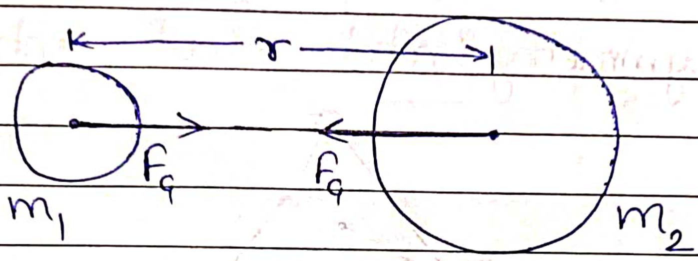
Gravitation

Newton's Law of Gravitation

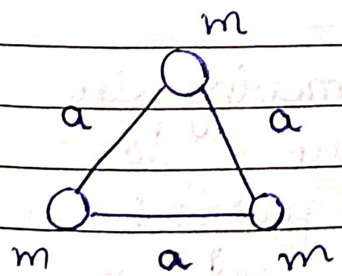


$$\boxed{F_G = G\frac{M_1 m_2}{r^2}}$$

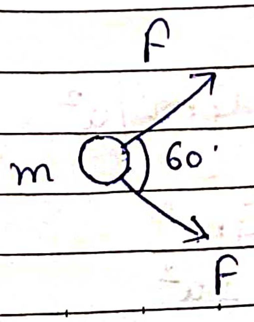
☆ For pt. masses, or Spherical masses with CoM at centre

$G - 6.67 \times 10^{-11}\ Nm^2 kg^{-2}$

(Universal Gravitational Const.)

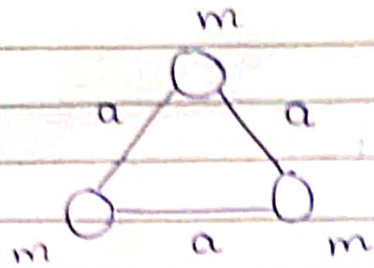
Q) 

Find force ~~of~~ on one of ~~f~~ the mass.

A) 

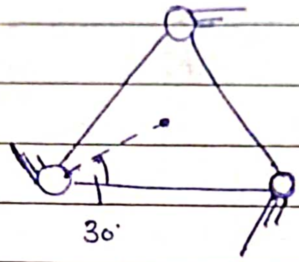
$F = Gm^2/a^2$

$$\Rightarrow \boxed{F_{net} = Gm^2\sqrt{3}/a^2}$$

Q) 

Masses moving in circle due to mutual gravitational force. Find vel.

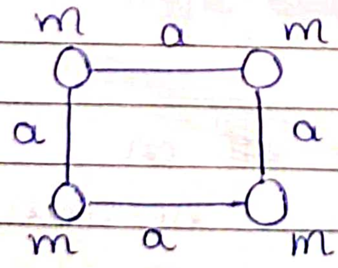
A) By symmetry they rotate about centre,



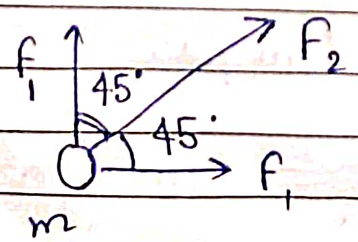
On a mass,

$$F_G = \frac{mv^2}{r} \Rightarrow \frac{Gm^2\sqrt{3}}{a^2} = \frac{mv^2}{(a/\sqrt{3})}$$

$$\Rightarrow \boxed{v = \sqrt{\frac{Gm}{a}}}$$

Q) 

Masses moving in circle due to mutual gravitational force. Find vel.

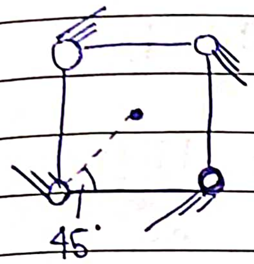
A) 

$$f_1 = Gm^2/a^2$$

$$f_2 = \frac{Gm^2}{2a^2}$$

$$f_{net} = f_2 + f_1\sqrt{2} \Rightarrow f_{net} = \left(\frac{2\sqrt{2}+1}{2}\right)\left(\frac{Gm^2}{a^2}\right)$$

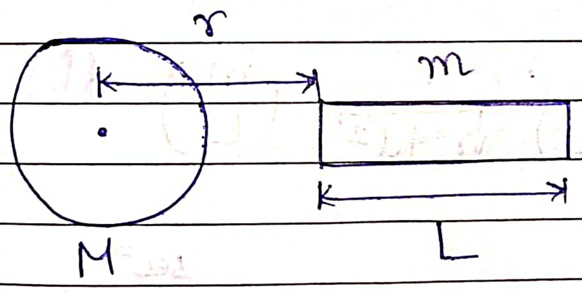
By Symmetry they rotate abt. centre,



$$f_c = \frac{mv^2}{r} \Rightarrow \left(\frac{2\sqrt{2}+1}{2}\right)\left(\frac{Gm^2}{a^2}\right) = \frac{mv^2}{(a/\sqrt{2})}$$

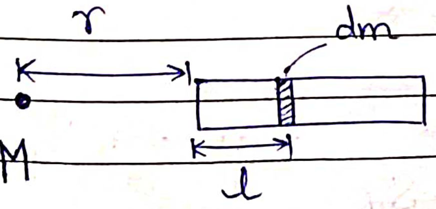
$$\Rightarrow \boxed{v = \sqrt{\left(1+\frac{1}{2\sqrt{2}}\right)\left(\frac{Gm}{a}\right)}}$$

Q) ☆



Find force b/w masses.

A)



$$df = \frac{GM\,dm}{(r+l)^2}$$

where $\rho = m/L = dm/dl \Rightarrow dm = dl\,(m/L)$

$$\Rightarrow f = \int_0^L \left(\frac{GMm}{L}\right)\frac{dl}{(r+l)^2} = \left(\frac{GMm}{L}\right)\left[\frac{-1}{(r+l)}\right]_0^L \Rightarrow \boxed{f = \frac{GMm}{r(r+L)}}$$

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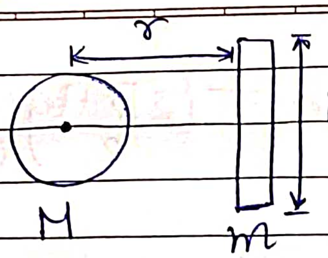
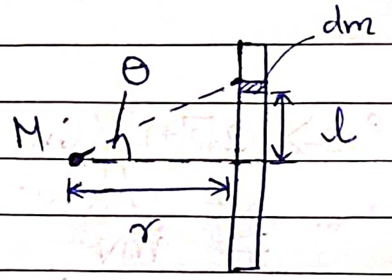
Q) 

find force b/w masses

A) 

$m/L = dm/dl$

$\Rightarrow dm = dl\,(m/L)$

By symmetry $F_y = 0$, so we only find F_x.

$$F_G = \int \frac{GM\,dm}{R^2} = \int \frac{GM}{(r^2+l^2)} \cos(\theta) \left(\frac{m}{L}\right) dl$$

$$= \int_{-L/2}^{L/2} \frac{GM}{(r^2+l^2)} \cdot \frac{r}{\sqrt{r^2+l^2}} \left(\frac{m}{L}\right) dl$$

Let $l = r\tan(\phi) \Rightarrow dl = r\sec^2(\phi)\,d\phi$

$$= \int_{-\tan^{-1}\left(\frac{L}{2r}\right)}^{\tan^{-1}\left(\frac{L}{2R}\right)} \frac{GM}{r^2 S_\phi^2} \cdot \frac{r}{r S_\phi} \cdot \frac{m}{L} \cdot r S_\phi^2 \, d\phi$$

$$= \left(\frac{GMm}{Lr}\right) \int_{-\tan^{-1}\left(\frac{L}{2r}\right)}^{\tan^{-1}(L/2r)} c_\phi \, d\phi = \left(\frac{2GMm}{Lr}\right) \int_{0}^{\tan^{-1}(L/2r)} c_\phi \, d\phi$$

$$= \left(\frac{2GMm}{Lr}\right)\sin\left(\tan^{-1}\left(\frac{L}{2r}\right)\right) = \boxed{\frac{2GMm}{r\sqrt{L^2+4r^2}}}$$

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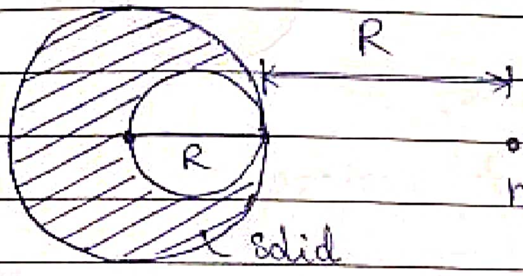
Q)



If mass of orig. sphere is M, find force on 'm'.

A)
$$\vec{F}_{m,\,⊙} = \vec{F}_{m,O} - \vec{F}_{m,o}$$

$$F_{m,O} = \frac{Gm}{(R+R)^2}\cdot M \quad , \quad F_{m,o} = \frac{Gm}{(R+R/2)^2}\cdot\frac{M}{8}$$

$$\Rightarrow F_{m,\,⊙} = \frac{GMm}{4R^2} - \frac{GMm}{\left(\frac{9R^2}{4}\right)(8)}$$

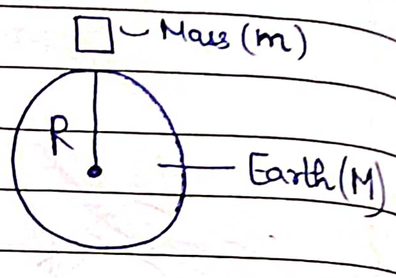
$$= \frac{GMm}{4R^2} - \frac{GMm}{18R^2}$$

$$\Rightarrow \boxed{F = \frac{7GMm}{36R^2}}$$

Gravitational Field :

Force on unit mass is termed gravitational field.

