MA-105
Problem Sheets
Autumn Semester 2024

MANOJ KESHARI
SWAPNEEL MAHAJAN

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai 400076, India

Contents

CHAPTER 1

Tutorial problems

1.0. Revision material on real numbers

Mark the following statements as True/False:

(1) $+\infty$ and $-\infty$ are both real numbers.
(2) The set of all even natural numbers is bounded.
(3) The set $\{x\}$ is an open interval for every $x \in \mathbb{R}$.
(4) The set $\{2/m | m \in \mathbb{N}\}$ is bounded above.
(5) The set $\{2/m | m \in \mathbb{N}\}$ is bounded below.
(6) Union of intervals is also an interval.
(7) Nonempty intersection of intervals is also an interval.
(8) Nonempty intersection of open intervals is also an open interval.
(9) Nonempty intersection of closed intervals is also a closed interval.
(10) Nonempty finite intersection of closed intervals is also a closed interval.
(11) For every $x \in \mathbb{R}$, there exists a rational $r \in \mathbb{Q}$, such that $r > x$.
(12) Between any two rational numbers there lies an irrational number.

1.1. Sequences

(1) Using (ϵ-N) definition prove the following:

(i) $\lim_{n\to\infty} \frac{10}{n} = 0.$

(ii) $\lim_{n\to\infty} \frac{5}{3n+1} = 0.$

(iii) $\lim_{n\to\infty} \frac{n^{2/3}\sin(n!)}{n+1} = 0.$

(iv) $\lim_{n\to\infty} \left(\frac{n}{n+1} - \frac{n+1}{n}\right) = 0.$

(2) Show that the following limits exist and find them :

(i) $\lim_{n\to\infty} \left(\frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n}\right).$

(ii) $\lim_{n\to\infty} \left(\frac{n!}{n^n}\right).$

(iii) $\lim_{n\to\infty} \left(\frac{n^3+3n^2+1}{n^4+8n^2+2}\right).$

(iv) $\lim_{n\to\infty} (n)^{1/n}.$

(v) $\lim_{n\to\infty} \left(\frac{\cos \pi\sqrt{n}}{n^2}\right).$

(vi) $\lim_{n\to\infty} \left(\sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)\right).$

(3) Show that the following sequences are not convergent :

(i) $\{\frac{n^2}{n+1}\}_{n\geq 1}.$

(ii) $\left\{(-1)^n\left(\frac{1}{2} - \frac{1}{n}\right)\right\}_{n\geq 1}.$

(4) Determine whether the sequences are increasing or decreasing:

(i) $\left\{\frac{n}{n^2+1}\right\}_{n\geq 1}.$

(ii) $\left\{\frac{2^n 3^n}{5^{n+1}}\right\}_{n\geq 1}.$

(iii) $\left\{\frac{1-n}{n^2}\right\}_{n\geq 2}.$

(5) Prove that the following sequences are convergent by showing that they are monotone and bounded. Also find their limits :

(i) $a_1 = 1, a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$ for all $n \geq 1$.

(ii) $a_1 = \sqrt{2}, a_{n+1} = \sqrt{2+a_n}$ for all $n \geq 1$.

(iii) $a_1 = 2,\ a_{n+1} = 3 + \frac{a_n}{2}$ for all $n \geq 1$.

(6) If $\lim_{n\to\infty} a_n = L$, find the following : $\lim_{n\to\infty} a_{n+1}$, $\lim_{n\to\infty} |a_n|$.

(7) If $\lim_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that

$$|a_n| \geq \frac{|L|}{2} \text{ for all } n \geq n_0.$$

(8) If $a_n \geq 0$ and $\lim_{n\to\infty} a_n = 0$, show that $\lim_{n\to\infty} a_n^{1/2} = 0$.

(**Optional:** State and prove a corresponding result if $a_n \to L > 0$.)

(9) For given sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, prove or disprove the following :

(i) $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent.

(ii) $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent and $\{b_n\}_{n\geq 1}$ is bounded.

(10) Show that a sequence $\{a_n\}_{n\geq 1}$ is convergent iff both the subsequences $\{a_{2n}\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ are convergent to the same limit.

Supplement

(1) A sequence $\{a_n\}_{n\geq 1}$ is said to be **Cauchy** if for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for all $m, n \geq n_0$.
In other words, the elements of a Cauchy sequence come arbitrarily close to each other after some stage. One can show that *every convergent sequence is also Cauchy and conversely, every Cauchy sequence in* $\mathbb{R}$ *is also convergent.* This is an equivalent way of stating the **Completeness property of real numbers**.

(2) To prove that a sequence $\{a_n\}_{n\geq 1}$ is convergent to L, one needs to find a real number L (not given by the sequences) and verify the required property. However the concept of 'Cauchyness' of a sequence is purely an 'intrinsic' property which can be verified purely for the given sequence. Still a sequence is Cauchy if and only if it is convergent.

(3) In problem 5(i) we defined

$$a_0 = 1, a_{n+1} = \frac{1}{2}(a_n + \frac{2}{a_n}) \text{ for all } n \geq 1.$$

The sequence $\{a_n\}_{n\geq 1}$ is a monotonically decreasing sequence of rational numbers which is bounded below. However, it cannot converge to a rational (why?). This exhibits the need to enlarge the concept of numbers beyond rational numbers. The sequence $\{a_n\}_{n\geq 1}$ converges to $\sqrt{2}$ and its elements a_n's are used to find rational approximation (in computing machines) of $\sqrt{2}$.

1.2. Limits, continuity, differentiability

(1) Let $a < c < b$ and $f, g : (a, b) \to \mathbb{R}$ be such that $\lim_{x\to c} f(x) = 0$. Prove or disprove the following statements:
 (i) $\lim_{x\to c} [f(x)g(x)] = 0$.
 (ii) $\lim_{x\to c} [f(x)g(x)] = 0$, if g is bounded.
 (iii) $\lim_{x\to c} [f(x)g(x)] = 0$, if $\lim_{x\to c} g(x)$ exists.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{x\to\alpha} f(x)$ exists for $\alpha \in \mathbb{R}$. Show that

$$\lim_{h\to 0} [f(\alpha + h) - f(\alpha - h)] = 0.$$

Analyze the converse.

(3) Discuss the continuity of the following functions :
 (i) $f(x) = \sin \frac{1}{x}$, if $x \neq 0$ and $f(0) = 0$.
 (ii) $f(x) = x \sin \frac{1}{x}$, if $x \neq 0$ and $f(0) = 0$.
 (iii) $f(x) = \begin{cases} \frac{x}{[x]} & \text{if } \ 1 \le x < 2 \\ 1 & \text{if } \ x = 2 \\ \sqrt{6-x} & \text{if } \ 2 \le x \le 3 \end{cases}$

(4) Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 0, show that f is continuous at every $c \in \mathbb{R}$.
(**Optional** Show that the function f satisfies $f(kx) = kf(x)$, for all $k \in \mathbb{R}$).

(5) Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable on $\mathbb{R}$. Is $f^{'}$ a continuous function?

(6) Let $f : (a, b) \to \mathbb{R}$ be a function such that

$$|f(x + h) - f(x)| \le C|h|^\alpha$$

for all $x, x + h \in (a, b)$, where C is a constant and $\alpha > 1$. Show that f is differentiable on (a, b) and compute $f^{'}(x)$ for $x \in (a, b)$.

(7) If $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$, then show that

$$\lim_{h\to 0^+} \frac{f(c + h) - f(c - h)}{2h}$$

exists and equals $f^{'}(c)$. Is the converse true ? [Hint: Consider $f(x) = |x|$.]

(8) Let $f : \mathbb{R} \to \mathbb{R}$ satisfy

$$f(x + y) = f(x)f(y) \text{ for all } x, y \in \mathbb{R}.$$

If f is differentiable at 0, then show that f is differentiable at every $c \in \mathbb{R}$ and $f^{'}(c) = f^{'}(0)f(c)$.
 (**Optional** Show that f has a derivative of every order on $\mathbb{R}$.)

(9) Using the theorem on derivative of inverse function. Compute the derivative of
 (i) $\cos^{-1} x, \ -1 < x < 1$.
 (ii) $\text{cosec}^{-1} x, |x| > 1$.

(10) Compute $\frac{dy}{dx}$, given

$$y = f\left(\frac{2x-1}{x+1}\right) \text{and} f^{'}(x) = \sin(x^2).$$

Optional Exercises:

(11) Construct an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous every where and is differentiable everywhere except at 2 points.

(12) Let $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$

Show that f is discontinuous at every $c \in \mathbb{R}$.

(13) Let $g(x) = \begin{cases} x, & \text{if } x \text{ is rational,} \\ 1-x, & \text{if } x \text{ is irrational.} \end{cases}$

Show that g is continuous only at $c = 1/2$.

(14) Let $f : (a, b) \to \mathbb{R}$ and $c \in (a, b)$ be such that $\lim_{x \to c} f(x) > \alpha$. Prove that there exists some $\delta > 0$ such that

$$f(c + h) > \alpha \text{ for all } 0 < |h| < \delta.$$

(See also question 7 of Tutorial Sheet 1.)

(15) Let $f : (a, b) \to \mathbb{R}$ and $c \in (a, b)$. Show that the following are equivalent :

(i) f is differentiable at c.

(ii) There exist $\delta > 0$ and a function $\epsilon_1 : (-\delta, \delta) \to \mathbb{R}$ such that $\lim_{h \to 0} \epsilon_1(h) = 0$ and

$$f(c + h) = f(c) + \alpha h + h\epsilon_1(h) \text{ for all } h \in (-\delta, \delta).$$

(iii) There exists $\alpha \in \mathbb{R}$ such that

$$\lim_{h \to 0} \left(\frac{|f(c + h) - f(c) - \alpha h|}{|h|}\right) = 0.$$

1.3. Rolle's and mean value theorems, Maximum/Minimum

(1) Show that the cubic $x^3 - 6x + 3$ has all roots real.

(2) Let p and q be two real numbers with $p > 0$. Show that the cubic $x^3 + px + q$ has exactly one real root.

(3) Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a)$ and $f(b)$ are of different signs and $f^{'}(x) \neq 0$ for all $x \in (a, b)$, show that there is a unique $x_0 \in (a, b)$ such that $f(x_0) = 0$.

(4) Consider the cubic $f(x) = x^3 + px + q$, where p and q are real numbers. If $f(x)$ has three distinct real roots, show that $4p^3 + 27q^2 < 0$ by proving the following:
 (i) $p < 0$.
 (ii) f has maximum/minimum at $\pm\sqrt{-p/3}$.
 (iii) The maximum/minimum values are of opposite signs.

(5) Use the MVT to prove $|\sin a - \sin b| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

(6) Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = a$ and $f(b) = b$, show that there exist distinct c_1, c_2 in (a, b) such that $f^{'}(c_1) + f^{'}(c_2) = 2$.

(7) Let $a > 0$ and f be continuous on $[-a, a]$. Suppose that $f^{'}(x)$ exists and $f^{'}(x) \leq 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, show that $f(0) = 0$.
Optional: Show that under the given conditions, in fact $f(x) = x$ for every x.

(8) In each case, find a function f which satisfies all the given conditions, or else show that no such function exists.
 (i) $f^{''}(x) > 0$ for all $x \in \mathbb{R}$, $f^{'}(0) = 1$, $f^{'}(1) = 1$
 (ii) $f^{''}(x) > 0$ for all $x \in \mathbb{R}$, $f^{'}(0) = 1$, $f^{'}(1) = 2$
 (iii) $f^{''}(x) \geq 0$ for all $x \in \mathbb{R}$, $f^{'}(0) = 1$, $f(x) \leq 100$ for all $x > 0$
 (iv) $f^{''}(x) > 0$ for all $x \in \mathbb{R}$, $f^{'}(0) = 1$, $f(x) \leq 1$ for all $x < 0$

(9) Let $f(x) = 1 + 12|x| - 3x^2$. Find the global maximum and the global minimum of f on $[-2, 5]$. Verify it from the sketch of the curve $y = f(x)$ on $[-2, 5]$.

(10) A window is to be made in the form of a rectangle surmounted by a semicircular portion with diameter equal to the base of the rectangle. The rectangular portion is of clear glass and the semicircular portion is to be of colored glass admitting only half as much light per square foot as the clear glass. If the total perimeter of the window frame is p feet, find the dimensions of the window which will admit the maximum light.

1.4. Curve sketching, Riemann integration

(1) Sketch the following curves after locating intervals of increase/decrease, intervals of concavity upward/downward, points of local maxima/minima, points of inflection and asymptotes. How many times and approximately where does the curve cross the x-axis?

(i) $y = 2x^3 + 2x^2 - 2x - 1$.

