cancel{-5t_3+4t_3} = (t_1+t_2+t_3) + t_1t_2t_3$$

$$\Rightarrow \boxed{\sum\left(1/t_1t_2\right) = (-1)}$$

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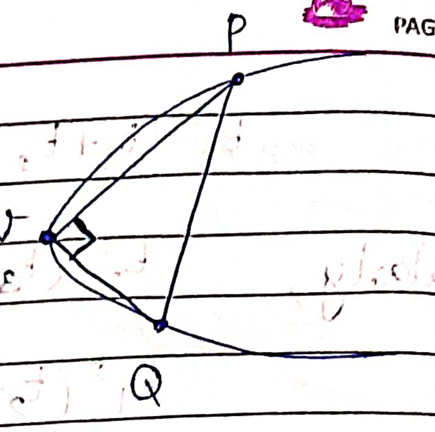

A) 90° at origin.

$P \equiv (at_1^2, 2at_1)$

$Q \equiv (at_2^2, 2at_2)$

Now, $\left(\frac{2}{t_1}\right)\left(\frac{2}{t_2}\right) = (-1) \Rightarrow t_1 t_2 = (-4)$

Area $= 20 = \frac{1}{2} \cdot PV \cdot VQ = \frac{1}{2} \cdot a(1+t_1^2) \cdot a(1+t_2)^2$

$\Rightarrow \left(\frac{40}{a^2}\right) = (1+t_1^2)\left(1+\frac{16}{t_1^2}\right)$

$\Rightarrow \boxed{t_1^4 + (17 - 40/a^2)t_1^2 + 16 = 0}$

'a' will be given, then solve for P.

Q) If $y^2 = 4ax$ & $y^2 = 4(x-1)$ do NOT have a common normal other than axis of parabola, then find 'a'.

A) Let us find condⁿ to have common normal. Let it ⊥ at $P(at_1^2, 2at_1)$ & $Q(t_2^2+1, 2t_2)$.

$$\text{Normal} \equiv \left(y = (-t_1)x + 2at_1 + at_1^3\right) \equiv \left(y = (-t_2)(x-1) + 2t_2 + t_2^3\right)$$

$\Rightarrow \quad t_1 = t_2 \quad$ & $\quad 2at_1 + at_1^3 = 3t_2 + t_2^3$

$\Rightarrow \quad (a-1)\, t_1^3 = (3-2a)\, t_1$

$\Rightarrow \quad \boxed{t_1^2 = \left(\dfrac{3-2a}{a-1}\right)}$

If $\quad a \in (1, 3/2) \Rightarrow$ Common Normal ✓

If $\quad a \notin [1, 3/2] \Rightarrow$ Common Normal ✗

Q) If normals at P, Q, R to $y^2 = 2x$ are concurrent at $(7,1)$; then centre of circle thru P, Q, R is (α, β) find $|\alpha| + 4\beta$.

A) Let $P \equiv (t_1^2/2, t_1)$; $Q \equiv (t_2^2/2, t_2)$; $R \equiv (t_3^2/2, t_3)$

$$\left(\begin{matrix}\perp \text{ bisectors} \\ \text{of } PQ\end{matrix}\right) \equiv \left[y - \left(\frac{t_1+t_2}{2}\right) = \left(\frac{-1}{2}\right)(t_1+t_2)\left[x - \left(\frac{t_1^2+t_2^2}{4}\right)\right]\right] \quad \text{— } 1$$

$$\left(\begin{matrix}\perp \text{ bisector} \\ \text{of } PR\end{matrix}\right) \equiv \left[y - \left(\frac{t_1+t_3}{2}\right) = \left(\frac{-1}{2}\right)(t_1+t_3)\left[x - \left(\frac{t_1^2+t_3^2}{4}\right)\right]\right] \quad \text{— } 2$$

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$$① - ② \Rightarrow \left(\frac{t_3 - t_2}{2}\right) = \left(\frac{t_3 - t_2}{2}\right)x + \left(\frac{1}{8}\right)\left[(t_1+t_2)(t_1^2+t_2^2) - (t_1+t_3)(t_1^2+t_3^2)\right]$$

$$\Rightarrow 4(t_3 - t_2) = 4(t_3 - t_2)x + \left[\cancel{t_1^3} + t_1t_2^2 + t_1^2t_2 + t_2^3 - \cancel{t_1^3} - t_1t_3^2 - t_1^2t_3 - t_3^3\right]$$

$$\Rightarrow 4(t_3 - t_2) = 4(t_3 - t_2)x + (t_2 - t_3)\left[t_1^2 + t_2^2 + t_3^2 + t_1t_2 + t_2t_3 + t_3t_1\right]$$

$$\Rightarrow 4 = 4x - \left[\underbrace{(\Sigma t_1)^2}_{\to (0)} - \Sigma(t_1t_2)\right]$$

$$\{\text{Using } ③\} \Rightarrow \boxed{x = 1 - \left(\frac{1}{4}\right)(\Sigma t_1t_2)} \xRightarrow{\{\text{Using } ④\}} \boxed{x = 4}$$

$$\text{Into } ①, \quad y - \left(\frac{t_1+t_2}{2}\right) = \left(\frac{-1}{2}\right)(t_1+t_2)\left[1 - \left(\frac{1}{4}\right)(\Sigma t_1t_2) - \left(\frac{t_1^2+t_2^2}{4}\right)\right]$$

$$\Rightarrow y = \left(\frac{t_1+t_2}{2}\right)\left[\frac{\Sigma t_1t_2 + t_1^2 + t_2^2}{4}\right]$$

$$\{\text{Using } ③\} \Rightarrow y = \left(\frac{-t_3}{8}\right)\left[(\Sigma t_1)^2 - \Sigma t_1t_2 - t_3^2\right]$$

$$\Rightarrow y = \left(\frac{1}{8}\right)\left[t_3 \Sigma t_1t_2 + t_3^2(t_3)\right]$$

$$\{\text{Using } ③\} \Rightarrow y = \left(\frac{1}{8}\right)\left[t_1t_2t_3 + t_3^2(t_1+t_2) - t_3^2(t_1+t_2)\right]$$

$$\Rightarrow \boxed{y = \left(\frac{1}{8}\right)(t_1t_2t_3)} \xRightarrow{\{\text{Using } ⑤\}} \boxed{y = 1/4}$$

Since normals thru $(7, 1)$, input in eq^n of normal.

$\Rightarrow \quad (1) = (-t)(7) + t + t^3/2$

$\Rightarrow \quad t^3 - 12t - 2 = 0 \; \begin{matrix} t_1 \\ t_2 \\ t_3 \end{matrix}$

$\Rightarrow \quad \sum t_1 = 0 \quad (3), \qquad \sum t_1 t_2 = (-12) \quad (4), \qquad t_1 t_2 t_3 = 2 \quad (5)$

Hence, $\alpha = 4$, $\beta = (1/4) \Rightarrow \boxed{\lfloor \alpha \rfloor + 4\beta = 5}$