Acc. due to gravity :



$$F_G = \frac{GMm}{R^2} = mg$$

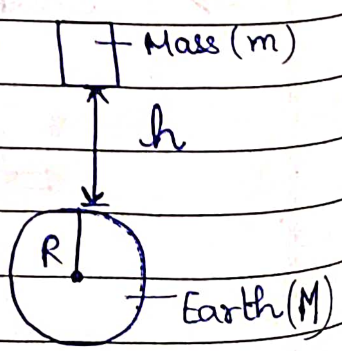
$$\Rightarrow \boxed{g = \frac{GM}{R^2}} \text{ — at surface.}$$

If Earth be spherical with uniform density 'ρ', then

$$\Rightarrow g = \left(\frac{G}{R^2}\right)\left(\frac{4}{3}\pi R^3 \rho\right) \Rightarrow \underbrace{\boxed{g = \frac{4}{3}\pi G R \rho}}_{\text{at surface}}$$

Variation in 'g' with height :

If 'h' height above surface,



$$mg' = \frac{GMm}{(R+h)^2} \Rightarrow g' = \frac{GM}{(R+h)^2}.$$

$$\Rightarrow \boxed{g' = g\left(\frac{R}{R+h}\right)^2}$$

If $h << R$, $\boxed{g' = g\left(1 - \frac{2h}{R}\right)}$

Q) Find height at which value of 'g' becomes 1 % of value of 'g' at Earth's surface.

A) $g' = g\left(\frac{R}{R+h}\right)^2 \Rightarrow \left(\frac{g'}{g}\right) = \frac{1}{100} = \left(\frac{R}{R+h}\right)^2$

$\Rightarrow \boxed{h = 9R}$

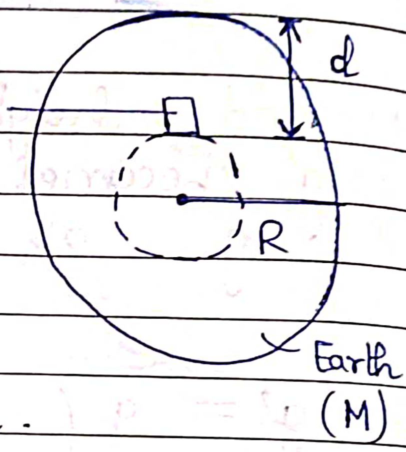
Q) In above Q, find height ~~is~~ if value of 'g' is decreased by 1 %.

A) $g' = g\left(\frac{R}{R+h}\right)^2 \Rightarrow \left(\frac{g'}{g}\right) = \left(\frac{99}{100}\right) = \left(\frac{1}{1+h/R}\right)^2$

$\Rightarrow \left(\frac{99}{100}\right) \sim \left(1 - \frac{2h}{R}\right) \Rightarrow \boxed{h = \left(\frac{R}{200}\right)}$

Variation in 'g' with depth :

If 'd' depth below surface,



Only sphere with radius (R−d) ~~with~~ will exert force.

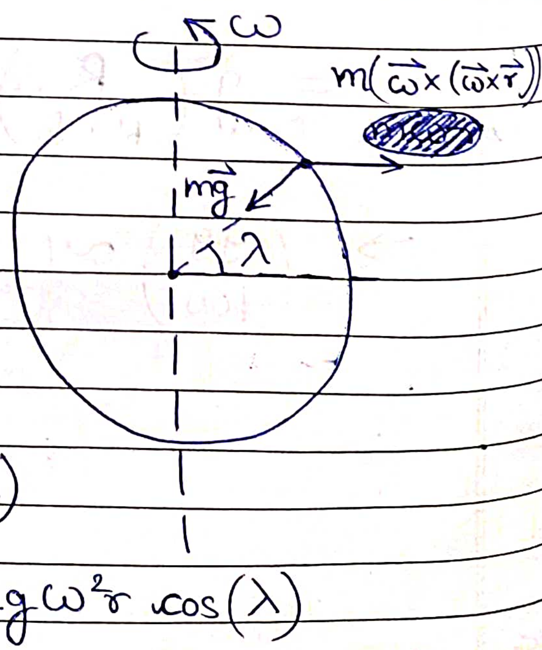
Force due to rest will cancel.

$$\Rightarrow mg' = \frac{Gm}{(R-d)^2} \cdot M\left(\frac{R-d}{R}\right)^3$$

$$\Rightarrow g' = \frac{GM(R-d)}{R^3} \Rightarrow \boxed{g' = g\left(1-\frac{d}{R}\right)}$$

Variation in 'g' with Earth's rotation

$$\boxed{g' = g - \omega^2 R \cos^2(\lambda)}$$



Proof: $m\vec{g'} = m\vec{g} + m\vec{\omega}\times(\vec{\omega}\times\vec{r})$

$$\Rightarrow (mg')^2 = (mg)^2 + (m\omega^2 r)^2$$
$$2(mg)(m\omega^2 r)\cos(\lambda)$$

$$\Rightarrow (g')^2 = g^2 + (\omega^2 r)^2 - 2g\omega^2 r\cos(\lambda)$$

Assuming ω small $\Rightarrow$ $\omega^4 \approx 0$ & $\omega^2 \ll 1$

$$\Rightarrow \quad g' = g\sqrt{1 - \frac{2\omega^2 r \cos(\lambda)}{g}} \sim g\left(1 - \frac{1}{2} \cdot \frac{2\omega^2 r \cos(\lambda)}{g}\right)$$

$$\{r = R\cos(\lambda)\}$$

$$\Rightarrow \quad \boxed{g' = g - \omega^2 R \cos^2(\lambda)}$$

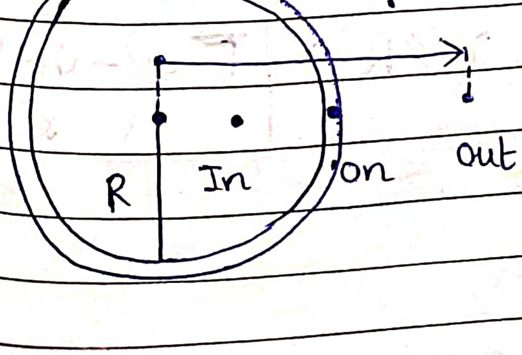
Now,

$$\boxed{g_{Equator} = g - \omega^2 R}$$

$$\boxed{g_{Poles} = g}$$

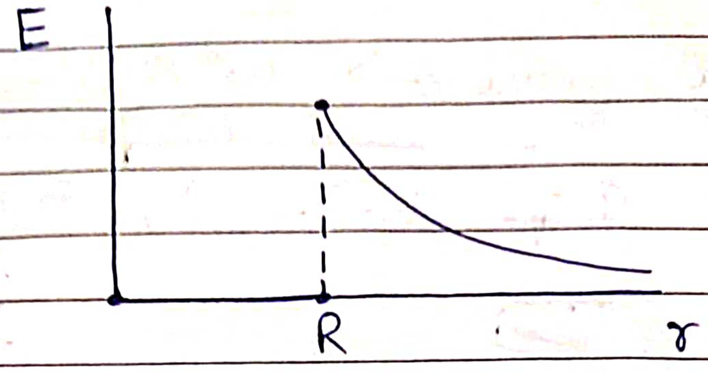
Gravitational field in Various ~~Sphe~~ Shapes

1) Hollow Sphere —

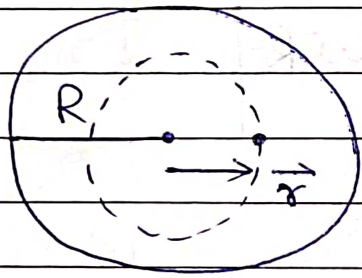


$r < R \Rightarrow$ $\boxed{\vec{E} = 0}$

$r \geq R \Rightarrow$ $\boxed{\vec{E} = \left(-\frac{GM}{r^2}\right)\hat{r}}$

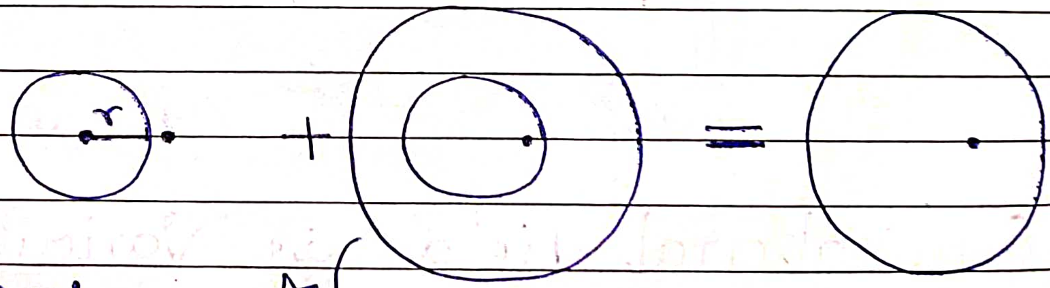


2) Solid Sphere —

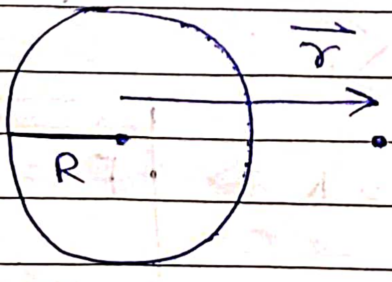


$r \leq R \Rightarrow \boxed{\vec{E} = \left(-\frac{GMr}{R^3}\right)\hat{r}}$

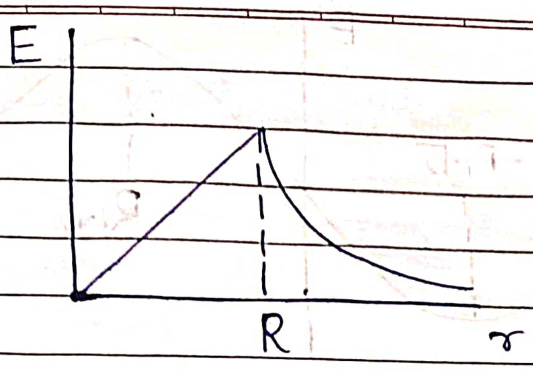
Proof:



