Differentiation

- First Principle: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

For $y = f(x)$; $\frac{dy}{dx}$ is $f'(x)$.

- Rules for differentiation:

(1) $y = u(x) \pm v(x) \pm w(x) \cdots$, then $\frac{dy}{dx} = u'(x) \pm v'(x) \pm w'(x) \pm$

(2) $y = k.f(x)$, then $\frac{dy}{dx} = k.f'(x)$.

(3) $y = u(x) \cdot v(x)$, then $\frac{dy}{dx} = u(x).v'(x) + u'(x).v(x)$

(4) $y = \frac{u(x)}{v(x)}$, then $\frac{dy}{dx} = \frac{u'(x).v(x) - u(x).v'(x)}{(v(x))^2}$

(5) Chain Rule (Diff of composite functions)

Let $y = f(g(x))$, then $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$

Let $y = f(g(h(x)))$, then $\frac{dy}{dx} = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$.

(6) Differentiation of Implicit function:

e.g. $x^2 + y^2 + xy = 2$, then find $\frac{dy}{dx} = ?$

Ex: If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \cdots \infty}}}$, where $\sin x > 0$

Then P.T. $\frac{dy}{dx} = \frac{\cos x}{2y-1}$.

(7) Diff. of Parametric function:

Let $y = f(t)$ and $x = g(t)$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{f'(t)}{g'(t)}.$$

e.g. If $x = e^{t^2}$ and $y = \tan^{-1}(2t+1)$, then P.T.

$$\frac{dy}{dx} = \frac{-e^{t^2}}{2t(2t^2+2t+1)}$$

Q: If $x = a(\theta - \sin\theta)$; $y = a(1 - \cos\theta)$, then

P.T. $\frac{dy}{dx} = \cot \theta/2$.

(8) Logarithmic Differentiation:

Let $y = (f(x))^{g(x)}$, then

$$\frac{dy}{dx} = (f(x))^{g(x)} \left[\frac{g(x)}{f(x)} \cdot f'(x) + \ln(f(x)) \cdot \frac{dg(x)}{dx} \right]$$

Q: If $x^y \cdot y^x = 1$, T.P.T. $\frac{dy}{dx} = -\frac{(y + x \cdot \ln y)}{(x + y \cdot \ln x)} \cdot \frac{y}{x}$.

Q: If $f(x) = (x+1)(x+2)(x+3) \cdots (x+n)$, then find $f'(0)$.

Ans: $(n!) \left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right]$.

(9) Higher Derivatives of a function:

d^2y/dx^2, d^3y/dx^3, ...

For Parametric function:

If $x = f(\theta)$ and $y = g(\theta)$, then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

and $\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}(dy/dx)}{dx/d\theta}$.

In general, $\frac{d^ny}{dx^n} = \frac{\frac{d}{d\theta}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)}{\frac{dx}{d\theta}}$.

Q1: If $x = a(t + \sin t)$ and $y = a(1 - \cos t)$, then P.T. $\frac{d^2y}{dx^2} = \frac{1}{4a} \sec^4\left(\frac{t}{2}\right)$.

Q2: Show that the function $y = f(x)$ defined by the parametric equations $x = e^t \cdot \sin t$, $y = e^t \cdot \cos t$ satisfies the relation $y''(x+y)^2 = 2(x \cdot y' - y)$

Exercise:

1. If $f(x) = \cos\left\{\frac{\pi}{2}[x] - x^3\right\}$, $1 < x < 2$, $[\cdot]$: G.I.F
   Then find $f'\left(\sqrt[3]{\frac{\pi}{2}}\right)$.

2. If $u = f(x^3)$, $v = g(x^2)$, $f'(x) = \cos x$ and $g'(x) = \sin x$
   Then find $\frac{du}{dv}$.

Ans: 1. 0   2. $\frac{3}{2} x \cdot \cos x^3 \cdot \operatorname{cosec} x^2$.

Differentiation

- List of formulae:
- Fundamental Theorems: Addition/sub . . . .

. . . . composite function.

Chain rule.

Ex: If $x = e^{\tan^{-1}\left(\frac{y-x^2}{x^2}\right)}$; Then P.T. $\frac{dy}{dx} = 2x + x\left[1+\tan(\ln x)\right]^2$.

- Logarithmic differentiation:

Ex: If $y = e^{x} \cdot e^{x^2} \cdot e^{x^3} \cdot e^{x^4}$; Ans: $\frac{dy}{dx} = y\left(1+2x+3x^2+4x^3\right)$

- Implicit f^n:

Q ① If $x^y + y^x = 2$; $\frac{dy}{dx} = -\frac{\left(y^x \ln y + x^y \cdot \frac{y}{x}\right)}{\left(x^y \cdot \ln x + y^x \cdot \frac{x}{y}\right)}$

Q ②: If $y = \cfrac{\sin x}{1 + \cfrac{\cos x}{1 + \cfrac{\sin x}{1+\cos x \cdots}}}$, $\frac{dy}{dx} = \frac{(1+y)\cos x + y\sin x}{1+2y+\cos x - \sin x}$.

- Parametric Differentiation: $y = f(t)$; $x = g(t)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \; ; \quad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} ,$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) \cdots\cdots \quad \frac{d^ny}{dx^n} = \frac{\frac{d}{dt}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)}{dx/dt}.$$

Ex ① Let $x = \frac{1+t}{t^3}$, $y = \frac{3}{2t^2} + \frac{2}{t}$, T.P.T. $x\left(\frac{dy}{dx}\right)^3 = 1 + \frac{dy}{dx}$.

② If $x = a(t+\sin t)$ and $y = a(1-\cos t)$, T.P.T.

$$\frac{d^2y}{dx^2} = \frac{1}{4a}\sec^4\left(\frac{t}{2}\right)$$

③ If $f(x) = x^3 + x^2 \cdot f'(1) + x \cdot f''(2) + f'''(3)$, for $\forall x \in \mathbb{R}$.

Then find $f(x)$ indep. of $f'(1)$, $f''(2)$ and $f'''(3)$.

Ans: $f(x) = x^3 - 5x^2 + 2x + 6$

Q: If 'g' is inverse of f and $f'(x) = \frac{1}{1+x^n}$, then find $g'(x)$.

Ans: $1 + (g(x))^n$.