14/09/2022

PROBABILITY

- Random Experiment – Exp. whose outcomes are known in advanced

- Sample Space – Totality of outcomes

- Event – Part/Subset of sample space.

Types of Events –

- Exhaustive – Events $A_1, A_2, ... A_n$ are said to be exhaustive if no event outside this set can result as an outcome of an experiment

$$\bigcup_{i=1}^{n} A_i = U$$

- Equally likely – When each event is as likely to occur as any other event.

- Disjoint / Mutually Exclusive – Events are said to be mutually exclusive if happening of any one of them precludes the happening of others.

→ Mathematical Def^n of Probability

Let S be a sample space. The probability of occurance of an event E is given by

$$P(E) = \frac{n(E)}{n(S)} = \left(\frac{\text{\# cases favourable to } E}{\text{\# Total cases}} \right)$$

→ Axiomatic (Set): Approach to Probability

	Statement	Meaning
1.	Complementary event of A	A^C
2.	If event A occurs, so does B	$A \subseteq B$
3.	At least one of the events A or B occurs	$A \cup B$
4.	Both A & B occur	$A \cap B$
5.	A & B are disjoint events	$A \cap B = \emptyset$
6.	Event A occurs & B does not occur	$A \cap B^C$
7.	Exactly one of the events A or B occurs	$(A \cap B^C) \cup (A^C \cap B)$

- Results

1. $P(\emptyset) = 0$
2. $P(S) = 1$
3. $P(A^c) = 1 - P(A)$
4. $P(A \cap B^c) = P(A) - P(A \cap B)$
5. $0 \leq P(E) \leq 1$

- Odds in favour & against an event

Let S be a sample space & E an event

Odds in favour of $E = \dfrac{n(E)}{n(E^c)} = \left(\dfrac{\text{\# cases in favour of } E}{\text{\# cases against } E} \right)$

Odds against $E = \dfrac{n(E^c)}{n(E)} = \left(\dfrac{\text{\# cases against } E}{\text{\# cases in favour of } E} \right)$

- Add^n Rule

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$P(E_1 \cup E_2 \cup E_3) = \sum P(E_i) - \sum P(E_i \cap E_j) + P(E_1 \cap E_2 \cap E_3)$$

Conditional Probability (Rule of Multiplication)

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \quad \& \quad P(B|A) = \frac{P(A\cap B)}{P(A)}$$

(Probability of occurance of A when B has already occured)

Q. A bag contains 5 white & 8 red balls. 2 draws of 3 balls each are made w/o replacement.
Find probability that Draw I gives 3 white balls & Draw II gives 3 red balls.

A. $P(DII \to 3R \mid DI \to 3W) = \dfrac{P(DII \to 3R \cap DI \to 3W)}{P(DI \to 3W)}$

$\Rightarrow P(DII \to 3R \cap DI \to 3W) = \left(\dfrac{{}^{8}C_3}{{}^{13}C_3}\right)\left(\dfrac{{}^{8}C_3}{{}^{(13-3)}C_3}\right) = \dfrac{7}{429}$

- Independent Events – Events A & B are said to be indep. if

$$P(A\cap B) = P(A)\,P(B)$$

If A & B are independent events, then :-

1. A & B^c
2. A^c & B
3. A^c & B^c

$\rightarrow$ (Also independent events)

Proof: $P(A^c \cap B^c) = P(A \cup B)^c = 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - P(A) - P(B) + P(A)P(B)$

$= (1 - P(A))(1 - P(B)) = P(A^c) \cdot P(B^c)$

Generalisation of this statement for multiple events is true as well.

- Pairwise & Mutual Independence -

Let A, B, C be 3 events

Pairwise ($^3C_2$)	Mutual
$P(A \cap B) = P(A)P(B)$	6 Pairwise
$P(B \cap C) = P(B)P(C)$	$P(A \cap B \cap C) = P(A)P(B)P(C)$
$P(C \cap A) = P(C)P(A)$	

n events : $^nC_2$ $\qquad$ $^nC_2 + {}^nC_3 + \dots + {}^nC_n$

$= 2^n - n - 1$

Q. 4 Tickets in an urn: 112, 121, 211, 222
Drawn at random.
A_i ($i = 1, 2, 3$) — i^{th} digit of ticket drawn is 1
Discuss independence of A_1, A_2, A_3

A.