$\left(M' = \frac{Mr^3}{R^3}\right)$ $E = \frac{GM'}{r^2}$ ☆ $E = 0$

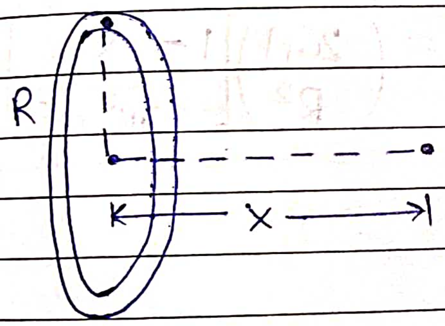


$r \geq R \Rightarrow \boxed{\vec{E} = \left(-\frac{GM}{r^2}\right)\hat{r}}$



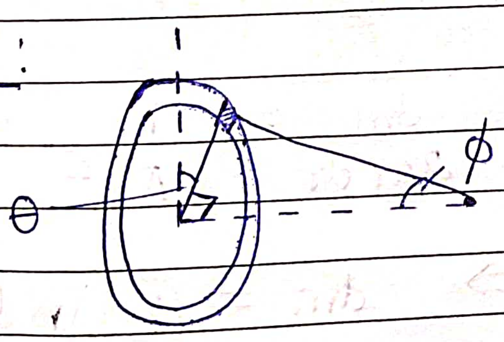
3) Ring –

For any pt. on axis of ring.



$$\boxed{\vec{E} = \left(\frac{-GMx}{(R^2+x^2)^{3/2}} \right) \hat{x}}$$

Proof :



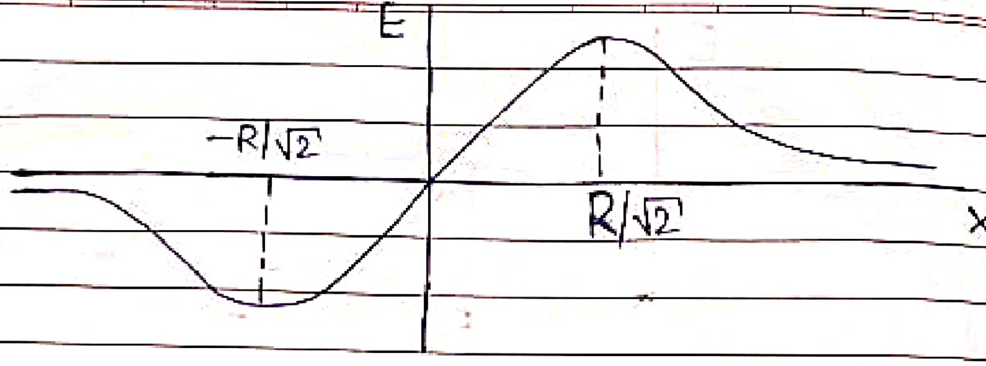
$$t\phi = R/x$$

$$\frac{dm}{R\,d\theta} = \frac{M}{2\pi R}$$

$$\Rightarrow \quad dm = M/2\pi \; d\theta$$

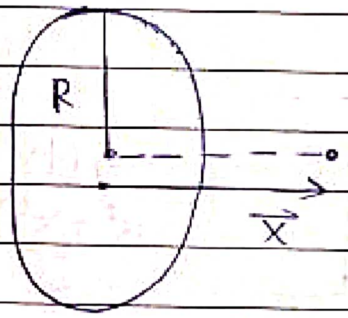
$$E_x = \int \frac{c\phi\, G\, dm}{(R^2+x^2)} = \frac{G\, c\phi}{(R^2+x^2)} \int_0^{2\pi} \left(\frac{M}{2\pi}\right) d\theta \Rightarrow \boxed{E_x = \frac{GMx}{(R^2+x^2)^{3/2}}}$$

$E_y = E_z = 0$ due to symmetry



4) Disc —

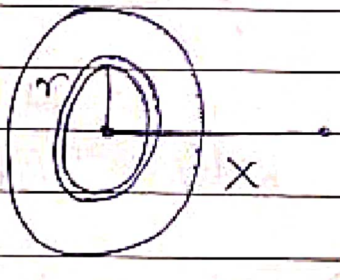
for any pt. on axis of disc.



$$\boxed{\vec{E} = \left(\frac{-2GM}{R^2}\right)\left[1 - \frac{x}{\sqrt{R^2+x^2}}\right]\hat{x}}$$

$(x > 0)$

Proof:

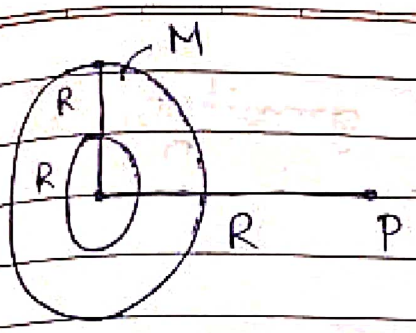


$$\frac{dm}{2\pi r\,dr} = \frac{M}{\pi R^2}$$

$$\Rightarrow \quad dm = \left(\frac{2M}{R^2}\right) r\,dr$$

$$E_x = \int \frac{Gx}{(r^2+x^2)^{3/2}}\,dm = \left(\frac{GMx}{R^2}\right)\int_0^R \frac{2r}{(r^2+x^2)^{3/2}}\,dr$$

$$= \left(\frac{GMx}{R^2}\right)\left[\frac{-2}{(r^2+x^2)^{1/2}}\right]_0^R = \boxed{\left(\frac{2GMx}{R^2}\right)\left[\frac{1}{x} - \frac{1}{\sqrt{x^2+R^2}}\right]}$$

Q)


find gravitational field at P.

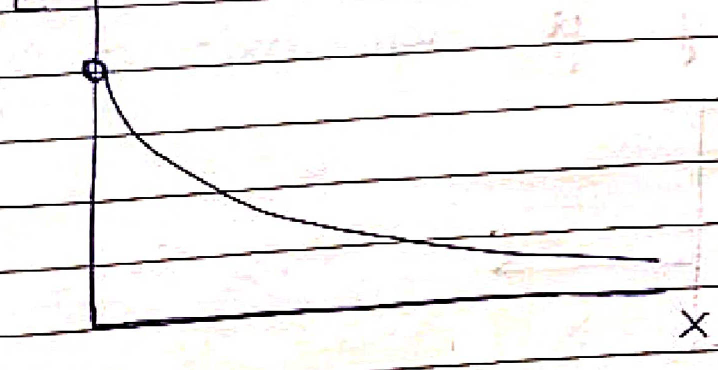
A) $\lambda = \dfrac{M}{\pi(2R)^2 - \pi R^2} \Rightarrow \lambda = \left(\dfrac{M}{3\pi R^2}\right)$

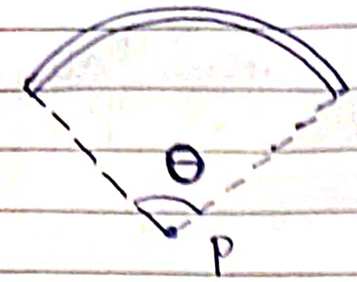
$$E_{0} = E_{0} - E_{0} = \left(\frac{2G}{(2R)^2}\right)\left[1 - \frac{R}{\sqrt{(2R)^2 + R^2}}\right]\pi(2R)^2\lambda$$

$$- \left(\frac{2G}{R^2}\right)\left[1 - \frac{R}{\sqrt{R^2+R^2}}\right]\pi R^2 \lambda$$

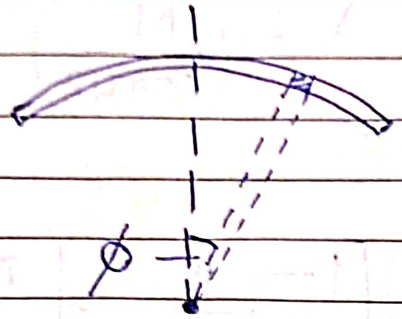
$$= (2\pi G\lambda)\left[\frac{R}{R\sqrt{2}} - \frac{R}{R\sqrt{5}}\right]$$

$$= \boxed{\left(\frac{2GM}{3R^2}\right)\left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{5}}\right]}$$



Q) 

Find gravitational field at P.

A) 

$E_x = 0$ (By Symmetry)

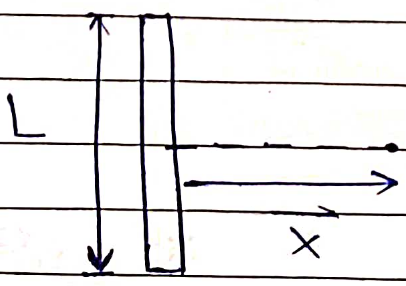
$$\frac{M}{R\theta} = \frac{dm}{R\,d\phi} \Rightarrow dm = \frac{M\,d\phi}{\theta}$$

$$E_y = \int \frac{G\cos\phi\,dm}{R^2} = \left(\frac{GM}{\theta R^2}\right)\int_{-\theta/2}^{\theta/2} \cos\phi\,d\phi$$

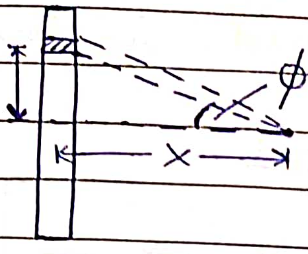
$$\Rightarrow \boxed{E_y = \left(\frac{2GM}{R^2}\right)\left(\frac{\sin\theta/2}{\theta}\right)}$$

5) Rod —

For any pt. on axis of symmetry of rod



Proof:



$E_y = 0$ (By Symmetry)

dm $\quad dm = \lambda\, dt$

$$E_x = \int \frac{G\, dm\, c_\phi}{(x^2+t^2)} = (G\lambda)\int \frac{c_\phi\, dt}{(x^2+t^2)}$$

Let us convert everything into $\phi \Rightarrow t = x\, t_\phi$

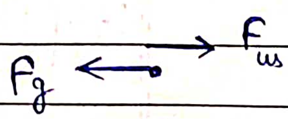
$$\Rightarrow \cancel{E_x}\ E_x = (G\lambda)\int_{-\tan^{-1}\left(\frac{L}{2x}\right)}^{\tan^{-1}(L/2x)} \frac{c_\phi}{S_\phi^2} S_\phi^2\, d\phi$$

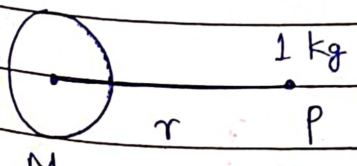
$$\Rightarrow E_x = (G\lambda)\left(s_\phi\right)_{-\tan^{-1}(L/2x)}^{\tan^{-1}(L/2x)} = \boxed{(2G\lambda)\sin\left(\tan^{-1}\left(\frac{L}{2x}\right)\right)}$$

Gravitational Potential

Work done in (by us) moving unit mass from ∞ to pt. P w/o change in K.E.