(ii) $y = \dfrac{x^2}{x^2+1}$.

(iii) $y = 1 + 12|x| - 3x^2, x \in [-2, 5]$.

(2) Sketch a continuous curve $y = f(x)$ having all the following properties:

$f(-2) = 8, f(0) = 4, f(2) = 0; f^{'}(2) = f^{'}(-2) = 0;$

$f^{'}(x) > 0$ for $|x| > 2, f^{'}(x) < 0$ for $|x| < 2$;

$f^{''}(x) < 0$ for $x < 0$ and $f^{''}(x) > 0$ for $x > 0$.

(3) Give an example of $f : (0, 1) \to \mathbb{R}$ such that f is

(i) strictly increasing and convex.

(ii) strictly increasing and concave.

(iii) strictly decreasing and convex.

(iv) strictly decreasing and concave.

(4) Let $f, g : \mathbb{R} \to \mathbb{R}$ satisfy $f(x) \geq 0$ and $g(x) \geq 0$ for all $x \in \mathbb{R}$. Define $h(x) = f(x)g(x)$ for $x \in \mathbb{R}$. Which of the following statements are true? Why?

(i) If f and g have a local maximum at $x = c$, then so does h.

(ii) If f and g have a point of inflection at $x = c$, then so does h.

(5) Let $f(x) = 1$ if $x \in [0, 1]$ and $f(x) = 2$ if $x \in (1, 2]$. Show from the first principle that f is Riemann integrable on $[0, 2]$ and find $\displaystyle\int_0^2 f(x)dx$.

(6) (a) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and $f(x) \geq 0$ for all $x \in [a, b]$. Show that $\displaystyle\int_a^b f(x)dx \geq 0$. Further, if f is continuous and $\displaystyle\int_a^b f(x)dx = 0$, show that $f(x) = 0$ for all $x \in [a, b]$.

(b) Give an example of a Riemann integrable function on $[a, b]$ such that $f(x) \geq 0$ for all $x \in [a, b]$ and $\displaystyle\int_a^b f(x)dx = 0$, but $f(x) \neq 0$ for some $x \in [a, b]$.

(7) Evaluate $\lim_{n\to\infty} S_n$ by showing that S_n is an approximate Riemann sum for a suitable function over a suitable interval:

(i) $S_n = \dfrac{1}{n^{5/2}} \displaystyle\sum_{i=1}^{n} i^{3/2}$.

(ii) $S_n = \displaystyle\sum_{i=1}^{n} \frac{n}{i^2 + n^2}$.

(iii) $S_n = \displaystyle\sum_{i=1}^{n} \frac{1}{\sqrt{in + n^2}}$.

(iv) $S_n = \dfrac{1}{n} \displaystyle\sum_{i=1}^{n} \cos \frac{i\pi}{n}$.

(v) $S_n = \dfrac{1}{n} \left\{ \displaystyle\sum_{i=1}^{n} \left(\frac{i}{n}\right) + \sum_{i=n+1}^{2n} \left(\frac{i}{n}\right)^{3/2} + \sum_{i=2n+1}^{3n} \left(\frac{i}{n}\right)^{2} \right\}$.

(8) Compute

(a) $\dfrac{d^2y}{dx^2}$, if $x = \displaystyle\int_0^y \frac{dt}{\sqrt{1+t^2}}$.

(b) $\dfrac{dF}{dx}$, if for $x \in \mathbb{R}$ (i) $F(x) = \displaystyle\int_1^{2x} \cos(t^2)dt$ (ii) $F(x) = \displaystyle\int_0^{x^2} \cos(t)dt$.

(9) Let p be a real number and let f be a continuous function on $\mathbb{R}$ that satisfies the equation $f(x+p) = f(x)$ for all $x \in \mathbb{R}$. Show that the integral $\displaystyle\int_a^{a+p} f(t)dt$ has the same value for every real number a. (Hint : Consider $F(a) = \displaystyle\int_a^{a+p} f(t)dt, a \in \mathbb{R}$.)

(10) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\lambda \in \mathbb{R}, \lambda \neq 0$. For $x \in \mathbb{R}$, let

$$g(x) = \frac{1}{\lambda}\int_0^x f(t)\sin\lambda(x-t)dt.$$

Show that $g^{''}(x) + \lambda^2 g(x) = f(x)$ for all $x \in \mathbb{R}$ and $g(0) = 0 = g^{'}(0)$.

1.5. Length, area, volume

(1) Find the area of the region bounded by the given curves in each of the following cases.
  (i) $\sqrt{x} + \sqrt{y} = 1$, $x = 0$ and $y = 0$.
  (ii) $y = x^4 - 2x^2$ and $y = 2x^2$.
  (iii) $x = 3y - y^2$ and $x + y = 3$.

(2) Let $f(x) = x - x^2$ and $g(x) = ax$. Determine a so that the region above the graph of g and below the graph of f has area 4.5.

(3) Find the area of the region inside the circle $r = 6a\cos\theta$ and outside the cardioid $r = 2a(1 + \cos\theta)$.

(4) Find the arc length of the each of the curves described below.
  (i) the cycloid $x = t - \sin t,\ y = 1 - \cos t,\ 0 \le t \le 2\pi$.
  (ii) $y = \displaystyle\int_0^x \sqrt{\cos 2t}\,dt,\ 0 \le x \le \pi/4$.

(5) For the following curve, find the arc length as well as the the area of the surface generated by revolving it about the line $y = -1$.

$$y = \frac{x^3}{3} + \frac{1}{4x},\ 1 \le x \le 3$$

(6) The cross sections of a certain solid by planes perpendicular to the x-axis are circles with diameters extending from the curve $y = x^2$ to the curve $y = 8 - x^2$. The solid lies between the points of intersection of these two curves. Find its volume.

(7) Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $y^2 + z^2 = a^2$.

(8) A fixed line L in 3-space and a square of side r in a plane perpendicular to L are given. One vertex of the square is on L. As this vertex moves a distance h along L, the square turns through a full revolution with L as the axis. Find the volume of the solid generated by this motion.

(9) Find the volume of the solid generated when the region bounded by the curves $y = 3 - x^2$ and $y = -1$ is revolved about the line $y = -1$, by both the Washer Method and the Shell Method.

(10) A round hole of radius $\sqrt{3}$ cms is bored through the center of a solid ball of radius 2 cms. Find the volume cut out.

1.6. Multivariables, limits, continuity

(1) Find the natural domains of the following functions of two variables:
(i) $\dfrac{xy}{x^2 - y^2}$
(ii) $\log(x^2 + y^2)$

(2) Describe the level curves and the contour lines for the following functions corresponding to the values $c = -3, -2, -1, 0, 1, 2, 3, 4:$
(i) $f(x, y) = x - y$
(ii) $f(x, y) = x^2 + y^2$
(iii) $f(x, y) = xy$

(3) Using definition, examine the following functions for continuity at $(0, 0)$. The expressions below give the value at $(x, y) \neq (0, 0)$. At $(0, 0)$, the value should be taken as zero: (i) $\dfrac{x^3 y}{x^6 + y^2}$
(ii) $xy\dfrac{x^2 - y^2}{x^2 + y^2}$
(iii) $||x| - |y|| - |x| - |y|$.

(4) Suppose $f,\ g : \mathbb{R} \to \mathbb{R}$ are continuous functions. Show that each of the following functions of $(x, y) \in \mathbb{R}^2$ are continuous:
(i) $f(x) \pm g(y)$
(ii) $f(x)g(y)$
(iii) $\max\{f(x),\ g(y)\}$
(iv) $\min\{f(x),\ g(y)\}$.

(5) Let

$$f(x, y) = \frac{x^2 y^2}{x^2 y^2 + (x - y)^2} \quad \text{for } (x, y) \neq (0, 0).$$

Show that the iterated limits

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x, y)\right] \text{ and } \lim_{y\to 0}\left[\lim_{x\to 0} f(x, y)\right]$$

exist and both are equal to 0, but $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

(6) Examine the following functions for the existence of partial derivatives at $(0, 0)$. The expressions below give the value at $(x, y) \neq (0, 0)$. At $(0, 0)$, the value should be taken as zero.
(i) $xy\dfrac{x^2 - y^2}{x^2 + y^2}$
(ii) $\dfrac{\sin^2(x + y)}{|x| + |y|}$

(7) Let $f(0, 0) = 0$ and

$$f(x, y) = (x^2 + y^2) \sin \frac{1}{x^2 + y^2} \quad \text{for} \quad (x, y) \neq (0, 0).$$

Show that f is continuous at $(0, 0)$, and the partial derivatives of f exist, but are not bounded in any disc (how so ever small) around $(0, 0)$.

(8) Let $f(0, 0) = 0$ and

$$f(x, y) = \begin{cases} x \sin(1/x) + y \sin(1/y), & \text{if } x \neq 0,\ y \neq 0 \\ x \sin 1/x, & \text{if } x \neq 0,\ y = 0 \\ y \sin 1/y, & \text{if } y \neq 0,\ x = 0. \end{cases}$$

Show that none of the partial derivatives of f exist at $(0, 0)$ although f is continuous at $(0, 0)$.

(9) Examine the following functions for the existence of directional derivatives and differentiability at $(0, 0)$. The expressions below give the value at $(x, y) \neq (0, 0)$. At $(0, 0)$, the value should be taken as zero:

(i) $xy\dfrac{x^2 - y^2}{x^2 + y^2}$

(ii) $\dfrac{x^3}{x^2 + y^2}$

(iii) $(x^2 + y^2)\sin\dfrac{1}{x^2 + y^2}$

(10) Let $f(x, y) = 0$ if $y = 0$ and

$$f(x, y) = \frac{y}{|y|}\sqrt{x^2 + y^2} \text{ if } y \neq 0.$$

Show that f is continuous at $(0, 0)$, $D_u f(0, 0)$ exists for every vector u, yet f is not differentiable at $(0, 0)$.

1.7. Maxima, minima, saddle points

(1) Let $F(x, y, z) = x^2 + 2xy - y^2 + z^2$. Find the gradient of F at $(1, -1, 3)$ and the equations of the tangent plane and the normal line to the surface $F(x, y, z) = 7$ at $(1, -1, 3)$.

(2) Find $D_u F(2, 2, 1)$, where $F(x, y, z) = 3x - 5y + 2z$, and u is the unit vector in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 9$ at $(2, 2, 1)$.

(3) Given $\sin(x + y) + \sin(y + z) = 1$, find $\dfrac{\partial^2 z}{\partial x \partial y}$, provided $\cos(y + z) \neq 0$.

(4) If $f(0, 0) = 0$ and

$$f(x, y) = xy\frac{x^2 - y^2}{x^2 + y^2} \text{ for } (x, y) \neq (0, 0),$$

show that both f_{xy} and f_{yx} exist at $(0, 0)$, but they are not equal. Are f_{xy} and f_{yx} continuous at $(0, 0)$?

(5) Show that the following functions have local minima at the indicated points.
  (i) $f(x, y) = x^4 + y^4 + 4x - 32y - 7$, $(x_0, y_0) = (-1, 2)$.
  (ii) $f(x, y) = x^3 + 3x^2 - 2xy + 5y^2 - 4y^3$, $(x_0, y_0) = (0, 0)$.

(6) Analyze the following functions for local maxima, local minima and saddle points:
  (i) $f(x, y) = (x^2 - y^2)e^{-(x^2+y^2)/2}$ (ii) $f(x, y) = x^3 - 3xy^2$

(7) Find the global maximum and the global minimum of

$$f(x, y) = (x^2 - 4x)\cos y$$

for $1 \leq x \leq 3$, $-\pi/4 \leq y \leq \pi/4$.

1.8. Multiple integrals

(1) For the following, write an equivalent iterated integral with the order of integration reversed:

(i) $\int_0^1 \left[\int_1^{e^x} dy\right] dx.$

(ii) $\int_0^1 \left[\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)dx\right] dy.$

(2) Evaluate the following integrals:

(i) $\int_0^\pi \left[\int_x^\pi \frac{\sin y}{y} dy\right] dx.$

(ii) $\int_0^1 \left[\int_y^1 x^2 e^{xy} dx\right] dy.$

(iii) $\int_0^2 (\tan^{-1} \pi x - \tan^{-1} x)dx.$

(3) Find $\iint_D f(x,y)d(x,y)$, where $f(x,y) = e^{x^2}$ and D is the region bounded by the lines $y = 0, x = 1$ and $y = 2x$.

(4) Evaluate the integral

$$\iint_D (x-y)^2 \sin^2(x+y)d(x,y),$$

where D is the parallelogram with vertices at $(\pi, 0), (2\pi, \pi), (\pi, 2\pi)$ and $(0, \pi)$.