A. A_1 : 121, 211 $\qquad P(A_1) = 1/2 \qquad P(A_1 \cap A_2) = 1/4$

A_2 : 112, 211 $\qquad P(A_2) = 1/2 \qquad P(A_2 \cap A_3) = 1/4$

A_3 : 112, 121 $\qquad P(A_3) = 1/2 \qquad P(A_3 \cap A_1) = 1/4$

$P(A_1 \cap A_2 \cap A_3) = 0$

$\Rightarrow$ Pairwise but not mutually independent events

Q. A & B alternately cut a pack of cards & the pack is shuffled after every cut. If A starts & the game is continued till one cuts a diamond. What are the respective chances of A & B first cutting a diamond.

A.

$$P(A \text{ win}) = \underbrace{\left(\frac{1}{4}\right)}_{\text{A wins on 1st try}} + \underbrace{\left(\frac{3}{4}\right)}_{\text{A loses on 1st try}}\underbrace{\left(\frac{3}{4}\right)}_{\text{B loses on 1st try}}\underbrace{\left(\frac{1}{4}\right)}_{\text{A wins on 2nd try}} + \left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right) + \cdots$$

$$= \left(\frac{1}{4}\right)\left(\frac{1}{1-\frac{3^2}{4^2}}\right) = \underline{\left(\frac{4}{7}\right)}$$

$$P(B \text{ win}) = 1 - P(A \text{ win}) = \underline{3/7}$$

TOTAL PROBABILITY THEOREM

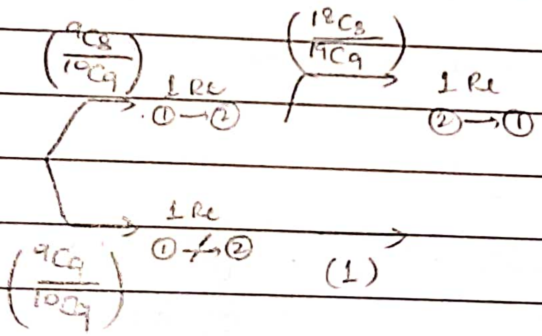
Let $E_1, E_2, \ldots, E_n$ be a set of mutually exclusive & exhaustive events and E be some event which is associated with $E_1, E_2, \ldots, E_n$.
Then

$$\boxed{\begin{aligned} P(E) &= \sum_{i=1}^{n} P(E \cap E_i) \\ &= \sum_{i=1}^{n} P(E_i)\, P(E|E_i) \end{aligned}}$$

Q. 2 purses
9 coins transferred
from ① → ② &
then ② → ①
Find probability of finding a 1 Re coin in ①
after the 2 transfers

Purse	50 p coins	1 Re coins
1	9	1
2	10	0

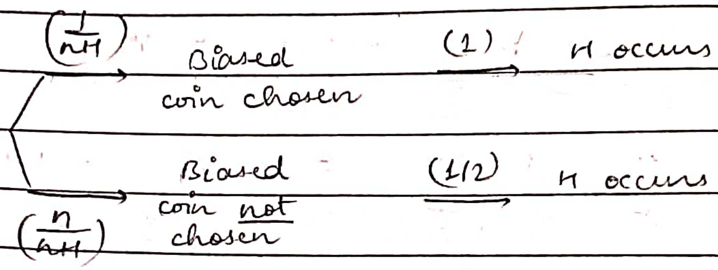
A.



$$P(E) = \left(\frac{^9C_8}{^{10}C_9}\right)\left(\frac{^{18}C_8}{^{19}C_9}\right) + \left(\frac{^9C_9}{^{10}C_9}\right)(1) = \underline{\frac{10}{19}}$$

Q. A bag contains $(n+1)$ coins. It is known that one of these coins shows H on both sides whereas others are fair. One coin is selected at random & tossed. If the probability that H occurs is $7/12$, find n?