We assume $V_\infty = 0$.





$F_{us} = GM/r^2$

P

$a = 0$

$dW_{us} = F_{us}\, dr \qquad \Rightarrow F_{us} = F_g$

$$\Rightarrow W_{us} = \int_\infty^r \frac{GM}{r^2}\, dr \Rightarrow \boxed{W = \left(-\frac{GM}{r}\right)}$$

$\Rightarrow$ $$\boxed{V = \left(\frac{-GM}{r}\right)}$$

Potential

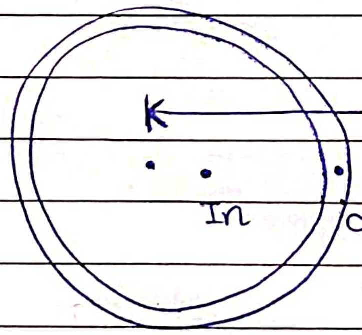
In general, $$\boxed{\vec{V} = \int -\vec{E}\cdot d\vec{r}}$$

ie. $$\boxed{\vec{E} = -\nabla V}$$

gravitational field

Gravitational Potential due to Various Shapes

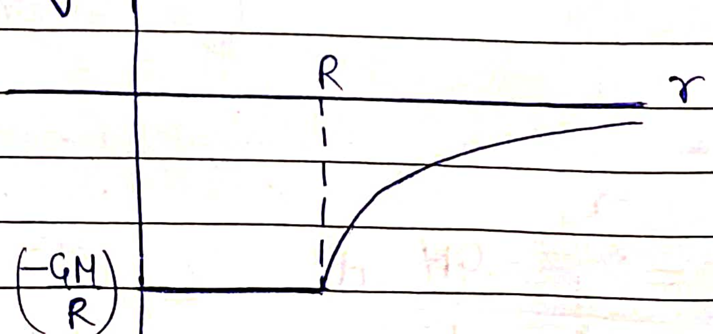
1) Hollow Sphere —



$r < R \Rightarrow$ $$\boxed{V = \left(\frac{-GM}{R}\right)}$$

$r \geq R \Rightarrow$ $$\boxed{V = \left(\frac{-GM}{r}\right)}$$

Proof: for $r < R$, $\vec{E} = 0 \Rightarrow -\int \vec{E}\cdot d\vec{r} = \text{Const.}$

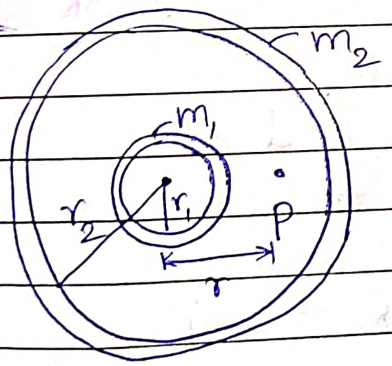


☆ Q) If $V_a = 0$, find expⁿ for V_r.

A) $\boxed{\text{We will bring obj. from pt. with 0 potential!}}$ ☆

$$dV = -\vec{E}\cdot d\vec{r} \Rightarrow \int_a^r dV = -\int_a^r \vec{E}\cdot d\vec{r}$$

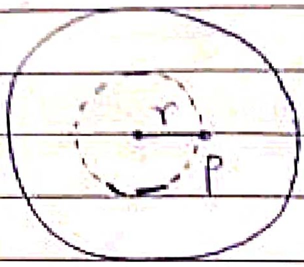
$$\Rightarrow V_r - \underbrace{\cancel{V_a}}_{(0)} = \boxed{V_r = -\int_a^r \vec{E}\cdot d\vec{r}}$$

Q) 

Find potential at P.

A) $$V_{P,\,\text{⭘}} = V_{P,O} + V_{P,\,\text{⭘}} = \boxed{-\frac{Gm_1}{r} - \frac{Gm_2}{r_2}}$$

2) Solid Sphere —

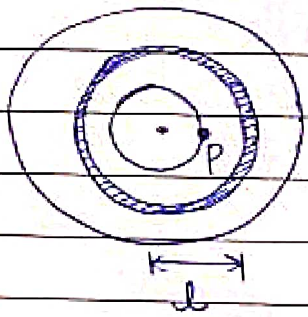


$r \leq R \Rightarrow \boxed{V = \left(\frac{-GM}{2R^3}\right)\left[3R^2 - r^2\right]}$

Proof : $V_{P,\bigcirc} = V_{P,o} + V_{P,\circledcirc}$

$$V_{P,o} = \frac{-G\,(Mr^3/R^3)}{r} = -\frac{GMr^2}{R^3}$$

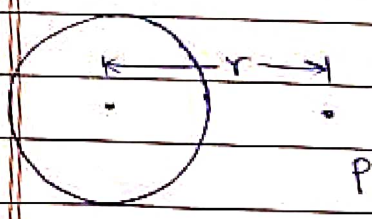
$$V_{P,\circledcirc} = \int dV_{P,\circledcirc} = \int -\frac{G\,dm}{l} = (-G)\int_r^R \frac{M}{\frac{4}{3}\pi R^3} \cdot \frac{4\pi l^2\,dl}{l}$$



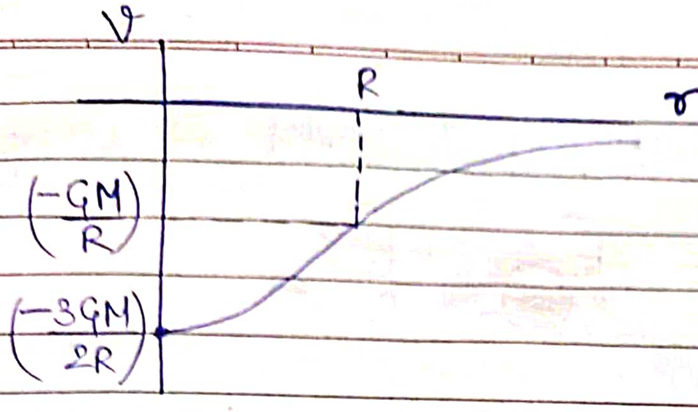
$$= \left(\frac{-3GM}{R^3}\right)\int_r^R l\,dl$$

$$= \left(\frac{-3GM}{2R^3}\right)\left(R^2 - r^2\right)$$

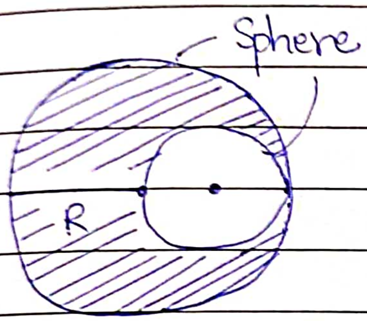
$$\Rightarrow \boxed{V_{P,\bigcirc} = \left(\frac{-GM}{2R^3}\right)\left[3R^2 - r^2\right]}$$



$r \geq R \Rightarrow \boxed{V = \left(\frac{-GM}{r}\right)}$



Q)



Find potential at centre of cavity.

Given mass of ⊘ = M

A) $V = V - V$

$$= \left(-\frac{GM}{2R^3}\right)\left[3R^2-(R/2)^2\right] - \left(-\frac{G(M/8)}{2(R/2)^3}\right)\left[3(R/2)^2-0^2\right]$$

$$= \left(-\frac{GM}{2R^3}\right)\left(\frac{11R^2}{4}\right) + \left(\frac{GM}{2R^3}\right)\left(\frac{3R^2}{4}\right)$$

$$= \left(-\frac{11GM}{8R}\right) + \left(\frac{3GM}{8R}\right) \Rightarrow \boxed{V = \left(-\frac{GM}{R}\right)}$$

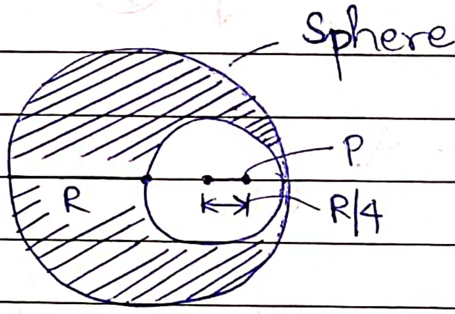
Q) In above Q, find field at centre of cavity.

A) $$\vec{E}_{P,} = \vec{E}_{P,} - \vec{E}_{P,}$$

$$= \left(-\frac{GM}{R^3}\right)\left(\frac{R}{2}\right) - \left(\frac{-G(M/8)}{(R/2)^3}\right)(0)$$

$$\Rightarrow \boxed{E = \left(-\frac{GM}{2R^2}\right)}$$

Q)



Find potential at P.
Given mass $= M$.

A) $$V_{P,} = V_{P,} - V_{P,}$$

$$= \left(-\frac{GM}{2R^3}\right)\left[3R^2 - \left(\frac{3R}{4}\right)^2\right] - \left(\frac{-G(M/8)}{2(R/2)^3}\right)\left[3\left(\frac{R}{2}\right)^2 - \left(\frac{R}{4}\right)^2\right]$$

$$= \left(-\frac{GM}{2R^3}\right)\left(\frac{39R^2}{16}\right) + \left(\frac{GM}{2R^3}\right)\left(\frac{11R^2}{16}\right)$$

$$= \left(\frac{-39GM}{32R}\right) + \left(\frac{11GM}{32R}\right) \Rightarrow \boxed{V = \left(\frac{-7GM}{8R}\right)}$$

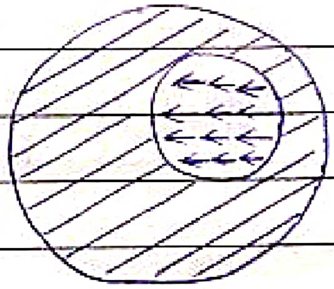
Q) In above Q, find field at P.