(5) Let D be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1, xy = 9$ and the lines $y = x, y = 4x$. Find $\iint_D d(x,y)$ by transforming it to $\iint_E d(u,v)$, where $x = \frac{u}{v}, y = uv, v > 0$.

(6) Find

$$\lim_{r\to\infty} \iint_{D(r)} e^{-(x^2+y^2)} d(x,y),$$

where $D(r)$ equals:

(i) $\{(x,y) : x^2 + y^2 \le r^2\}$.

(ii) $\{(x,y) : x^2 + y^2 \le r^2, x \ge 0, y \ge 0\}$.

(iii) $\{(x,y) : |x| \le r, |y| \le r\}$.

(iv) $\{(x,y) : 0 \le x \le r, 0 \le y \le r\}$.

(7) Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ using double integral over a region in the plane. (Hint: Consider the part in the first octant.)

(8) Express the solid $D = \{(x,y,z) | \sqrt{x^2+y^2} \le z \le 1\}$ as

$$\{(x,y,z) | a \le x \le b, \phi_1(x) \le y \le \phi_2(x), \xi_1(x,y) \le z \le \xi_2(x,y)\}.$$

(9) Evaluate

$$I = \int_0^{\sqrt{2}} \left(\int_0^{\sqrt{2-x^2}} \left(\int_{x^2+y^2}^2 x dz \right) dy \right) dx.$$

Sketch the region of integration and evaluate the integral by expressing the order of integration as $dxdydz$.

(10) Using suitable change of variables, evaluate the following:

(i)

$$I = \iiint_D (z^2x^2 + z^2y^2)dxdydz$$

where D is the cylindrical region $x^2 + y^2 \leq 1$ bounded by $-1 \leq z \leq 1$.

(ii)

$$I = \iiint_D \exp(x^2 + y^2 + z^2)^{3/2} dxdydz$$

over the region enclosed by the unit sphere in $\mathbb{R}^3$.

1.9. Vector fields, curves, parameterization

(1) Let $\mathbf{a}, \mathbf{b}$ be two fixed vectors, $\mathbf{r} = (x, y, z)$ and $r^2 = x^2 + y^2 + z^2$. Prove the following:
  (i) $\nabla(r^n) = nr^{n-2}\mathbf{r}$ for any integer n.
  (ii) $\mathbf{a} \cdot \nabla \left(\frac{1}{r}\right) = -\left(\frac{\mathbf{a} \cdot \mathbf{r}}{r^3}\right)$.
  (iii) $\mathbf{b} \cdot \nabla \left(\mathbf{a} \cdot \nabla \left(\frac{1}{r}\right)\right) = \frac{3(\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})}{r^5} - \frac{\mathbf{a} \cdot \mathbf{b}}{r^3}$.

(2) For any two scalar functions f, g on $\mathbb{R}^m$ establish the relations:
  (i) $\nabla(fg) = f\nabla g + g\nabla f$.
  (ii) $\nabla f^n = nf^{n-1}\nabla f$.
  (iii) $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$ whenever $g \neq 0$.

(3) Prove the following:
  (i) $\nabla \cdot (f\mathbf{v}) = f\nabla \cdot \mathbf{v} + (\nabla f) \cdot \mathbf{v}$.
  (ii) $\nabla \times (f\mathbf{v}) = f(\nabla \times \mathbf{v}) + \nabla f \times \mathbf{v}$.
  (iii) $\nabla \times \nabla \times \mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - (\nabla \cdot \nabla)\mathbf{v}$,
  where $\nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the Laplacian operator.
  (iv) $\nabla \cdot (f\nabla g) - \nabla \cdot (g\nabla f) = f\nabla^2 g - g\nabla^2 f$.
  (v) $\nabla \cdot (\nabla \times \mathbf{v}) = 0$
  (vi) $\nabla \times (\nabla f) = 0$.
  (vii) $\nabla \cdot (g\nabla f \times f\nabla g) = 0$.

(4) Let $\mathbf{r} = (x, y, z)$ and $r = |\mathbf{r}|$. Show that
  (i) $\nabla^2 f = \text{div } (\nabla f(r)) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$.
  (ii) $\text{div } (r^n \mathbf{r}) = (n + 3)r^n$.
  (iii) $\text{curl}(r^n \mathbf{r}) = 0$
  (iv) $\text{div } (\nabla \frac{1}{r}) = 0$ for $r \neq 0$.

(5) Prove that
  (i) $\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v})$
  Hence, if $\mathbf{u}, \mathbf{v}$ are irrotational, $\mathbf{u} \times \mathbf{v}$ is solenoidal. (**Def:** A vector-field $\mathbf{u}$ is said to be irrotational if $\nabla \times \mathbf{u} = 0$. A vector-field $\mathbf{u}$ is said to be solenoidal if $\nabla \cdot \mathbf{u} = 0$.)
  (ii) $\nabla \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} + (\nabla \cdot \mathbf{v})\mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{v}$.
  (iii) $\nabla(\mathbf{u} \cdot \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{u}) + \mathbf{u} \times (\nabla \times \mathbf{v})$.
  Hint: Write $\nabla = \sum \mathbf{i}\frac{\partial}{\partial x}$, $\nabla \times \mathbf{v} = \sum \mathbf{i}\frac{\partial}{\partial x} \times \mathbf{v}$ and $\nabla \cdot \mathbf{v} = \sum \mathbf{i}\frac{\partial}{\partial x} \cdot \mathbf{v}$.

(6) (i) If $\mathbf{w}$ is a vector field of constant direction and $\nabla \times \mathbf{w} \neq 0$, prove that $\nabla \times \mathbf{w}$ is always orthogonal to $\mathbf{w}$.
  (ii) If $\mathbf{v} = \mathbf{w} \times \mathbf{r}$ for a constant vector $\mathbf{w}$, prove that $\nabla \times \mathbf{v} = 2\mathbf{w}$.
  (iii) If $\rho\mathbf{v} = \nabla p$ where $\rho(\neq 0)$ and p are continuously differentiable scalar functions, prove that

$$\mathbf{v} \cdot (\nabla \times \mathbf{v}) = 0.$$

(7) Calculate the line integral of the vector field

$$F(x, y) = (x^2 - 2xy, y^2 - 2xy)$$

from $(-1, 1)$ to $(1, 1)$ along $y = x^2$.

(8) Calculate the line integral of the vector field

$$F(x, y) = (x^2 + y^2, x - y)$$

once around the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ in the anticlockwise direction.

(9) Calculate the value of the line integral

$$\oint_C \frac{(x+y)dx - (x-y)dy}{x^2+y^2}$$

where C is the curve $x^2 + y^2 = a^2$ traversed once in the anticlockwise direction.

(10) Calculate

$$\oint_C ydx + zdy + xdz$$

where C is the intersection of two surfaces $z = xy$ and $x^2 + y^2 = 1$ traversed once in a direction that appears anticlockwise when viewed from high above the xy-plane.

1.10. Line integrals and applications

(1) Consider the helix
$$\gamma(t) = (a\cos t, a\sin t, ct)$$
lying on the cylinder $x^2 + y^2 = a^2$. Parameterize this in terms of arc length.

(2) Evaluate the line integral
$$\oint_C \frac{x^2 y dx - x^3 dy}{(x^2+y^2)^2}$$
where C is the square with vertices $(\pm 1, \pm 1)$ oriented in the anticlockwise direction.

(3) Find
$$\oint_C \text{grad } (x^2 - y^2) \cdot ds$$
where C is the curve $x^2 + y^2 = 1$ oriented in the anticlockwise direction.

(4) Evaluate
$$\int_{(0,0)}^{(2,8)} \text{grad } (x^2 - y^2) \cdot ds$$
where C is $y = x^3$. The notation is written to suggest that $(0,0)$ is the initial point and $(2,8)$ is the final point of C.

(5) Compute the line integral
$$\oint_C \frac{dx + dy}{|x| + |y|}$$
where C is the square with vertices $(1,0), (0,1)$, $(-1,0)$ and $(0,-1)$ traversed once in the anticlockwise direction.

(6) A force $F = (xy, x^6 y^2)$ moves a particle from $(0,0)$ onto the line $x = 1$ along $y = ax^b$ where $a, b > 0$. If the work done is independent of b, find the value of a.

(7) Calculate the work done by the force field $F(x,y,z) = (y^2, z^2, x^2)$ along the curve C of intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and cylinder $x^2 + y^2 = ax$ where $z \geq 0, a > 0$ (specify the orientation of C that you use.)

(8) Determine whether or not the vector field $F(x,y) = (3xy, x^3 y)$ is a gradient field on any open subset of $\mathbb{R}^2$.

(9) Let $S = \mathbb{R}^2 \setminus \{(0,0)\}$. Let
$$F(x,y) = (-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}) = (P(x,y), Q(x,y)).$$
Show that $\dfrac{\partial}{\partial y} P(x,y) = \dfrac{\partial}{\partial x} Q(x,y)$ on S while F is not the gradient of a scalar field on S.

(10) For $F = (2xy + z^3, x^2, 3xz^2)$. Show that $\nabla\phi = F$ for some ϕ and calculate $\oint_C F \cdot ds$ where C is any arbitrary smooth closed curve.

(11) A radial force field is one which can be expressed as $F = f(r)\mathbf{r}$ where $\mathbf{r} = (x,y,z)$ is the position vector and $r = \|\mathbf{r}\|$. Show that F is conservative if f is continuous.

1.11. Green's theorem and applications

(1) Verify Green's theorem in each of the following cases:
  (i) $f(x,y) = -xy^2$; $g(x,y) = x^2y$; $D: x \geq 0,\ 0 \leq y \leq 1 - x^2$;
  (ii) $f(x,y) = 2xy$; $g(x,y) = e^x + x^2$; where D is the triangle with vertices $(0,0), (1,0)$, and $(1,1)$.

(2) Use Green's theorem to evaluate the integral $\oint_{\partial D} y^2\, dx + x\, dy$ where:
  (i) D is the square with vertices $(0,0), (2,0), (2,2), (0,2)$.
  (ii) D is the square with vertices $(\pm 1, \pm 1)$.
  (iii) D is the disc of radius 2 and center $(0,0)$ (specify the orientation you use for the curve.)

(3) For a simple closed curve given in polar coordinates $r = p(\theta)$ show using Green's theorem that the area enclosed is given by

$$A = \frac{1}{2} \oint_C p(\theta)^2 d\theta.$$

Use this to compute the area enclosed by the following curves:
  (i) The cardioid: $r = p(\theta) = a(1 - \cos\theta)$, $0 \leq \theta \leq 2\pi$.
  (ii) The lemniscate: $r^2 = p(\theta)^2 = a^2 \cos 2\theta$, $-\pi/4 \leq \theta \leq \pi/4$.

(4) Find the area of the following regions:
  (i) The area lying in the first quadrant of the cardioid $r = p(\theta) = a(1 - \cos\theta)$.
  (ii) The region under one arch of the cycloid

$$C(t) = (a(t - \sin t), a(1 - \cos t)),\ 0 \leq t \leq 2\pi.$$

  (iii) The region bounded by the limaçon

$$r = p(\theta) = 1 - 2\cos\theta,\ \ 0 \leq \theta \leq \pi/2$$

  and the two axes.

(5) Evaluate

$$\oint_C xe^{-y^2} dx + [-x^2 y e^{-y^2} + 1/(x^2 + y^2)] dy$$

around the square determined by $|x| \leq a$, $|y| \leq a$ traced in the anticlockwise direction.

(6) Let C be a simple closed curve in the xy-plane. Show that

$$3I_0 = \oint_C x^3 dy - y^3 dx,$$

where I_0 is the polar moment of inertia of the region D enclosed by C. It is defined by $I_0 = \iint_D x^2 + y^2\, dxdy$.

(7) Consider $a = a(x,y)$, $b = b(x,y)$ having continuous partial derivatives on the unit disc D. If

$$a(x,y) \equiv 1,\ b(x,y) \equiv y$$

on the boundary circle C, and

$$u = (a,b),\ v = ((a_x - a_y), (b_x - b_y)),\ w = ((b_x - b_y), (a_x - a_y)),$$

find

$$\iint_D u \cdot v\, dxdy \ \text{ and } \ \iint_D u \cdot w\, dxdy.$$

(8) Let C be any closed curve in the plane. Compute $\oint_C \nabla(x^2 - y^2) \cdot n |ds|$.

(9) Green's Identities are as follows:

(i) $\iint_D \nabla^2 w \, dxdy = \oint_{\partial D} \frac{\partial w}{\partial n} \, |ds|$.

(ii) $\iint_D [w\nabla^2 w + \nabla w \cdot \nabla w] \, dxdy = \oint_{\partial D} w \frac{\partial w}{\partial n} \, |ds|$.