A.



$$P(H) = \left(\frac{1}{n+1}\right)(1) + \left(\frac{n}{n+1}\right)\left(\frac{1}{2}\right) = \frac{7}{12} \Rightarrow \frac{n+2}{n+1} = \frac{7}{6}$$

$$\Rightarrow \underline{n = 5}$$

BAYES' THEOREM

Let $E_1, E_2, \ldots, E_n$ be a set of mutually exclusive & exhaustive events and an event E occurs. Then

$$\boxed{P(E_i|E) = \frac{P(E_i \cap E)}{P(E)} = \frac{P(E_i)\, P(E|E_i)}{\sum_{i=1}^{n} P(E_i)\, P(E|E_i)}}$$

Q. In a test, an examinee either guesses, or copies or knows the answer to a SCQ with 4 choices. The $P(\text{Guess}) = 1/3$, $P(\text{Copies}) = 1/6$, $P(\text{correct}|\text{Copied}) = 1/8$. Find $P(\text{Knows}|\text{Correct})$.

$1 - P(G) - P(C)$

A. $P(K|CA) = \dfrac{P(K\cap CA)}{P(CA)} = \dfrac{P(K)\,P(CA|K)}{P(K)P(CA|K) + P(G)P(CA|G) + P(C)P(CA|C)}$

$= \dfrac{(\frac{1}{2})(1)}{(\frac{1}{2})(1) + (\frac{1}{3})(\frac{1}{4}) + (\frac{1}{6})(\frac{1}{8})} = \dfrac{24}{29}$ $\left(\frac{1}{4}\right)$

BERNOULLI TRIALS & BINOMIAL PROBABILITY

Consider a series of 'n' independent Bernoulli Trials (Events) s.t for each trial,

$$\left.\begin{array}{l} P(\text{Success}) = p \\ P(\text{Failure}) = q \end{array}\right\} \Rightarrow p + q = 1$$

Then, the probability of 'r' successes in a series of 'n' independent trials is given by

$$P(X=r) = {}^nC_r\, p^r q^{(n-r)} \quad ; \quad r = 0, 1, \ldots, n$$

$X \to$ Binomial Variate

- Mean : $E(X) = \sum_{r=0}^{n} r\, {}^nC_r\, p^r q^{(n-r)} = np$

- Variance : $V(x) = V(x^2) - (V(x))^2$

$$= \sum_{r=0}^{n} r^2\, {}^nC_r\, p^r q^{(n-r)} - (np)^2$$

$$= npq$$

$$\boxed{V(x) = q\,E(x)}$$

NOTE: $P(X \leq r) = P(X=0) + P(X=1) + \ldots + P(X=r) = 1 - P(X > r) = 1 - [P(X=r+1) + \ldots + P(X=n)]$

Q. A die is thrown 7 times. What is the chance that an odd no. turns up

(i) exactly 4 times
(ii) at least 4 times

A. (i) $P(4 \text{ odd}) = {}^7C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^3 = \underline{35/128}$

(ii) $P(\text{atleast } 4) = P(4 \text{ odd}) + P(5 \text{ odd}) + P(6 \text{ odd}) + P(7 \text{ odd})$

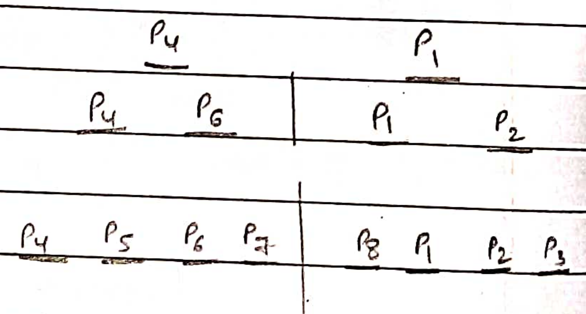
$$= {}^7C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^3 + {}^7C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^2 + {}^7C_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right) + {}^7C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)$$

$$= \underline{\left(\frac{1}{2}\right)}$$

Q. 8 Players $P_1, P_2, \ldots, P_8$ play a knockout tournament. It is known that whenever the players P_i & P_j play, P_i will win if $i < j$.
Assuming that players are paired at random in each round, find probability that P_4 reaches final.

A. P_4 will win only if paired with P_5, P_6, P_7, P_8.
(the losers L_i)

Example of a favourable tournament



So, initial pairing should be of the form
$(P_4, L_1), (L_2, L_3), (L_4, W_1), (W_2, W_3)$

W_i — P_1, P_2, P_3

$$P(P_4 \text{ in final}) = \frac{{}^4C_3 \cdot \frac{3!}{2!} \left(\frac{4!}{(2!)^3}\right)}{\frac{8!}{(2!)^4 4!} \left(\frac{4!}{2!\,2!}\right)} = \left(\frac{4}{35}\right)$$

ways of arranging tournaments