A) $$\vec{E}_{P,} = \vec{E}_{P,} - \vec{E}_{P,}$$

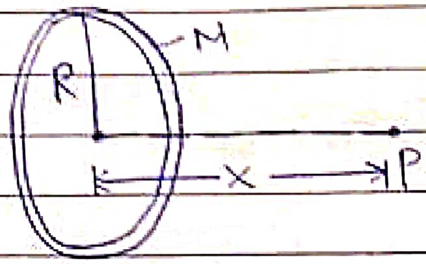
$$= \left(-\frac{GM}{R^3}\right)\left(\frac{3R}{4}\right) - \left(\frac{-G(M/8)}{(R/2)^3}\right)\left(\frac{R}{4}\right)$$

$$= \left(-\frac{3GM}{4R^2}\right) + \left(\frac{GM}{4R^2}\right) \Rightarrow \boxed{E = \left(\frac{-GM}{2R^2}\right)}$$

☆ Inside a spherical body, if there is a spherical cavity, then at any pt. inside cavity gravitation field is same, in both magnitude & dirxn

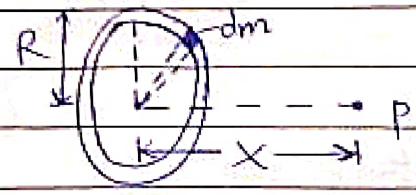


3) Ring —



$$\boxed{V = \left(\frac{-GM}{\sqrt{x^2+R^2}}\right)}$$

Proof :

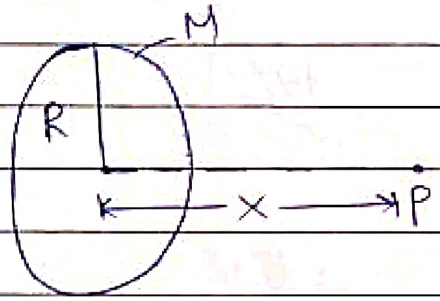


$$dV = \frac{-G\,dm}{\sqrt{R^2+x^2}}$$

$$\Rightarrow \boxed{V = \left(\frac{-GM}{\sqrt{R^2+x^2}}\right)}$$

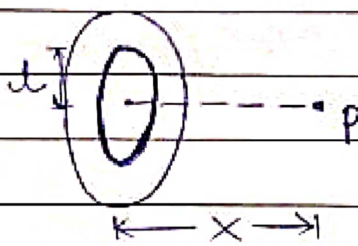
---

4) Disc —



$$\boxed{V = \left(\frac{-2GM}{R^2}\right)\left[\sqrt{R^2+x^2} - x\right]}$$

Proof :



$$dV = \frac{-G\,dm}{\sqrt{l^2+x^2}}$$

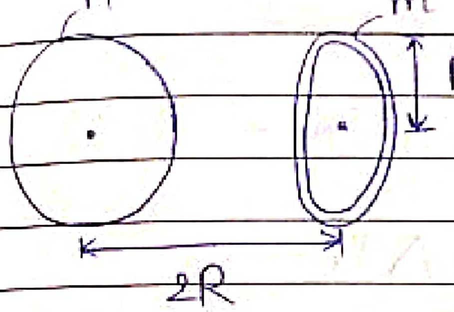
$$\Downarrow$$

$$\frac{dm}{2\pi l\,dl} = \frac{M}{\pi R^2} \Rightarrow dm = \left(\frac{2M}{R^2}\right) l\,dl \qquad dV = \left(\frac{-2GM}{R^2}\right)\frac{l\,dl}{\sqrt{l^2+x^2}}$$

$$\Rightarrow V = \left(\frac{-2GM}{R^2}\right)\int_0^R \frac{l}{\sqrt{l^2+x^2}}\,dl$$

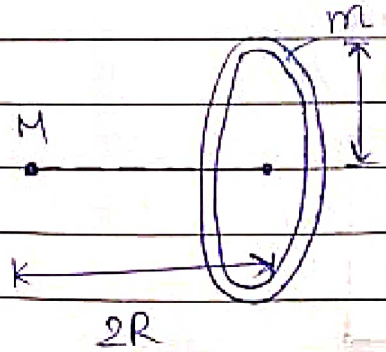
$$= \left(\frac{-2GM}{R^2}\right)\left[\sqrt{l^2+x^2}\right]_0^R \Rightarrow \boxed{V = \left(\frac{-2GM}{R^2}\right)\left[-\sqrt{R^2+x^2}-x\right]}$$

Q)



Find force b/w masses

A) Consider M as pt. mass.



$$E_{ring} = \frac{Gm(2R)}{\left(\sqrt{R^2+(2R)^2}\right)^3}$$

$$= \left(\frac{2Gm}{5^{3/2}R^2}\right)$$

$$\Rightarrow \boxed{F = \left(\frac{2GMm}{5^{3/2}R^2}\right)}$$

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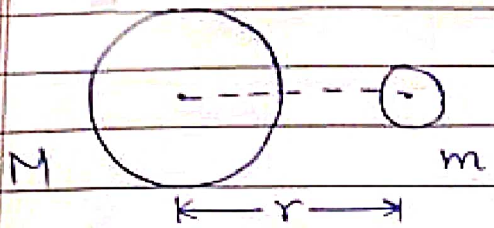
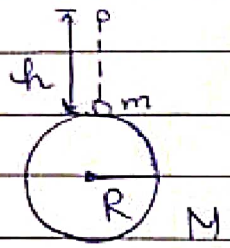
Potential Energy



$$\boxed{U = \left(-\frac{GMm}{r}\right)}$$

Q) 

find ΔU

A) $\Delta U = \left(-\frac{GMm}{(R+h)}\right) - \left(-\frac{GMm}{R}\right)$

$$\Rightarrow \quad \boxed{\Delta U = \left(\frac{GMmh}{R(R+h)}\right)}$$

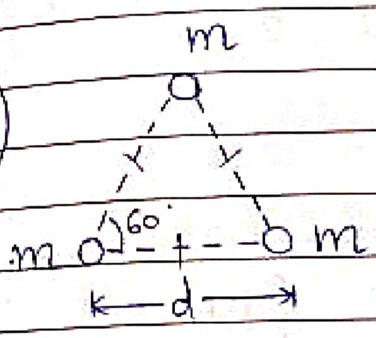
$$\Rightarrow \quad \Delta U = \left(\frac{GM}{R^2}\right)\left(\frac{mh}{1+h/R}\right)$$

$$\Rightarrow \quad \Delta U = \underbrace{g}_{\text{at surface}}\left(\frac{mh}{1+h/R}\right)$$

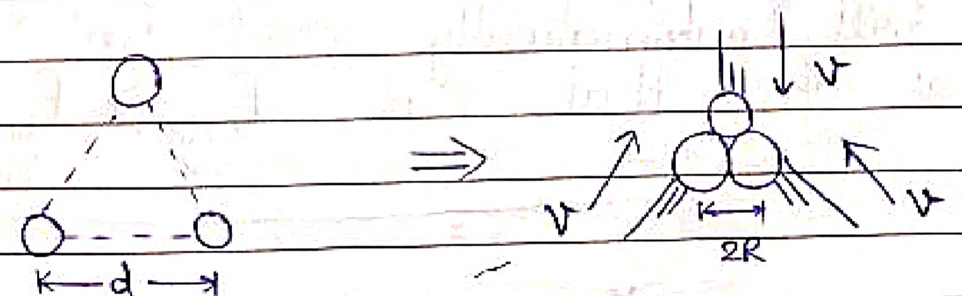
If $h \ll R \Rightarrow \boxed{\Delta U = mgh}$

Conservation of Energy

$$\boxed{U_i + K_i = U_f + K_f}$$ for isolated system

1) 

Masses are allowed to move. Find speed of each mass when they are about to collide. Radius of each is R.

A) 

for system,
$$3\left(-\frac{Gm^2}{d}\right) + 0 = 3\left(-\frac{Gm^2}{2R}\right) + \frac{3mv^2}{2}$$

$$\Rightarrow \boxed{v = \sqrt{(2Gm)\left(\frac{1}{2R} - \frac{1}{d}\right)}}$$

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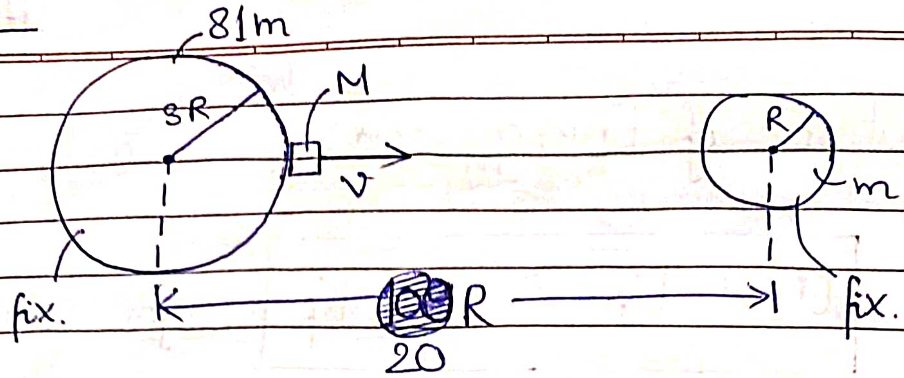
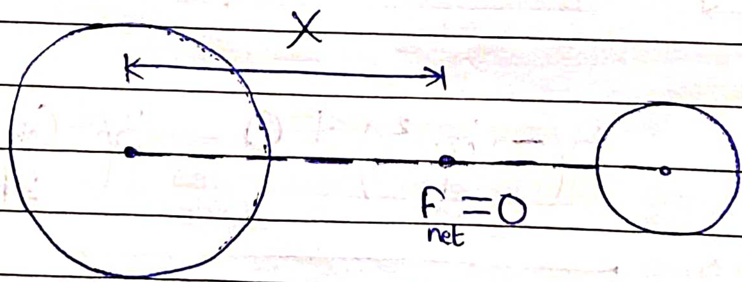
☆ Q1)



A Find min. vel. s.t. obj. just reach.