(iii) $\oint_{\partial D} \left(v\frac{\partial w}{\partial n} - w\frac{\partial v}{\partial n} \right) |ds| = \iint_D (v\nabla^2 w - w\nabla^2 v) \, dxdy$.

Here $\nabla^2 w$ is the divergence of the gradient of w. Also, $\frac{\partial w}{\partial n}$ is the same as $\nabla w \cdot n$.

(a) Use (i) to compute

$$\oint_C \frac{\partial w}{\partial n} \, |ds|$$

for $w = e^x \sin y$, and D the triangle with vertices $(0,0), (4,2), (0,2)$.

(b) Let D be a plane region bounded by a simple closed curve C and let $F, G : U \longrightarrow \mathbb{R}^2$ be smooth functions where U is a region containing $D \cup C$ such that

$$\operatorname{curl} F = \operatorname{curl} G, \operatorname{div} F = \operatorname{div} G \text{ on } D \cup C$$

and

$$F \cdot n = G \cdot n \text{ on } C,$$

where n is the unit normal to the curve. Show that $F = G$ on D.

(10) Evaluate the following line integrals where the loops are traced in the anti-clockwise sense

(i)

$$\oint_C \frac{y\,dx - x\,dy}{x^2 + y^2}$$

where C is any simple closed curve not passing through the origin.

(ii)

$$\oint_C \frac{x^2 y dx - x^3 dy}{(x^2 + y^2)^2},$$

where C is the square with vertices $(\pm 1, \pm 1)$.

(iii) Let C be a smooth simple closed curve lying in the annulus $1 < x^2 + y^2 < 2$. Find

$$\oint_C \frac{\partial(\log r)}{\partial y} dx - \frac{\partial(\log r)}{\partial x} dy.$$

1.12. Surface area and surface integrals

(1) Find a suitable parameterization $\Phi(u,v)$ and the normal vector $\Phi_u \times \Phi_v$ for the following surface:
  (i) The plane $x - y + 2z + 4 = 0$.
  (ii) The cylinder $y^2 + z^2 = a^2$.
  (iii) The cylinder of radius 1 whose axis is along the line $x = y = z$.

(2) (a) For a surface S let the unit normal n at every point make the same acute angle α with z-axis. Let SA_{xy} denote the area of the projection of S onto the xy plane. Show that SA, the area of the surface S satisfies the relation: $SA_{xy} = SA\cos\alpha$.
(b) Let S be a parallelogram not parallel to any of the coordinate planes. Let S_1, S_2, and S_3 denote the areas of the projections of S on the three coordinate planes. Show that the area of S is $\sqrt{S_1^2 + S_2^2 + S_3^2}$.

(3) Compute the surface area of that portion of the sphere $x^2 + y^2 + z^2 = a^2$ which lies within the cylinder $x^2 + y^2 = ay$, where $a > 0$.

(4) A surface S is parametrized by

$$\Phi(u,v) = (u\cos v, u\sin v, u^2),$$

where $0 \le u \le 4$ and $0 \le v \le 2\pi$.
  (i) Show that S is a portion of a surface of revolution. Make a sketch and indicate the geometric meanings of the parameters u and v on the surface.
  (ii) Compute the vector $\Phi_u \times \Phi_v$ in terms of u and v.
  (iii) The area of S is $\frac{\pi}{n}(65\sqrt{65} - 1)$ where n is an integer. Compute the value of n.

(5) Compute the area of that portion of the paraboloid $x^2 + z^2 = 2ay$ which is between the planes $y = 0$ and $y = a$.

(6) A sphere is inscribed in a cylinder. The sphere is sliced by two parallel planes perpendicular the axis of the cylinder. Show that the portions of the sphere and the cylinder lying between these planes have equal surface areas.

(7) Let S denote the plane surface whose boundary is the triangle with vertices at $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and let $F(x,y,z) = (x,y,z)$. Let n denote the unit normal to S having a nonnegative z-component. Evaluate the surface integral $\iint_S F \cdot n\,|dS|$, using
  (i) The parametrization $\Phi(u,v) = ((u+v),(u-v),(1-2u))$.
  (ii) A parametrization of the form $\Phi(x,y) = (x,y,f(x,y))$.

(8) If S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$, compute the value of the surface integral (with the choice of outward unit normal)

$$\iint_S xz dy \wedge dz + yz dz \wedge dx + x^2 dx \wedge dy.$$

Choose a parametrization in which the fundamental vector product points in the direction of the outward normal.

(9) A fluid flow has flux density vector

$$F(x,y,z) = (x, -(2x+y), z).$$

Let S denote the hemisphere $x^2 + y^2 + z^2 = 1$, $z \ge 0$, and let n denote the unit normal that points out of the sphere. Calculate $\iint_S F \cdot n|dS|$, which is the mass of the fluid flowing through S in unit time in the direction of n.

(10) Solve the previous exercise when S includes the planar base of the hemisphere also with the outward unit normal on the base being $(0, 0, -1)$.

1.13. Divergence theorem and applications

(1) Verify the divergence theorem for the vector field
$$F(x,y,z) = (xy^2, yz^2, zx^2)$$
for the region W defined by
$$y^2 + z^2 \le x^2; 0 \le x \le 4.$$

(2) Verify the divergence theorem for the vector field
$$F(x,y,z) = (xy, yz, zx)$$
for the region W in the first octant bounded by the plane
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

(3) Let W be a region bounded by a piecewise smooth closed surface S with outward unit normal
$$n = (n_x, n_y, n_z).$$
Let $u, v : W \to \mathbb{R}$ be continuously differentiable. Show that
$$\iiint_W u\frac{\partial v}{\partial x} d(x,y,z) = -\iiint_W v\frac{\partial u}{\partial x} d(x,y,z) + \iint_{\partial W} u\, v\, n_x\, |dS|.$$
[Hint: Consider $F = u\, v\,(1,0,0)$.]

(4) Suppose a scalar field ϕ, which is never zero has the properties
$$\|\nabla\phi\|^2 = 4\phi \text{ and } \nabla\cdot(\phi\nabla\phi) = 10\phi.$$
Evaluate $\iint_S \frac{\partial\phi}{\partial n}\,|dS|$, where S is the surface of the unit sphere.

(5) Let V be the volume of a region W bounded by a closed surface S and $n = (n_x, n_y, n_z)$ be its outer unit normal. Prove that
$$V = \iint_S x\, n_x\, |dS| = \iint_S y\, n_y\, |dS| = \iint_S z\, n_z\, |dS|.$$

(6) Compute $\iint_S (x^2 dy\wedge dz + y^2\, dz\wedge dx + z^2 dx\wedge dy)$, where S is the surface of the cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$. The surface S is oriented by the normal going out of the solid cube.

(7) Compute $\iint_S yzdy\wedge dz + zxdz\wedge dx + xydx\wedge dy$, where S is the unit sphere oriented by the outward normal.

(8) Let $F = (-x^3, y^3 + 3z^2\sin z, e^y\sin z + x^4)$ and S be the portion of the sphere $x^2+y^2+z^2 = 1$ with $z \ge \frac{1}{2}$ and n is the unit normal with positive z-component. Use divergence theorem to compute $\iint_S (\operatorname{curl} F)\cdot n\,|dS|$.

(9) Let p denote the distance from the origin to the tangent plane at the point (x,y,z) to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Prove that
   (a) $\iint_S p\,|dS| = 4\pi abc$.
   (b) $\iint_S \frac{1}{p}\,|dS| = \frac{4\pi}{3abc}(b^2c^2 + c^2a^2 + a^2b^2)$.

(10) Interpret Green's theorem as a divergence theorem in the plane.

1.14. Stokes theorem and applications

(1) Consider the vector field $F = (x - y, x + z, y + z)$. Verify Stokes theorem for F where S is the surface of the cone: $z^2 = x^2 + y^2$ intercepted by
 (a) $x^2 + (y - a)^2 + z^2 = a^2 : z \geq 0$
 (b) $x^2 + (y - a)^2 = a^2$

(2) Evaluate using Stokes theorem, the line integral
$$\oint_C yz\,dx + xz\,dy + xy\,dz,$$
where C is the curve of intersection of $x^2 + 9y^2 = 9$ and $z = y^2 + 1$ with clockwise orientation when viewed from the origin.

(3) Compute
$$\iint_S (\operatorname{curl} F) \cdot n|dS|,$$
where $F = (y, xz^3, -zy^3)$ and n is the outward unit normal to S, the surface of the cylinder $x^2 + y^2 = 4$ between $z = 0$ and $z = -3$.

(4) Compute $\oint_C F \cdot ds$ for
$$F = \frac{(-y, x)}{x^2 + y^2},$$
where C is the circle of unit radius in the xy plane centered at the origin and oriented clockwise. Can the above line integral be computed using Stokes theorem?

(5) Compute
$$\oint_C (y^2 - z^2)dx + (z^2 - x^2)dy + (x^2 - y^2)dz,$$
where C is the curve cut out of the boundary of the cube
$$0 \leq x \leq a,\ 0 \leq y \leq a,\ 0 \leq z \leq a$$
by the plane $x + y + z = \frac{3}{2}a$ (specify the orientation of C.)

(6) Calculate
$$\oint_C ydx + zdy + xdz,$$
where C is the intersection of the surface $bz = xy$ and the cylinder $x^2 + y^2 = a^2$, oriented anticlockwise as viewed from a point high upon the positive z-axis.

(7) Consider a plane with unit normal (a, b, c). For a closed curve C lying in this plane, show that the area enclosed by C is given by
$$A(C) = \frac{1}{2}\oint_C (bz - cy)dx + (cx - az)dy + (ay - bx)dz,$$
where C is given the anticlockwise orientation. Compute $A(C)$ for the curve C given by
$$\mathbf{u}\cos t + \mathbf{v}\sin t,\ 0 \leq t \leq 2\pi.$$
.

CHAPTER 2

Answers

2.0. Tutorial sheet 0

(1) **False** (2) **False** (3) **False** (4) **True** (5) **True**. (6) **False** (7) **True** (8) **False** (9) **True** (10) **True** (11) **True**

2.1. Tutorial sheet 1

(1) (i) $N > 10/\epsilon$, (ii) $N > \dfrac{5-\epsilon}{3\epsilon}$, (iii) $N > \dfrac{1}{\epsilon^3}$ (iv) $N > \dfrac{2}{\epsilon}$

(2) (i) $\lim_{n\to\infty} a_n = 1$. (ii) $\lim_{n\to\infty} \dfrac{n!}{n^n} = 0$. (iii) $\lim_{n\to\infty} a_n = 0$.
(iv) $\lim_{n\to\infty} n^{1/n} = 1$. (v) $\lim_{n\to\infty} a_n = 0$. (vi) $\frac{1}{2}$.

(3) (i) Not convergent (ii) Not convergent

(4) (i) Decreasing
(ii) Increasing
(iii) Increasing

(5) (i) Using induction show that
$a_n \geq \sqrt{2}, (n \geq 2)$ and $a_{n+1} - a_n < 0$ for all n. $\lim_{n\to\infty} a_n = \sqrt{2}$.
(ii) Using induction show that $a_n \leq 2$ and $\left(\frac{a_{n+1}}{a_n}\right)^2 > 1$ for all n. $\lim_{n\to\infty} a_n = 2$.
(iii) Using induction show that $a_n \leq 6$ and $a_{n+1} - a_n > 0$ for all n. $\lim_{n\to\infty} a_n = 6$.

(6) $\lim_{n\to\infty} |a_n| = |L|$.

(7) Hint: Consider $\epsilon = |L|/2$.

(8) Follows from definition.
Optional: Use the inequality: For $x > y > 0$,
$$\sqrt{x} - \sqrt{y} \leq \sqrt{x-y}$$

(9) Both the statements are **False**.

2.2. Tutorial sheet 2

(1) (i) **False** (ii) **True** (iii) **True**
(2) The converse is **False**.
(3) (i) Not continuous at $\alpha = 0$. (ii) Continuous everywhere. (iii) Continuous everywhere except at $x = 2$.
(5) Continuous for $x \neq 0$, not continuous at $x = 0$.
(7) The converse is **False** .
(9) (i) $\dfrac{-1}{\sin(x)} = \dfrac{-1}{\sqrt{1-\cos^2(x)}} = \dfrac{-1}{\sqrt{1-y^2}}$.

(ii) $\csc^{-1}(x) = \dfrac{(\frac{-1}{x^2})}{\sqrt{(1-\frac{1}{x^2})}}, |x| > 1$.

(10) $\dfrac{3}{(x+1)^2}\sin\left(\dfrac{2x-1}{x+1}\right)^2$.