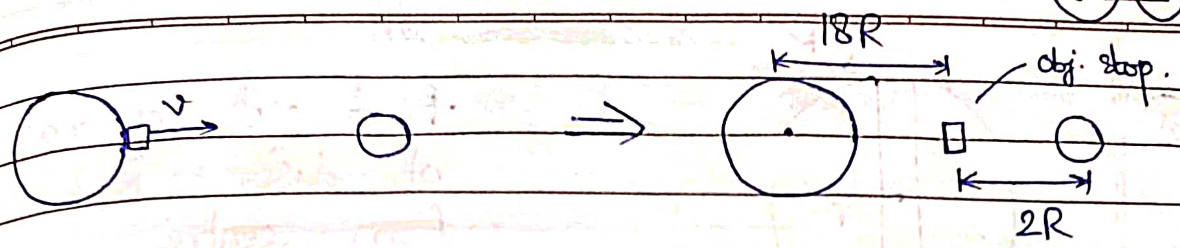
A) ~~By Energy Conserv.~~

☆ We find pt. where force due to 'm' just exceeds force due to '81m'.
If obj. just reach there, obj will automatically reach 'm' as after that pt $F_{\text{due to 'm'}} > F_{\text{due to '81m'}}$.



$$F_{net} = 0 \Rightarrow \frac{G(81m)M}{x^2} = \frac{G(m)M}{(20R-x)^2}$$

$$\Rightarrow 81 = \left(\frac{x}{20R-x}\right)^2 \Rightarrow \boxed{x = 18R}$$



$$U_i = -\frac{G(81)mM}{3R} - \frac{G(m)M}{17R} - \frac{G(81m)m}{20R}$$

$$K_i = \frac{1}{2}Mv^2$$

$$U_f = -\frac{G(81m)M}{18R} - \frac{G(m)M}{2R} - \frac{G(81m)m}{20R}$$

$$K_f = 0$$

Now, $K_i + U_i = K_f + U_f$

$$\Rightarrow \frac{1}{2}Mv^2 - 27\frac{GMm}{R} - \frac{GMm}{17R} = \left(\frac{-9}{2}\right)\frac{GMm}{R} - \frac{GMm}{2R}$$

$$\Rightarrow \frac{1}{2}Mv^2 = \left(\frac{GMm}{2R}\right)\left[\frac{2}{17} + 54 - 9 - 1\right]$$

$$\Rightarrow \boxed{v = \left(\frac{750Gm}{17R}\right)^{1/2}}$$

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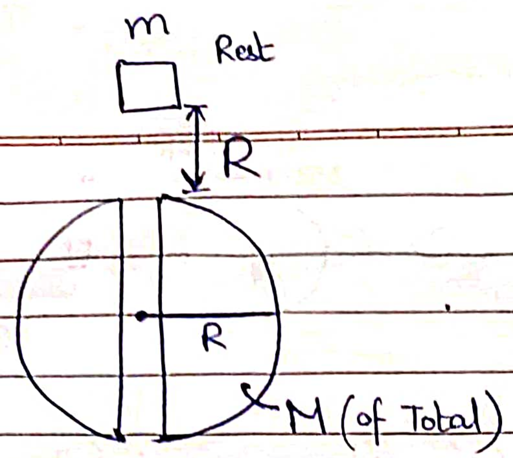
Q) 

Find vel. of mass at dist. R/2 from centre.

A) $U_i = \left(-\frac{GMm}{2R}\right)$

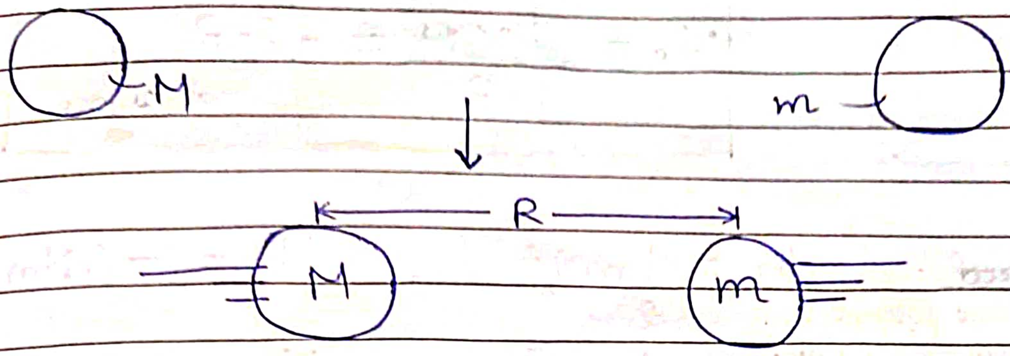
$U_f = Vm = \left(-\frac{GMm}{2R^3}\right)\left[3R^2 - (R/2)^2\right]$

$K_f = U_i + \cancel{K_i}^{(0)} - U_f = \left(\frac{-GMm}{2R}\right) + \left(\frac{GMm}{2R^3}\right)\left(\frac{11R^2}{4}\right)$

$\Rightarrow \frac{1}{2}mv^2 = \left(\frac{GMm}{2R}\right)\left(\frac{11}{4} - 1\right)$

$\Rightarrow \boxed{v = \sqrt{\frac{7GM}{4R}}}$

Q) Masses initially at rest. They start moving towards each other from large dist. due to mutual force. Find rel. vel. when seperation b/w them is R.



A) By Energy Consrv., $U_i + K_i = U_f + K_f$

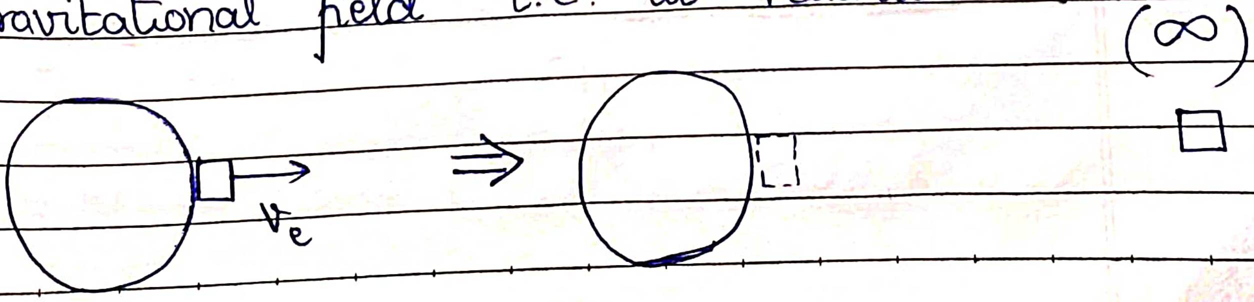
$$\Rightarrow 0 + 0 = \left(-\frac{GMm}{R}\right) + K_{f\ \underset{\text{wrt CoM}}{\text{system}}} + K_{f\ \underset{\text{wrt space}}{\text{CoM}}}$$

$$\Rightarrow \left(\frac{GMm}{R}\right) = \frac{1}{2}\left(\frac{Mm}{M+m}\right)v_{rel}^2 + 0 \left(\begin{matrix}\text{as no}\\ \text{ext. force}\end{matrix}\right)$$

$$\Rightarrow \boxed{v_{rel.} = \sqrt{\frac{2G(M+m)}{R}}}$$

Escape Velocity

Min. vel. to be given to an obj. s.t. it escapes a planet's ~~vel~~ gravitational field i.e. it reaches ∞



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$$\boxed{v_e = \sqrt{\frac{2GM}{R}} = \underbrace{\sqrt{2gR}}_{\text{at surface}}}$$

Proof: $K_i = \frac{1}{2}mv_e^2$, $U_i = \left(-\frac{GMm}{R}\right)$

$K_f = \frac{1}{2}mu^2$, $U_f = \left(-\frac{GMm}{d}\right)$

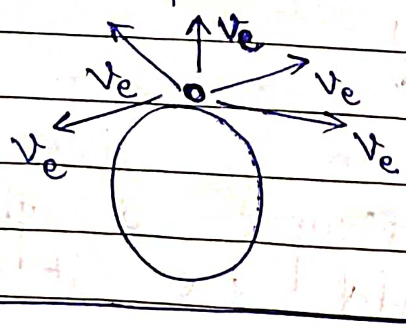
By Energy Consrv., $K_i + U_i = K_f + U_f$

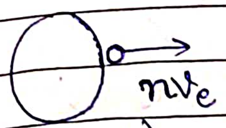
$$\Rightarrow \frac{1}{2}mv_e^2 - \frac{GMm}{R} = \frac{1}{2}mu^2 - \frac{GMm}{d}$$

To escape $d \to \infty$, and for v_{min}, $u \to 0$.

$$\Rightarrow \boxed{v_e = \sqrt{\frac{2GM}{R}}}$$

☆ 'v_e' is <u>Independent</u> of dirx^n of proj.



Q) 

$(n>1)$

Body proj. with vel. 'nv_e'. Find vel. when it has escaped.

A) By Energy Consrv., $\frac{1}{2}mv^2 = \frac{1}{2}m(nv_e)^2 - \frac{GMm}{R}$

$\Rightarrow \frac{1}{2}mv^2 = \frac{1}{2}m(nv_e)^2 - \frac{1}{2}mv_e^2$

$\Rightarrow \boxed{v = v_e\sqrt{n^2-1}}$

Q) In above Q, if $n<1$, find max. height attained by obj.