2.3. Tutorial sheet 3

(8) (i) Not possible (ii) Possible (iii) Not possible (iv) Possible
(9) Global max $= 13$ at $x = \pm 2$ and global min $= -14$ at $x = 5$.
(10) $h = \dfrac{p(4+\pi)}{2(8+3\pi)}$.

2.4. Tutorial sheet 4

(i) $f(x)$ is strictly increasing in the intervals $(-\infty,-1)\cup(1/3,\infty)$ and $f(x)$ is strictly decreasing in that interval $(-1,1/3)$.

$f(x)$ has a local maximum at $x=-1$, a local minimum at $x=\frac{1}{3}$ and a point of inflection at $x=-\dfrac{1}{3}$.

(ii) $y=1$ is an asymptote.

$f(x)$ is increasing in $(0,\infty)$ and decreasing in $(-\infty,0)$.

$f(x)$ has points of inflection at $x=\pm\dfrac{1}{\sqrt{3}}$.

(iii) f is not differentiable at $x=0$.

$f(0)=1$,

$$f \text{ is concave in}(-2,0)\cup(0,5),$$
$$\text{decreasingin}(-2,0)\cup(2,5),$$

and

$$\text{increasingin}(0,2).$$

Further, f has a global maximum at $x=\pm 2$.

(2) f has a local max at $x=-2$ and a local min at $x=2$, f is concave in $(-\infty,0)$ and convex in $(0,\infty)$, and $x=0$ is a point of inflection.

(3) **Motivate students to discover their own examples**.

(i) $f'>0, f''>0$. Example: $f(x)=x^2;\ 0<x<1$.

(ii) $f'>0, f''<0$. Example: $f(x)=\sqrt{x};\ 0<x<1$.

(iii) $f'<0, f''>0$. Example: $f(x)=-\sqrt{x};\ 0<x<1$.

(iv) $f'<0, f''<0$. Example: $f(x)=-x^2;\ 0<x<1$.

(4) (i) **True** .

(ii) **False**; consider $f(x)=g(x)=\sin(x), c=0$.

(7) (i) $\dfrac{2}{5}$

(ii) $\dfrac{\pi}{4}$

(iii) $2(\sqrt{2}-1)$

(iv) 0.

(v) $\dfrac{1}{2}+\dfrac{2}{5}(4\sqrt{2}-1)+\dfrac{19}{3}$

(8) (a) $\dfrac{y}{\sqrt{1+y^2}}\dfrac{dy}{dx}=y$

(b) (i) $F'(x)=\cos((2x)^2)2=2\cos(4x^2)$.

(ii) $F'(x)=\cos(x^2)2x=2x\cos(x^2)$.

2.5. Tutorial Sheet 5

(1) (i) $\dfrac{1}{6}$

(ii) $\dfrac{128}{15}$

(iii) $\dfrac{4}{3}$

(2) $a = -2$

(3) $4\pi a^2$

(4) (i) 8

(ii) 1

(5) $L = \dfrac{53}{6}$, the surface area is $\left(101\dfrac{5}{18}\right)\pi$

(6) $\frac{512\pi}{15}$

(7) Volume is $\dfrac{16a^3}{3}$

(8) Volume is r^2h

(9) $2^9\pi/15$

(10) $\frac{28\pi}{3}$

2.6. Tutorial Sheet 6

(1) (i) $\{(x, y) \in \mathbb{R}^2 \mid x \neq \pm y\}$.
 (ii) $\mathbb{R}^2 - \{(0, 0)\}$

(2) (i) Level curves are parallel lines $x - y = c$. Contours are the same lines shifted to $z = c$, (some c).
 (ii) Level curves do not exist for $c \leq -1$. It is just a point for $c = 0$ and are concentric circles for $c = 1, 2, 3, 4$. Contours are the sections of paraboloid of revolution $z = x^2 + y^2$ by $z = c$, i.e., concentric circles in the plane $z = c$.
 (iii) Level curves are rectangular hyperbolas. Branches are in first and third quadrant for for $c > 0$ and in second and fourth quadrant for $c < 0$. For $c = 0$ it is the union of x-axis and y-axis.

(3) (i) Discontinuous at $(0, 0)$
 (ii) Continuous at $(0, 0)$
 (iii) Continuous at $(0, 0)$

(6) (i) $f_x(0, 0) = 0 = f_y(0, 0)$.
 (ii) f is continuous at $(0, 0)$. Both $f_x(0, 0)$ and $f_y(0, 0)$ do not exist.

(8) Does not exist.

(9) (i) $(D_v f)(0, 0)$ exists and equals 0 for every unit vector $v \in \mathbb{R}^2$; f is also differentiable at $(0, 0)$.
 (ii) It is not differentiable, but for every unit vector $v = (a, b)$, $D_v f(0, 0)$ exists.
 (iii) $(D_v f)(0, 0) = 0$; f is differentiable at $(0, 0)$.

2.7. Tutorial sheet 7

(1) Tangent plane:

$$0.(x-1)+4(y+1)+6(z-3)=0, \text{i.e.,} 2y+3z=7$$

Normal line: $x=1,\ 3y-2z+9=0$

(2) $-\dfrac{2}{3}$.

(3) $\dfrac{\sin(x+y)}{\cos(y+z)}+\tan(y+z)\dfrac{\cos^2(x+y)}{\cos^2(y+z)}$.

(6) (i) $(0,0)$is a saddle point; $(\pm\sqrt{2},0)$are local maxima; $((0,\pm\sqrt{2})$are local minima.

(ii) $(0,0)$ is a saddle point.

(7) $f_{min}=-4$ at $(2,0)$ and $f_{max}=-\frac{3}{\sqrt{2}}$ at $(3,\pm\frac{\pi}{4})$

2.8. Tutorial sheet 8

(1) (i) $\int_1^e \left(\int_{\log y}^1 dx\right) dy$

 (ii) $\int_{-1}^1 \left(\int_{x^2}^1 f(x,y)dy\right) dx$

(2) (i) 2

 (ii) $\frac{1}{2}(e-2))$

 (iii) $\frac{\pi-1}{2\pi}\log 5 + 2(\tan^{-1} 2\pi - \tan^{-1} 2) - \frac{1}{2\pi}\left[\log\frac{(4\pi^2+1)}{5}\right]$.

(3) $(e-1)$.

(4) $\frac{\pi^4}{3}$.

(5) $8\log 2$.

(6) (i) π.

 (ii) $\pi/4$.

 (iii) π.

 (iv) $\pi/4$.

(7) $\frac{16a^3}{3}$.

(8) $\{(x,y,z)| -1 \leq x \leq 1,\ -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2},\ \sqrt{x^2+y^2} \leq z \leq 1\}$

(9) $\frac{8\sqrt{2}}{15}$. We can also write D as

$\{(x,y,z)|0 \leq z \leq 2,\ 0 \leq x \leq \sqrt{z-y^2},\ 0 \leq y \leq \sqrt{z}\}$.

(10) (i) $\pi/3$.

 (ii) $\frac{4\pi(e-1)}{3}$.

2.9. Tutorial sheet 9

(7) $-\frac{14}{15}$. (8) πab. (9) -2π. (10) $-\pi$.

2.10. Tutorial sheet 10

(1) The arc length parametrization is

$$\tilde{\gamma}(u) = \left(a\cos\left(\frac{u}{\sqrt{a^2+c^2}}\right), a\sin\left(\frac{u}{\sqrt{a^2+c^2}}\right), \frac{cu}{\sqrt{a^2+c^2}}\right).$$

(2) $\displaystyle\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = -\pi.$

(3) $\displaystyle\oint_C \nabla(x^2-y^2)\cdot ds = 0.$

(4) $\displaystyle\oint_C \nabla(x^2-y^2)\cdot ds = -60.$

(5) $\displaystyle\int_C \frac{dx}{dy}|x| + |y| = 2-2 = 0.$

(6) $\sqrt{3/2}.$

(7) $\dfrac{-\pi a^3}{4}.$

2.11. Tutorial sheet 11

(1) (i) $\iint_D (g_x - f_y)\,dxdy = \frac{1}{3} = \oint_{\partial D}(fdx + gdy)$.
(ii) LHS $= 1 =$ RHS.
(2) (i) -4.
(ii) 4.
(iii) 4π.
(3) (i) $A = \dfrac{3\pi a^2}{2}$.
(ii) $A = a^2/2$.
(4) (i) $\frac{a^2}{8}(3\pi - 8)$.
(ii) $2\pi a^2$.
(iii) $\frac{1}{2}\left(\frac{3\pi-8}{2}\right)$.
(5) 0.
(8) 0.
(6) $3I_0$.
(7) $-\pi$.
(10) (i) Case (a): If the curve does not enclose the origin, the integral vanishes.
Case (b): If the curve encloses the origin, it is equal to -2π.
(ii) $-\pi/4$.
(iii) -2π.

2.12. Tutorial sheet 12

(1) (i) $\Phi(x,y) = (x, y, \frac{1}{2}(4 + y - x)), (x,y) \in \mathbb{R}^2$ as a parametrization.
The normal vector is $\Phi_x \times \Phi_y = (\frac{1}{2}, -\frac{1}{2}, 1)$.
(ii) $\Phi(u,v) = (u, a\sin v, a\cos v), u \in \mathbb{R}, 0 \le v \le 2\pi$,
normal is $(0, a\sin v, a\ \cos v)$.
(iii) $\Phi_\theta \times \Phi_t = \cos\theta \mathbf{u} + \ \sin\theta \mathbf{v}$, where $\mathbf{u}$ is a vector in the cross-section by a plane through the origin, and $\mathbf{v} = \mathbf{e} \times \mathbf{u}$.
(3) $2(\pi - 2)a^2$.
(5) $\frac{2\pi}{3}(3\sqrt{3} - 1)a^2$.
(7) (i) $\iint\limits_S F \cdot n|dS| = \dfrac{1}{2}$.
(ii) Explicitly, surface is $z = 1 - x - y$. Area($S_1^* = 1/2$).
(8) 0.
(9) $\frac{2\pi}{3}$.
(10) $\dfrac{2\pi}{3}$.

2.13. Tutorial sheet 13

(4) 8π.
(6) 3.
(7) 0.

2.14. Tutorial sheet 14

(2) $\iint_S \text{curl } F \cdot n|dS| = 0$.
(3) -108π.
(4) Stokes theorem cannot be applied to F directly.
(5) $\oint_C (y^2 - z^2)dx + (z^2 - x^2)dy + (x^2 - y^2)dz = -\dfrac{9a^3}{2}$.
(6) $-\pi a^2$.
(7) $\pi||\mathbf{u} \times \mathbf{v}||$.

CHAPTER 3

Solutions

3.0. Tutorial sheet 0

(1) **False**. $+\infty$ and $-\infty$ are just symbols to represent infinite intervals.
(2) **False**. The set of all even natural numbers is bounded below but not above.
(3) **False**. Any nonempty open interval has at least two distinct points. In fact, $\{x\} = [x, x]$ is a closed interval.
(4) **True**. Note that $\frac{2}{m} \leq 2$ for all $m \in \mathbb{N}$.
(5) **True**. Note that $\frac{2}{m} > 0$ for all $m \in \mathbb{N}$.
(6) **False**. For example, $(0, 1) \bigcup (2, 3)$ is not an interval.
(7) **True**. Let $I_\alpha,\ \alpha \in J$, be intervals and let $I = \bigcap_{\alpha \in J} I_\alpha \neq \emptyset$. Suppose $x, y \in I$. Then $x, y \in I_\alpha$ for every α, and if $x < y$ then $[x, y] \subset I_\alpha$ for every α. Thus $[x, y] \subset I$. This shows that I is an interval.
(8) **False**. For example, $\bigcap_{n=1}^{\infty} (\frac{-1}{n}, \frac{1}{n}) = \{0\}$ is not an open interval.
(9) **True**. Consult the solution of (10) below. That solution can be adapted here by using 'sup' in place of 'max' and using 'inf' in place of 'min'. Since 'sup' and 'inf' may not be discussed in the class, the student can only be expected to believe the statement in an intuitive way.
(10) **True**. Consider $I = \bigcap_{n=1}^{m} I_n$ where each I_n is a closed interval, and suppose $I \neq \emptyset$. We provide a solution only in the case where each I_n is a finite closed interval $[a_n, b_n]$; the solution can easily be modified to apply to the case where some of the I_n are infinite closed intervals. Let

$$a = \max\{a_1, a_2, \ldots, a_m\}$$

and

$$b = \min\{b_1, b_2, \ldots, b_m\}.$$

Note that $a \leq b$. (Indeed, here $a = a_k$ for some k and $b = b_l$ for some b_l; if $a_k = a > b = b_l$, then $[a_k, b_k]$ would not intersect $[a_l, b_l]$ forcing I to be empty). Now one has, for any n, $a_n \leq a \leq b \leq b_n$ so that $[a, b] \subset [a_n, b_n]$ for every n; thus

$$[a, b] \subset \bigcap_{n=1}^{m} [a_n, b_n].$$

Next, any number p that is strictly less than $a = a_k$ does not belong to the interval $[a_k, b_k]$; also, any number q that is strictly greater than $b = b_l$ does not belong to the interval $[a_l, b_l]$; this shows that

$$\bigcap_{n=1}^{m} [a_n, b_n] \subset [a, b].$$

Hence one has

$$\bigcap_{n=1}^{m} [a_n, b_n] = [a, b].$$

(11) **True**. Recall the Archimedean property of $\mathbb{R}$: For every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$.