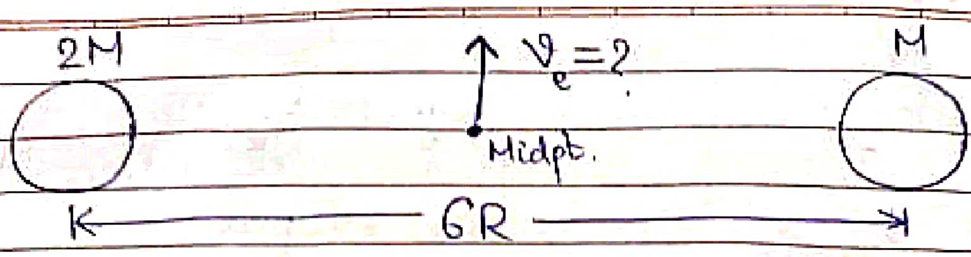
A) By Energy Consrv., $\frac{1}{2}m(nv_e)^2 - \frac{GMm}{R} = 0 - \frac{GMm}{(R+h)}$ (as obj. stop.

$\Rightarrow \frac{1}{2}m(nv_e)^2 - \frac{1}{2}mv_e^2 = \frac{1}{2}mv_e^2\left(\frac{R}{R+h}\right)$

$\Rightarrow (R+h) = \left(\frac{R}{n^2-1}\right)$

$\Rightarrow \boxed{h = \frac{n^2R}{(1-n^2)}}$

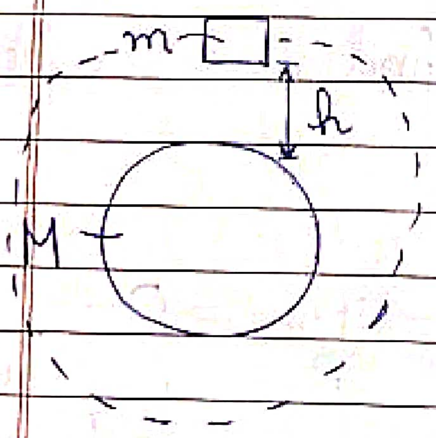
Q)



A) By Energy Conserv., $-\frac{G(2M)m}{3R} - \frac{G(M)m}{3R} + \frac{1}{2}mv_e^2 = 0$

$$\Rightarrow \boxed{v_e = \sqrt{\frac{2GM}{R}}}$$

Satellite



Satellite moving with vel. 'v_o'.

$$\boxed{v_o = \sqrt{\frac{GM}{R+h}}}$$

Orbital vel.

Proof: F_G acts as centripetal force.

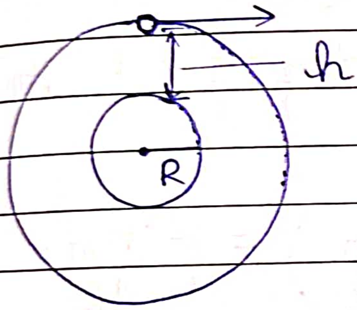
$$\frac{GMm}{(R+h)^2} = \frac{mv_o^2}{(R+h)} \Rightarrow \boxed{v_o = \sqrt{\frac{GM}{R+h}}}$$

Near Earth's surface, $h \approx 0 \Rightarrow \boxed{v_o = \sqrt{gR}}$

Energy : $\boxed{U = -\dfrac{GMm}{(R+h)}}$, $\boxed{K = \dfrac{GMm}{2(R+h)}}$

$$\boxed{T.E. = -\dfrac{GMm}{2(R+h)}}$$

☆ $\boxed{U = 2(T.E.) = (-2)(K.E)}$

Q) 

By how much % its K.E. should be increased so that it escapes from its ~~or~~ orbit.

A) $K = \dfrac{1}{2} m v_0^2 = \dfrac{GMm}{2(R+h)}$

for escape $v = v_e \Rightarrow K' = \dfrac{1}{2} m \cdot \left(\dfrac{2GM}{R+h}\right)$

$\Rightarrow K' = \left(\dfrac{GMm}{R+h}\right)$

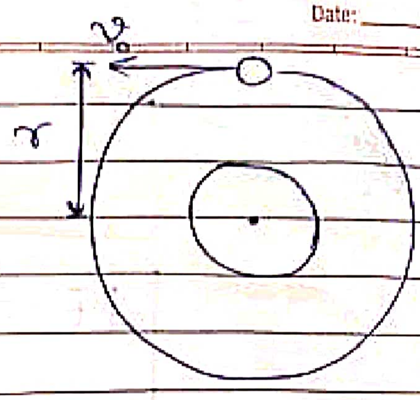
$(\% \text{ added}) = \left(\dfrac{K'-K}{K}\right) \times 100\% = \boxed{100\%}$

Better Method : $T.E_1 = K + U = (-K)$ as $(TE = (-K))$

$T.E_2 = K' + U = 0$ (as obj escape)

$\Rightarrow (\text{Energy added}) = TE_2 - TE_1 = K' - K = 0 - (-K) \Rightarrow \boxed{K' = 2K}$

Time Period :



$$V_0 = \sqrt{\frac{GM}{r}}$$

$$T = \left(\frac{2\pi r}{V_0}\right)$$

$$\Rightarrow \boxed{T = \left(\frac{2\pi}{\sqrt{GM}}\right)(r)^{3/2}}$$

Q) Find work done in moving a satellite from $h = R$ to $h = 2R$.

A) $$TE_1 = \left(\frac{U_1}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{-GMm}{R+R}\right) = \left(\frac{-GMm}{4R}\right)$$

$$TE_2 = \left(\frac{U_2}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{-GMm}{R+2R}\right) = \left(\frac{-GMm}{6R}\right)$$

$$\boxed{(\text{Work done})} = (TE_2 - TE_1) = \boxed{\frac{GMm}{12R}}$$

Q) Find work done in putting a satellite into orbit of $h = R$

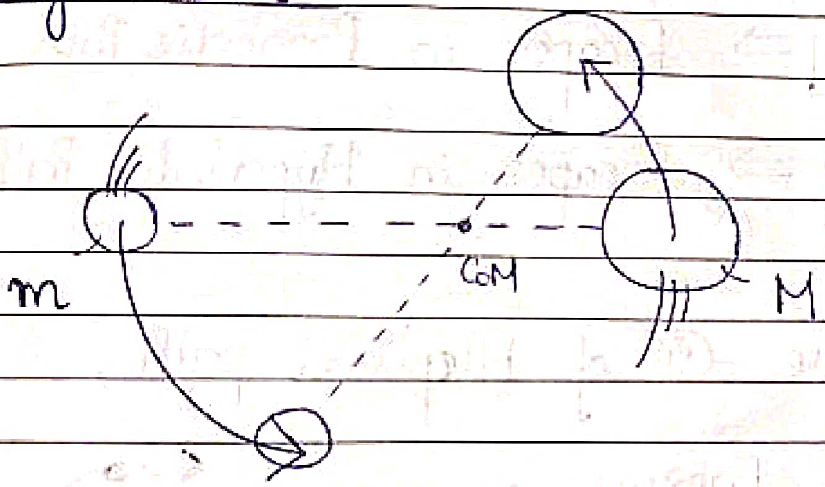
A) $$W = \Delta K + \Delta U = TE_2 - TE_1 = \left(\frac{1}{2}\right)(U_2) - (U_1)$$

as not in orbit.

$$\Rightarrow \quad W = \left(\frac{1}{2}\right)\left(\frac{-GMm}{R+R}\right) + \left(\frac{GMm}{R}\right)$$

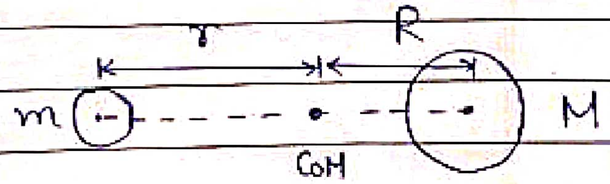
$$\Rightarrow \quad \boxed{W = \left(\frac{3GMm}{4R}\right)}$$

Binary Stars



Both rotate about CoM. 'm', 'M' and CoM always collinear.

F_G acts as centripetal for both.

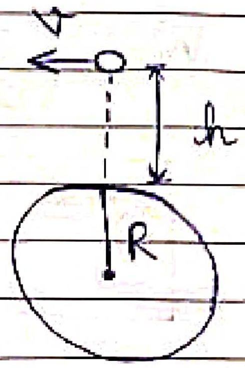


$$F_G = \frac{GMm}{(R+r)^2} = m\omega r^2$$

$$F_G = \frac{GMm}{(R+r)^2} = M\omega R^2$$

(Both rotate with same ω!)

Paths of Projection



$\boxed{v < v_o} \Rightarrow$ Body will fall on Earth

$\boxed{v = v_o} \Rightarrow$ Circular Path

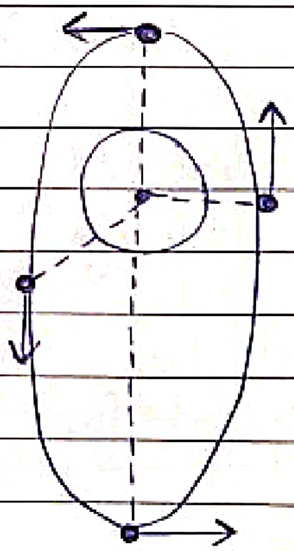
$\boxed{v_o < v < v_e} \Rightarrow$ Elliptical Path

$\boxed{v = v_e} \Rightarrow$ Escapes in Parabolic Path

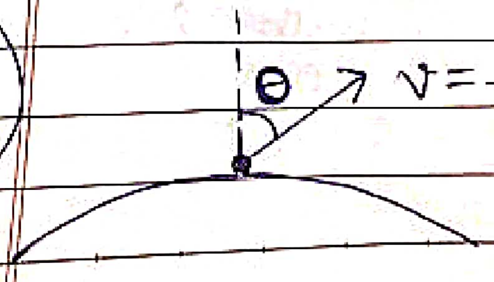
$\boxed{v > v_e} \Rightarrow$ Escapes in Hyperbolic Path

To solve Q of Elliptical path,

1) Consrv. Energy
2) Consrv. Angular Momentum.