(12) **True**. Note that $\sqrt{2}$ is an irrational number between 1 and 2. Let $r_1, r_2 \in \mathbb{Q}$ be such that $r_1 < r_2$. Then $r = r_2 - r_1 > 0$ and $r < r\sqrt{2} < 2r$ so that

$$r_1 = (2r_1 - r_2) + r < (2r_1 - r_2) + r\sqrt{2} < (2r_1 - r_2) + 2r = r_2.$$

Thus $s = (2r_1 - r_2) + r\sqrt{2}$ is an irrational number between r_1 and r_2.

3.1. Tutorial sheet 1

(1) For a given ϵ select
 (i) $N > \frac{10}{\epsilon}$,
 (ii) $N > \frac{5-\epsilon}{3\epsilon}$,
 (iii) $N > \frac{1}{\epsilon^3}$ as $\frac{1}{n^{\frac{1}{3}}} > \frac{n^{\frac{2}{3}}}{n+1} > |a_n|$,
 (iv) $N > \frac{2}{\epsilon}$ as $\frac{2}{n} > \frac{2-\frac{1}{n+1}}{n} = |a_n|$.

(2) (i) $\dfrac{n^2}{n^2+n} \le a_n \le \dfrac{n^2}{n^2+1} \Rightarrow \lim\limits_{n\to\infty} a_n = 1$.
 (ii) $0 < \dfrac{n!}{n^n} = \left(\dfrac{n-1}{n}\right)\left(\dfrac{n-2}{n}\right)\cdots\left(\dfrac{2}{n}\right)\left(\dfrac{1}{n}\right) \le \dfrac{1}{n} \Rightarrow \lim\limits_{n\to\infty} \dfrac{n!}{n^n} = 0$.
 (iii) $0 < \dfrac{n^3+3n^2+1}{n^4+8n^2+2} < \dfrac{4}{n} \Rightarrow \lim\limits_{n\to\infty} a_n = 0$.
 (iv) Let $n^{\frac{1}{n}} = 1 + h_n$. Then, for $n \ge 2$, one has

$$n = (1+h_n)^n \ge 1 + nh_n + \binom{n}{2}h_n^2 > \binom{n}{2}h_n^2.$$

 Thus $0 < h_n^2 < \frac{2}{n-1}$ $(n \ge 2)$ giving $\lim_{n\to\infty} h_n = 0$. So $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.
 (v) As

$$0 < \left|\frac{\cos(\pi\sqrt{n})}{n^2}\right| \le \frac{1}{n^2},$$

 $\lim_{n\to\infty} a_n = 0$.
 (vi) $\sqrt{n}(\sqrt{n+1} - \sqrt{n}) = \dfrac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \dfrac{1}{\sqrt{1+\frac{1}{n}}+1} \to \frac{1}{2}$ as $n \to \infty$.

(3) (i) $\{\frac{n^2}{n+1} = (n-1) + \frac{1}{n+1}\}_{n\ge1}$ is not convergent since $\frac{1}{n+1} \to 0$ as $n \to \infty$.
 (ii) $\{(-1)^n\left(\frac{1}{2} - \frac{1}{n}\right) = \frac{(-1)^n}{2} - \frac{(-1)^n}{n}\}_{n\ge1}$ is not convergent since $\frac{(-1)^n}{n} \to 0$ as $n \to \infty$.

(4) (i) Decreasing as $a_n = \frac{1}{n+\frac{1}{n}}$ and $\{n + \frac{1}{n}\}_{n\ge1}$ is increasing.
 (ii) Increasing as $\dfrac{a_{n+1}}{a_n} = \dfrac{6}{5} > 1$.
 (iii) Increasing as $a_{n+1} - a_n = \dfrac{n(n-1)-1}{n^2(1+n)^2} > 0$ for $n \ge 2$.

(5) (i) Using induction show that $a_n - \sqrt{2} > 0$ for $n \ge 2$, and note that $a_{n+1} - a_n = \frac{2-a_n^2}{2a_n} < 0$ $n \ge 2$; $\lim_{n\to\infty} a_n = \sqrt{2}$.
 (ii) Using induction show that $a_n < 2$ for all n, and note that
 $a_{n+1} - a_n = \dfrac{(2-a_n)(1+a_n)}{\sqrt{2+a_n}+a_n} > 0$ for all n; $\lim\limits_{n\to\infty} a_n = 2$.
 (iii) Using induction show that $a_n < 6$ for all n, and note that
 $a_{n+1} - a_n = \dfrac{6-a_n}{2} > 0$ for all n; $\lim\limits_{n\to\infty} a_n = 6$.

(6) $\lim_{n\to\infty} a_{n+1} = L$; and

$$||a_n| - |L|| \le |a_n - L|$$

 implies that $\lim_{n\to\infty} |a_n| = |L|$.

(7) Given $\epsilon = |L|/2$, there exists n_0 such that

$$(||a_n| - |L|| \leq)\ |a_n - L| < |L|/2 \ \text{ for all } n \geq n_0.$$

Thus $|a_n| > |L| - \frac{|L|}{2} = \frac{|L|}{2}$ for all $n \geq n_0$.

(8) Follows from the definition of 'limit of a sequence'.

(9) Both the statements are false. Consider, for example, $a_n = 1$ and $b_n = (-1)^n$.

(10) The implication $\Rightarrow$ is obvious. For the converse, let both $\{a_{2n}\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ $\blacksquare$ be convergent to ℓ, and let $\epsilon > 0$ be given. Choose $n_1, n_2 \in \mathbb{N}$ such that

$$|a_{2n} - \ell| < \epsilon \text{ for all } n \geq n_1, \text{ and} |a_{2n+1} - \ell| < \epsilon \text{ for all } n \geq n_2.$$

Let $n_0 = \max\{n_1, n_2\}$. Then

$$|a_n - \ell| < \epsilon \text{ for all } n \geq 2n_0 + 1.$$

3.2. Tutorial sheet 2

(1) (i) The statement is **false**. For example consider $a = -1, b = 1, c = 0$ and define $f, g : (-1, 1) \to \mathbb{R}$ by

$$f(x) = x, g(x) = \begin{cases} 1 & \text{if} \quad x = 0 \\ 1/x^2 & \text{if} \quad x \neq 0. \end{cases}$$

(ii) The statement is **true** since $0 \leq |f(x)g(x)| \leq M|f(x)|$ if $|g(x)| \leq M$ for x in (a, b).

(iii) The statement is **true** since $\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x) \lim_{x\to c} g(x)$.

(2) Let $\lim_{x\to\alpha} f(x) = L$. It follows from

$$|f(\alpha + h) - f(\alpha - h)| \leq |f(\alpha + h) - L| + |f(\alpha - h) - L|$$

that

$$\lim_{h\to 0} |f(\alpha + h) - f(\alpha - h)| = 0.$$

The converse is **false**; e.g. consider $\alpha = 0$ and

$$f(x) = \begin{cases} 1 & \text{if} \quad x = 0 \\ \frac{1}{|x|} & \text{if} \quad x \neq 0. \end{cases}$$

(3) (i) Continuous everywhere except at $x = 0$. This can be seen in part by considering the sequences $\{x_n\}_{n\geq 1}$, $\{y_n\}_{n\geq 1}$ where

$$x_n := \frac{1}{n\pi} \text{and} y_n := \frac{1}{2n\pi + \frac{\pi}{2}}.$$

Note that both $x_n, y_n \to 0$, but

$$f(x_n) \to 0, \text{and} f(y_n) \to 1.$$

(ii) Continuous everywhere. For ascertaining the continuity of f at $x = 0$, note that $|f(x)| \leq |x|$ and $f(0) = 0$.

(iii) Continuous everywhere on $[1, 3]$ except at $x = 2$.

(4) Taking $x = 0 = y$, we get $f(0+0) = 2f(0)$ so that $f(0) = 0$. By the assumption of the continuity of f at 0, $\lim_{x\to 0} f(x) = 0$. Thus,

$$\lim_{h\to 0} f(c + h) = \lim_{h\to 0} [f(c) + f(h)] = f(c)$$

showing that f is continuous at $x = c$.

Optional: First verify the equality for all $k \in \mathbb{Q}$ and then use the continuity of f to establish it for all $k \in \mathbb{R}$.

(5) Clearly, f is differentiable for all $x \neq 0$ and the derivative is

$$f'(x) = 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x}), x \neq 0.$$

Also,

$$f'(0) = \lim_{h\to 0} \frac{h^2 \sin(\frac{1}{h}) - 0}{h} = 0.$$

Clearly, f' is continuous at any $x \neq 0$. However, $\lim_{x\to 0} f'(x)$ does not exist. Indeed, for any $\delta > 0$, we can choose $n \in \mathbb{N}$ such that

$$x := \frac{1}{n\pi},\ y := \frac{1}{(n+1)\pi} \in (-\delta, \delta);$$

note that x and y satisfy $|f'(x) - f'(y)| = 2$.

(6)

$$0 \le \left| \frac{f(x+h) - f(x)}{h} \right| \le c|h|^{\alpha - 1}$$

implies by the Sandwich Theorem that

$$\lim_{h \to 0} \left| \frac{f(x+h) - f(x)}{h} \right| = 0 \quad \text{for all } x \in (a, b).$$

(7)

$$\lim_{h \to 0+} \frac{f(c+h) - f(c-h)}{2h} = \lim_{h \to 0+} \frac{1}{2} \left[\frac{f(c+h) - f(c)}{h} + \frac{f(c-h) - f(c)}{-h} \right]$$

$$= \frac{1}{2}[f'(c) + f'(c)] = f'(c).$$

The converse is **false**; consider, for example, $f(x) = |x|$ and $c = 0$.

(8)

$$f(x+y) = f(x)f(y) \Rightarrow f(0) = f(0)^2 \Rightarrow f(0) = 0 \text{ or } 1.$$

If $f(0) = 0$, then

$$f(x+0) = f(x)f(0) \Rightarrow f(x) = 0 \quad \text{for all } x.$$

Thus, trivially, f is differentiable. If $f(0) = 1$, then

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = f(c) \left(\lim_{h \to 0} \frac{f(h) - f(0)}{h} \right) = f'(0)f(c).$$

(9) (i) Let $f(x) = \cos(x)$. Then $f'(x) = -\sin(x) \neq 0$ for $x \in (0, \pi)$. Thus $g(y) = f^{-1}(y) = \cos^{-1}(y), \quad -1 < y < 1$ is differentiable and

$$g'(y) = \frac{1}{f'(x)}, \text{ where } x \text{ is such that } f(x) = y.$$

Therefore,

$$g'(y) = \frac{-1}{\sin(x)} = \frac{-1}{\sqrt{1 - \cos^2(x)}} = \frac{-1}{\sqrt{1 - y^2}}.$$

(ii) Note that

$$\operatorname{cosec}^{-1}(x) = \sin^{-1} \frac{1}{x} \text{ for } |x| > 1.$$

Since

$$\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}} \text{ for } |x| < 1,$$

one has, by the chain rule,

$$\frac{d}{dx} \operatorname{cosec}^{-1}(x) = \frac{1}{\sqrt{(1 - \frac{1}{x^2})}} (\frac{-1}{x^2}), |x| > 1.$$

(10)

$$\frac{dy}{dx} = f'\left(\frac{2x-1}{x+1}\right)\frac{d}{dx}\left(\frac{2x-1}{x+1}\right)$$
$$= \sin\left(\frac{2x-1}{x+1}\right)^2\left[\frac{3}{(x+1)^2}\right] = \frac{3}{(x+1)^2}\sin\left(\frac{2x-1}{x+1}\right)^2.$$

(11) $f(x) : |x| + |1-x|$

(12) For $c \in \mathbb{R}$, select a sequence $\{a_n\}_{n\geq 1}$ of rational numbers and a sequence $\{b_n\}_{n\geq 1}$ of irrational numbers, both converging to c. Then $\{f(a_n)\}_{n\geq 1}$ converges to 1 while $\{f(b_n)\}_{n\geq 1}$ converges to 0, showing that limit of f at c does not exist.

(13) Let $c \neq 1/2$.If $\{a_n\}_{n\geq 1}$ is a sequence of rational numbers and $\{b_n\}_{n\geq 1}$ a sequence of irrational numbers, both converging to c, then $g(a_n) = a_n \to c$, while $g(b_n) = 1 - b_n \to 1 - c$, and $c \neq 1 - c$. Thus g is not continuous at any $c \neq 1/2$. Further, if $\{a_n\}_{n\geq 1}$ is any sequence converging to $c = 1/2$, then $g(a_n) \to 1/2 = g(1/2)$. Hence, g is continuous at $c = 1/2$.

(14) Let $L = \lim_{x\to c} f(x)$. For $\epsilon = L - \alpha$, choose a δ such that

$$|f(c+h) - L| < \epsilon \text{for } 0 < |h| < \delta;$$

then $f(c+h) > L - \epsilon = \alpha$ for $0 < |h| < \delta$.

(15) $(i) \Rightarrow (ii):$ Take $\alpha = f'(c)$ and $\epsilon_1(h) = \begin{cases} \frac{f(c+h)-f(c)-\alpha h}{h}, & \text{if } h \neq 0 \\ 0, & \text{if } h = 0. \end{cases}$

$(ii) \Rightarrow (iii):$ $\displaystyle\lim_{h\to 0}\frac{|f(c+h)-f(c)-\alpha h|}{|h|} = \lim_{h\to 0}|\epsilon_1(h)| = 0$

$(iii) \Rightarrow (i):$ $\displaystyle\lim_{h\to 0}\left|\frac{f(c+h)-f(c)}{h} - \alpha\right| = 0 \Rightarrow \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ exists and in fact is equal to α.

3.3. Tutorial sheet 3

(1) $f(x) = x^3 - 6x + 3$ has stationary points at $x = \pm\sqrt{2}$.
Note that $f(-\sqrt{2}) = 4\sqrt{2} + 3 > 0, f(+\sqrt{2}) = -4\sqrt{2} + 3 < 0$. Therefore f has a root in $(-\sqrt{2}, \sqrt{2})$. Also, $f \to -\infty$ as $x \to -\infty$ implying that f has a root in $(-\infty, -\sqrt{2})$. Similarly, $f \to +\infty$ as $x \to +\infty$ implying that f has a root in $(\sqrt{2}, \infty)$. Since f has at most three roots, all its root are real.

(2) For $f(x) = x^3 + px + q,\ \ p > 0,\ \ f'(x) = 3x^2 + p > 0$. Therefore f is strictly increasing and can have **at most one** root. Since

$$\lim_{x \to \pm\infty} \left(\frac{p}{x^2} + \frac{q}{x^3} \right) = 0,$$

$$\frac{f(x)}{x^3} = 1 + \frac{p}{x^2} + \frac{q}{x^3} > 0$$

for $|x|$ very large. Thus $f(x) > 0$ if x is large positive and $f(x) < 0$ if x is large negative. By the Intermediate Value Property (IVP) f must have **at least one** root.

(3) By the IVP, there exists **at least one** $x_0 \in (a, b)$ such that $f(x_0) = 0$. If there were another $y_0 \in (a, b)$ such that $f(y_0) = 0$, then by Rolle's theorem there would exist some c between x_0 and y_0 (and hence between a and b) with $f'(c) = 0$, leading to a contradiction.

(4) Since f has 3 distinct roots say $r_1 < r_2 < r_3$, by Rolle's theorem $f'(x)$ has **at least two** real roots, say, x_1 and x_2 such that $r_1 < x_1 < r_2$ and $r_2 < x_2 < r_3$. Since $f'(x) = 3x^2 + p$, this implies that $p < 0$, and $x_1 = -\sqrt{-p/3}$, $x_2 = \sqrt{-p/3}$. Now, $f''(x_1) = 6x_1 < 0 \implies f$ has a local maximum at $x = x_1$. Similarly, f has a local minimum at $x = x_2$. Since the quadratic $f'(x)$ is negative between its roots x_1 and x_2 (so that f is decreasing over $[x_1, x_2]$) and f has a root r_2 in (x_1, x_2), we must have $f(x_1) > 0$ and $f(x_2) < 0$. Further,

$$f(x_1) = q + \sqrt{\frac{-4p^3}{27}}, f(x_2) = q - \sqrt{\frac{-4p^3}{27}}$$

so that

$$\frac{4p^3 + 27q^2}{27} = f(x_1)f(x_2) < 0.$$

(5) For some c between a and b, one has

$$\left| \frac{\sin(a) - \sin(b)}{a - b} \right| = |\cos(c)| \leq 1.$$

(6) By Lagrange's Mean Value Theorem (MVT) there exists $c_1 \in \left(a, \frac{(a+b)}{2}\right)$ such that

$$\frac{f\left(\frac{a+b}{2}\right) - f(a)}{\left(\frac{b-a}{2}\right)} = f'(c_1)$$

and there exists $c_2 \in \left(\frac{a+b}{2}, b\right)$ such that

$$\frac{f(b) - f\left(\frac{a+b}{2}\right)}{\left(\frac{b-a}{2}\right)} = f'(c_2).$$

Clearly one has $c_1 < c_2$, and adding the above equations one obtains

$$f'(c_1) + f'(c_2) = \frac{f(b) - f(a)}{\left(\frac{b-a}{2}\right)} = 2 \text{ (as } f(b) = b, f(a) = a).$$

(7) By Lagrange's MVT, there exists $c_1 \in (-a, 0)$ and there exists $c_2 \in (0, a)$ such that

$$f(0) - f(-a) = f'(c_1)a \text{ and } f(a) - f(0) = f'(c_2)a.$$

Using the given conditions, we obtain

$$f(0) + a \leq a \text{ and } a - f(0) \leq a$$

which implies $f(0) = 0$.

Optional: Consider $g(x) = f(x) - x$, $x \in [-a, a]$. Since $g'(x) = f'(x) - 1 \leq 0$, g is decreasing over $[-a, a]$. As $g(-a) = g(a) = 0$, we have $g \equiv 0$.

(8) (i) No such function exists in view of Rolle's theorem.

(ii) Possible, $f(x) = \frac{x^2}{2} + x$

(iii) $f'' \geq 0 \Rightarrow f'$ increasing. As $f'(0) = 1$, by Lagrange's MVT we have $f(x) - f(0) \geq x$ for $x > 0$. Hence f with the required properties cannot exist.

(iv) Possible,

$$f(x) = \begin{cases} \frac{1}{1-x} & \text{if } x \leq 0 \\ 1 + x + x^2 & \text{if } x > 0. \end{cases}$$

(9) The points to check are the end points $x = -2$ and $x = 5$, the point of non-differentiability $x = 0$, and the stationary point $x = 2$. The values of f at these points are given by

$$f(-2) = f(2) = 13, f(0) = 1, f(5) = -14.$$

Thus, global max $= 13$ at $x = \pm 2$, and global min $= -14$ at $x = 5$.

(10) Let $2a$ be the width of the window and h be its height. Then $2a + 2h + \pi a = p$, and $0 \leq a \leq \dfrac{p}{2+\pi}$. As the area of the colored glass is $\frac{\pi a^2}{2}$ and the area of the plane glass is $2ah$, the total light admitted is

$$L(a) = 2ah + \frac{\pi a^2}{4} = 2a\left[\frac{p - (\pi + 2)a}{2}\right] + \frac{\pi a^2}{4} \quad (0 \leq a \leq \frac{p}{2+\pi}).$$

Since

$$L'(a) = 0 \Rightarrow a = \frac{2p}{8 + 3\pi}$$

and

$$L'(a) > 0 \text{ in } [\,0, \frac{2p}{8+3\pi}\,) \text{ and } L'(a) < 0 \text{ in } (\frac{2p}{8+3\pi}, \frac{p}{2+\pi}],$$

$a = \dfrac{2p}{8+3\pi}$ must give the global maximum. That yields $h = \dfrac{p(4+\pi)}{2(8+3\pi)}$.

3.4. Tutorial sheet 4

(1) (i) $f(x) = 2x^3 + 2x^2 - 2x - 1 \Rightarrow f'(x) = 6x^2 + 4x - 2 = 2(x+1)(3x-1)$. Thus, $f'(x) > 0$ in $(-\infty, -1) \cup (1/3, \infty)$ so that $f(x)$ is strictly increasing in those intervals, and $f'(x) < 0$ in $(-1, 1/3)$ so that $f(x)$ is strictly decreasing in that interval.

4.1.1-eps-converted-to.pdf

Graph of f

Thus, $f(x)$ has a local maximum at $x = -1$, and a local minimum at $x = \frac{1}{3}$. As $f''(x) = 12x + 4$ we have that $f(x)$ is convex in $\left(-\frac{1}{3}, \infty\right)$ and concave in $\left(-\infty, -\frac{1}{3}\right)$, with a point of inflection at $x = -\dfrac{1}{3}$.

(ii) $y = \dfrac{x^2}{x^2+1} \Rightarrow \lim_{x \to \pm\infty} y = 1 \Rightarrow y = 1$ is an asymptote.

4.1.2-eps-converted-to.pdf

Graph of f

$y' = \dfrac{2x}{(x^2+1)^2} \Rightarrow y$ is increasing in $(0, \infty)$ and decreasing in $(-\infty, 0)$.

Further, $y'' = -\frac{2(3x^2-1)}{(x^2+1)^3}$ implies that $y'' > 0$ if $|x| < \frac{1}{\sqrt{3}}$, and $y'' < 0$ if $|x| > \frac{1}{\sqrt{3}}$. Therefore,

y is convex in $\left(-\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$ and concave in $\mathbb{R} \backslash \left[-\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right]$ with the points $x = \pm\dfrac{1}{\sqrt{3}}$ being the points of inflection.

(iii) $f(x) = 1 + 12|x| - 3x^2$; f is not differentiable at $x = 0$; $f(0) = 1$.

4.1-eps-converted-to.pdf

Graph of f

Further,

$$f'(x) = 0 \text{ at } x = \pm 2,$$
$$f'(x) < 0 \text{ in } (-2, 0) \cup (2, 5],$$
$$f'(x) > 0 \text{ in } (0, 2),$$

and

$$f''(x) = -6 \text{ in } (-2, 0) \cup (0, 5).$$

Thus

$$f \text{ is concave in} (-2, 0) \cup (0, 5),$$
$$\text{decreasingin} (-2, 0) \cup (2, 5),$$

and

$$\text{increasingin} (0, 2);$$

further, f has a global maximum at $x = \pm 2$.

(2) In view of the given conditions, f has a local max at $x = -2$ and a local min at $x = 2$, f is concave in $(-\infty, 0)$ and convex in $(0, \infty)$, and $x = 0$ is a point of inflection.

(3) **Motivate students to discover their own examples**.

(i) $f' > 0, f'' > 0$. Example: $f(x) = x^2$; $0 < x < 1$.
(ii) $f' > 0, f'' < 0$. Example: $f(x) = \sqrt{x}$; $0 < x < 1$.
(iii) $f' < 0, f'' > 0$. Example: $f(x) = -\sqrt{x}$; $0 < x < 1$.
(iv) $f' < 0, f'' < 0$. Example: $f(x) = -x^2$; $0 < x < 1$.

(4) (i) The statement is **true**. In $(c - \delta, c + \delta)$, $f(x) \le f(c), g(x) \le g(c)$. As all the quantities are non-negative, $f(x)g(x) \le f(c)g(c)$ in $(c - \delta, c + \delta)$.
(ii) The statement is **false.** E.g. $f(x) = g(x) =! + \sin(x)$, $c = 0$.

(5) Let P_n be the partition of $[0, 2]$ into 2×2^n equal parts. Then $U(P_n, f) = 3$ and

$$L(P_n, f) = 1 + 1 \times \frac{1}{2^n} + 2 \times \frac{(2^n - 1)}{2^n} \to 3$$

as $n \to \infty$. Thus, $\int_0^2 f(x)dx = 3$.

(6) $f(x) \ge 0 \Rightarrow U(P, f) \ge 0, L(P, f) \ge 0 \Rightarrow \int_a^b f(x)dx \ge 0$.
Suppose, moreover, f is continuous and $\int_a^b f(x)dx = 0$. Assume $f(c) > 0$ for some c in $[a, b]$. Then $f(x) > \frac{f(c)}{2}$ in a δ-nbhd of c for some $\delta > 0$. This implies that

$$U(P, f) > \delta \times \frac{f(c)}{2}$$

for any partition P, and hence, $\int_a^b f(x)dx \ge \delta f(c)/2 > 0$, a contradiction.