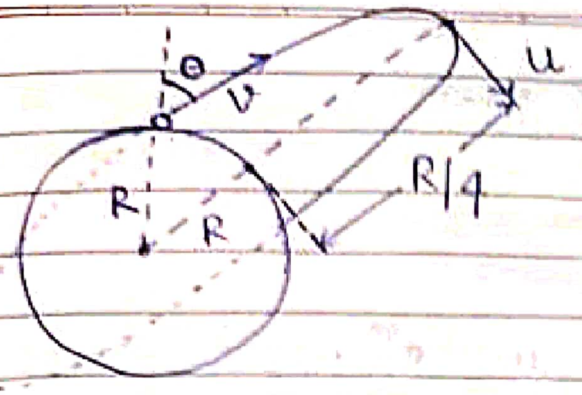
☆ Q)



$v = \sqrt{gR/2}$

Max. height reached is $R/4$. Find angle of proj. from vertical & vel. at max. height.

A)



At highest pt: vel. || to Earth's surface.

Energy Consrv.,
$$\frac{1}{2}mv^2 - \frac{GMm}{R} = \frac{1}{2}mu^2 - \frac{GMm}{(5R/4)}$$

$$\Rightarrow \frac{1}{2}m\cdot\frac{gR}{2} - mgR = \frac{1}{2}mu^2 - \frac{4}{5}mgR$$

$$\Rightarrow \frac{1}{2}mu^2 = \frac{mgR}{4} - mgR + \frac{4}{5}mgR \Rightarrow \boxed{u = \sqrt{\frac{gR}{10}}}$$

Angular Momentum Consrv. (abt centre of Earth),
$$mv\sin\theta\, R = mu\,(5R/4)$$

$$\Rightarrow \sqrt{\frac{gR}{2}}\sin\theta\, R = \sqrt{\frac{gR}{10}}\cdot\frac{5R}{4}$$

$$\Rightarrow \boxed{\theta = \sin^{-1}\left(\frac{\sqrt{5}}{4}\right)}$$

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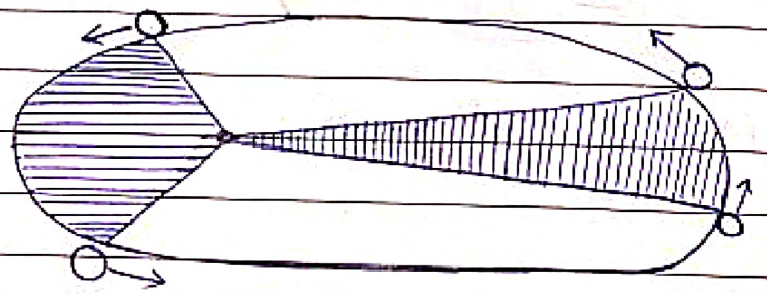
Kepler's Laws of Motion

First Law –

All planets move in elliptical orbits with Sun at one of the foci.

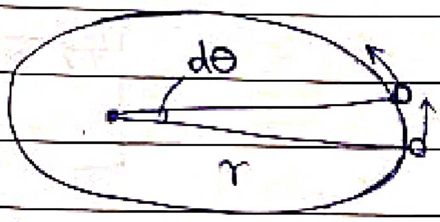
Second law –

Areal vel. is const.



$$\boxed{\left(\frac{A_1}{t_1}\right) = \left(\frac{A_2}{t_2}\right)}$$

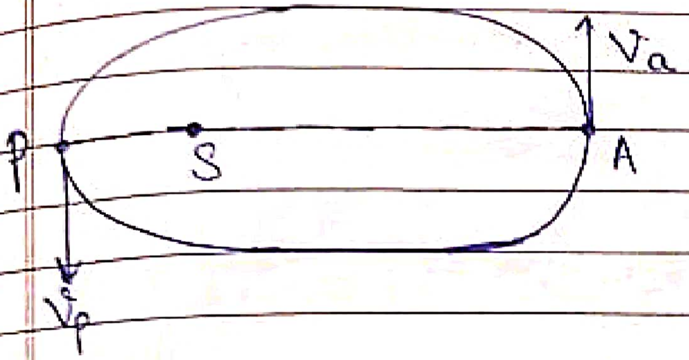
Proof: $dA = \frac{1}{2} r^2 \, d\theta$



$$\Rightarrow \left(\frac{dA}{dt}\right) = \frac{1}{2} r^2 \left(\frac{d\theta}{dt}\right) = \frac{\omega r^2}{2}$$

$$\Rightarrow \left(\frac{dA}{dt}\right) = \frac{1}{2}\omega\left(\frac{mr^2}{m}\right) = \frac{1}{2}\omega\left(\frac{I}{m}\right) = \frac{(I\omega)}{2m} = \frac{L}{2m}$$

L is const. as $\tau_{ext} = 0 \Rightarrow$ $dA/dt = \text{Const.}$



A = Apogee (farthest away)

P = Perigee (closest to S)

$$\boxed{V_p = \sqrt{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)}}$$

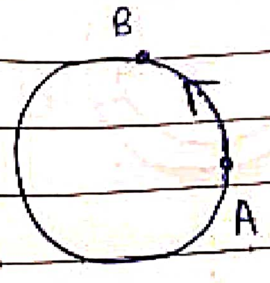
$$\boxed{V_a = \sqrt{\frac{GM}{a}\left(\frac{1-e}{1+e}\right)}}$$

a = Length of Semi-Major axis

e = Eccentricity of Ellipse.

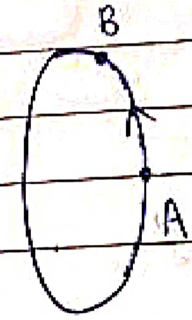
Circular	Elliptical
1) Speed Const.	1) Speed changes
$\boxed{v \propto 1/\sqrt{r}}$ at A & P	$\boxed{v \propto 1/r}$ at A & P

2)



$W_{AB} = 0$

2)



$W_{AB} \neq 0$ but $W_{\text{Closed Loop}} = 0$

Third Law —

$$\boxed{T^2 \propto a^3}$$

T — Time Period a — Semi major axis

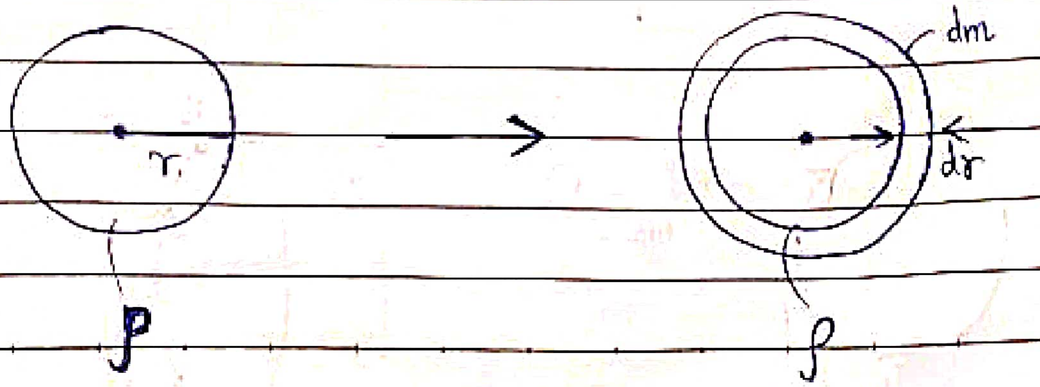
Self Energy

Potential Energy stored in the body during its formation, i.e. by virtue of its existence

For a sphere, —

Let us assume $\exists$ a sphere of ρ density and radius r.

We will find work done to make it into a sphere with radius $(r+dr)$, and integrate these small works to find total work done.



$$dW = \left(V_{\substack{\text{due to } \odot \\ \text{of radius 'r'}}}\right)\overbrace{(dm)}^{\text{added mass}} - \underbrace{0}_{\left(\substack{\text{as obj.} \\ \text{brought from } \infty}\right)}$$

$$\Rightarrow dW = \underbrace{\left(\frac{-G\left(\frac{4}{3}\pi r^3 \rho\right)}{r}\right)}_{V}\underbrace{\left(\rho 4\pi r^2 dr\right)}_{dm}$$

$$\Rightarrow \int dW = \int\left(\frac{-G(16\pi^2\rho^2)}{3}\right)\left(r^4\, dr\right)$$

$$\Rightarrow W = \left(\frac{-16G\pi^2\rho^2}{3}\right)\int_0^R r^4\, dr = \left(\frac{-16G\pi^2\rho^2 R^5}{15}\right)$$

$$\Rightarrow W = \left(\frac{-16G\pi^2R^5}{15}\right)\left(\frac{M}{\frac{4}{3}\pi R^3}\right)^2$$

$$\Rightarrow \boxed{W = \left(\frac{3GM^2}{5R}\right)}$$

for a hollow sphere —

Let us assume $\exists$ a hollow sphere of mass 'm' and radius 'R'

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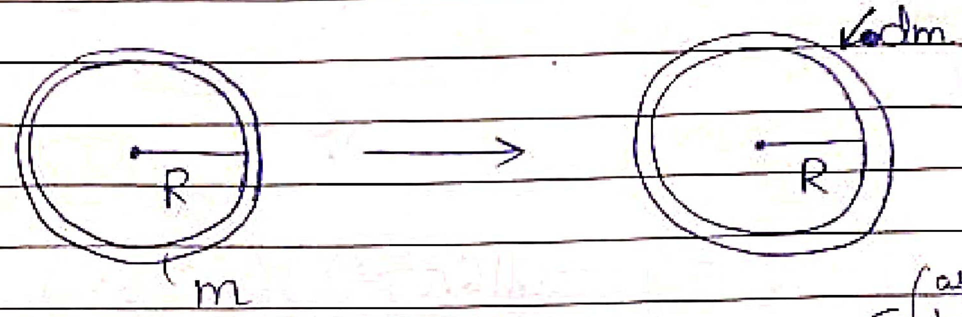
We will find work done to increase its mass by 'dm' and integrate these small works to find total work done.



(as obj. brought from ∞)

$$dW = \left(V_{\text{due to hollow } \odot \text{ of radius 'R'}}\right)(dm) - 0$$

added mass

$$\Rightarrow \quad dW = \left(-\frac{Gm}{R}\right)(dm)$$

$$\Rightarrow \quad W = \left(\frac{-G}{R}\right)\int_0^M m\, dm$$

$$\Rightarrow \quad \boxed{W = \left(-\frac{GM^2}{2R}\right)}$$