(7) (i) $S_n = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{\frac{3}{2}} \longrightarrow \int_0^1 (x)^{3/2}dx = \frac{2}{5}$

(ii) $S_n = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\left(\frac{i}{n}\right)^2+1} \longrightarrow \int_0^1 \frac{dx}{x^2+1} = \frac{\pi}{4}$

(iii) $S_n = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{\frac{i}{n}+1}} \longrightarrow \int_0^1 \frac{dx}{\sqrt{x+1}} = 2(\sqrt{2}-1)$

(iv) $S_n = \frac{1}{n}\sum_{i=1}^{n}\cos\frac{i\pi}{n} \longrightarrow \int_0^1 \cos\pi x dx = 0$

(v) $S_n \longrightarrow \int_0^1 x dx + \int_1^2 x^{3/2}dx + \int_2^3 x^2 dx = \frac{1}{2} + \frac{2}{5}(4\sqrt{2}-1) + \frac{19}{3}$

(8) Let $F(x) = \int_a^x f(t)dt$. Then $F'(x) = f(x)$. Note that

$$\int_{u(x)}^{v(x)} f(t)dt = \int_a^{v(x)} f(t)dt - \int_a^{u(x)} f(t)dt = F(v(x)) - F(u(x)).$$

By the Chain Rule one has

$$\begin{aligned} \frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt &= F'(v(x))v'(x) - F'(u(x))u'(x) \\ &= f(v(x))v'(x) - f(u(x))u'(x). \end{aligned}$$

(a) $\frac{dy}{dx} = \frac{1}{dx/dy} = \sqrt{1+y^2}, \frac{d^2y}{dx^2} = \frac{y}{\sqrt{1+y^2}}\frac{dy}{dx} = y.$

(b) (i) $F'(x) = \cos((2x)^2)2 = 2\cos(4x^2).$

(ii) $F'(x) = \cos(x^2)2x = 2x\cos(x^2)$

(9) Define

$$F(x) = \int_x^{x+p} f(t)dt, x \in \mathbb{R}.$$

Then $F'(x) = 0$ for every x.

(10) Expand $\sin\lambda(x-t)$ in the integrand, evaluate $g'(x), g''(x)$, and simplify to show lhs=rhs.

3.5. Tutorial sheet 5

(1) (i) $\int_0^1 y\ dx = \int_0^1 (1 + x - 2\sqrt{x})\ dx = \frac{1}{6}$
(ii) $2\int_0^2 (2x^2 - (x^4 - 2x^2))\ dx = 2\int_0^2 (4x^2 - x^4)\ dx = \frac{128}{15}$
(iii) $\int_1^3 (3y - y^2 - (3 - y))\ dy = \int_1^3 (4y - y^2 - 3)\ dy = \frac{4}{3}$

(2) $\int_0^{1-a} (x - x^2 - ax)\ dx = \int_0^{1-a} ((1-a)x - x^2)\ dx = 4.5$ gives $\frac{(1-a)^3}{6} = 4.5$ so that $a = -2$.

(3) Required area $= 2 \times \int_0^{\pi/3} \frac{1}{2}(r_2^2 - r_1^2) d\theta$
$= 4a^2 \int_0^{\pi/3} (8\cos^2\theta - 2\cos\theta - 1) d\theta = 4\pi a^2$.

(4) (i) Length $= \displaystyle\int_0^{2\pi} \sqrt{(1 - \cos(t))^2 + \sin^2(t)} dt$
$= \displaystyle\int_0^{2\pi} 2|\sin(t/2)| dt = 4\int_0^{\pi} |\sin(u)| du = 8.$

(ii) Length $= \displaystyle\int_0^{\pi/4} \sqrt{1 + y'^2} dx = \int_0^{\pi/4} \sqrt{1 + \cos(2x)} dx$
$= \sqrt{2}\int_0^{\pi/4} |\cos(x)| dx = 1.$

(5) $\displaystyle\frac{dy}{dx} = x^2 + \left(-\frac{1}{4x^2}\right).$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + x^4 + \frac{1}{16x^4} - \frac{1}{2}} = x^2 + \frac{1}{4x^2}.$$

Therefore,

$$\text{Length} = \int_1^3 \left(x^2 + \frac{1}{4x^2}\right) dx = \left[\frac{x^3}{3} - \frac{1}{4x}\right]_1^3 = \frac{53}{6}.$$

The surface area is

$$S = \int_1^3 2\pi(y + 1)\frac{ds}{dx} dx = \int_1^3 2\pi \left(\frac{x^3}{3} + \frac{1}{4x} + 1\right)\left(x^2 + \frac{1}{4x^2}\right) dx$$

$$= 2\pi \left[\frac{x^6}{18} + \frac{x^3}{3} + \frac{x^2}{6} - \frac{1}{32x^2} - \frac{1}{4x}\right]_1^3 = \left(101 + \frac{5}{18}\right)\pi.$$

(6) The diameter of the circle at a point x is given by

$$(8 - x^2) - x^2, \ \ -2 \le x \le 2.$$

So the area of the cross-section at x is $A(x) = \pi(4 - x^2)^2$. Thus

$$\text{Volume} = \int_{-2}^2 \pi(4 - x^2)^2 dx = 2\pi\int_0^2 (4 - x^2)^2 dx = \frac{512\pi}{15}.$$

(7) In the first octant, the sections perpendicular to the y-axis are squares with

$$0 \le x \le \sqrt{a^2 - y^2}, \ \ 0 \le z \le \sqrt{a^2 - y^2}, \ \ 0 \le y \le a.$$

Since the squares have sides of length $\sqrt{a^2 - y^2}$, the area of the cross-section at y is $A(y) = 4(a^2 - y^2)$. Thus the required volume is

$$\int_{-a}^a A(y) dy = 8\int_0^a (a^2 - y^2) dy = \frac{16a^3}{3}.$$

(8) Let the line be along z-axis, $0 \le z \le h$. For any fixed z, the section is a square of area r^2. Hence the required volume is $\int_0^h r^2 dz = r^2 h$.

(9) **Washer Method**

Area of washer $= \pi(1+y)^2 = \pi(1+(3-x^2))^2 = \pi(4-x^2)^2$ so that

Volume $= \int_{-2}^{2} \pi(4-x^2)^2 dx = 512\pi/15$.

(This is the same integral as in (6) above).

Shell method

Area of shell $= 2\pi(y-(-1))2x = 4\pi(1+y)\sqrt{3-y}$ so that

$$\text{Volume} = \int_{-1}^{3} 4\pi(1+y)\sqrt{3-y}dy = 512\pi/15.$$

(10) **Washer Method**

Required volume = Volume of the sphere -Volume generated by revolving the shaded region is

$32\pi/3 - [\int_{-1}^{1} \pi x^2 dy - \pi(\sqrt{3})^2 2] = 32\pi/3 - 2\pi[\int_0^1 (4-y^2)dy - 3]$

$= 32\pi/3 - 2\pi[11/3 - 3] = 28\pi/3$

Shell Method

Required volume = Volume of the sphere -Volume generated by revolving the shaded region is

$= 32\pi/3 - \int_{\sqrt{3}}^{2} 2\pi x(2y)dx = 32\pi/3 - 4\pi \int_{\sqrt{3}}^{2} x\sqrt{4-x^2}dx$

$= 32\pi/3 - 4\pi(1/3) = 28\pi/3$

3.6. Tutorial sheet 6

(1) (i) $\{(x,y) \in \mathbb{R}^2 \mid x \neq \pm y\}$.
(ii) $\mathbb{R}^2 - \{(0,0)\}$.

(2) (i) A level curve corresponding to any of the given values of c is the straight line $x - y = c$ in the xy-plane. A contour line corresponding to any of the given values of c is the same line shifted to the plane $z = c$ in $\mathbb{R}^3$.

(ii) Level curves do not exist for $c = -3, -2, -1$. The level curve corresponding to $c = 0$ is the point $(0,0)$. The level curves corresponding to $c = 1, 2, 3, 4$ are concentric circles centered at the origin in the xy-plane. Contour lines corresponding to $c = 1, 2, 3, 4$ are the cross-sections in $\mathbb{R}^3$ of the paraboloid $z = x^2 + y^2$ by the plane $z = c$, i.e., circles in the plane $z = c$ centered at $(0, 0, c)$.

(iii) For $c = -3, -2, -1$, level curves are rectangular hyperbolas $xy = c$ in the xy-plane with branches in the second and fourth quadrant. For $c = 1, 2, 3, 4$, level curves are rectangular hyperbolas $xy = c$ in the xy-plane with branches in the first and third quadrant. For $c = 0$, the corresponding level curve (resp. the contour line) is the union of the x-axis and the y-axis in the xy-plane (resp. in the xyz-space). A contour line corresponding to a non-zero c is the cross-section of the hyperboloid $z = xy$ by the plane $z = c$, i.e., a rectangular hyperbola in the plane $z = c$.

(3) (i) Discontinuous at $(0,0)$. (Check $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ using $y = mx^3$).

(ii) Continuous at $(0,0)$:

$$\left| xy\frac{x^2 - y^2}{x^2 + y^2} \right| \leq |xy|\frac{x^2 + y^2}{x^2 + y^2} = |xy|.$$

(iii) Continuous at $(0,0)$:

$$|f(x,y)| \leq 2(|x| + |y|) \leq 4\sqrt{x^2 + y^2}.$$

(4) (i) Use the sequential definition of limit: $(x_n, y_n) \to (a,b) \implies x_n \to a \text{and} y_n \to b \implies f(x_n) \to f(a) \text{and} g(y_n) \to g(b) \implies f(x_n) \pm g(y_n) \to f(a) \pm g(b)$ by the continuity of f, g and limit theorems for sequences.

(ii) $(x_n, y_n) \to (a,b) \implies x_n \to a \text{and} y_n \to b \implies f(x_n) \to f(a) \text{and} g(y_n) \to g(b) \implies f(x_n)g(y_n) \to f(a)g(b)$ by the continuity of f, g and limit theorems for sequences.

(iii) Follows from (i) above and the following:

$$\min\{f(x), g(y)\} = \frac{f(x) + g(y)}{2} - \frac{|f(x) - g(y)|}{2},$$

$$\max\{f(x), g(y)\} = \frac{f(x) + g(y)}{2} + \frac{|f(x) - g(y)|}{2}.$$

(5) Note that limits are different along different paths: $f(x,x) = 1$ for every x and $f(x,0) = 0$.

(6) (i) $f_x(0,0) = 0 = f_y(0,0)$.
(ii)

$$f_x(0,0) = \lim_{h\to 0} \frac{\sin^2(h)/|h|}{h} = \lim_{h\to 0} \frac{\sin^2(h)}{h|h|}$$

does not exist (Left Limit $\neq$ Right Limit). Similarly, $f_y(0,0)$ does not exist.

(7) $|f(x,y)| \leq x^2+y^2 \Rightarrow f$ is continuous at$(0,0)$.
It is easily checked that $f_x(0,0)=f_y(0,0)=0$.
Now,

$$f_x = 2x\left(\sin\left(\frac{1}{x^2+y^2}\right) - \frac{1}{x^2+y^2}\cos\left(\frac{1}{x^2+y^2}\right)\right).$$

The function $2x\sin\left(\frac{1}{x^2+y^2}\right)$ is bounded in any disc centered at $(0,0)$, while $\frac{2x}{x^2+y^2}\cos\left(\frac{1}{x^2+y^2}\right)$ is unbounded in any such disc.
(To see this, consider $(x,y)=\left(\frac{1}{\sqrt{n\pi}},0\right)$ for n a large positive integer.)
Thus f_x is unbounded in any disc around $(0,0)$.

(8) $f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\sin\frac{1}{h}$ does not exist. Similarly $f_y(0,0)$ does not exist. Clearly, f is continuous at $(0,0)$.

(9) (i) Let $v=(a,b)$ be any unit vector in $\mathbb{R}^2$. We have

$$\left(D_v f\right)(0,0) = \lim_{h\to 0}\frac{f(hv)}{h} = \lim_{h\to 0}\frac{f(ha,hb)}{h} = \lim_{h\to 0}\frac{h^2ab\left(\frac{a^2-b^2}{a^2+b^2}\right)}{h} = 0.$$

Therefore $\left(D_v f\right)(0,0)$ exists and equals 0 for every unit vector $v\in\mathbb{R}^2$.
For considering differentiability, note that $f_x(0,0) = \left(D_{\hat{i}}f\right)(0,0) = 0 = f_y(0,0) = \left(D_{\hat{j}}f\right)(0,0)$. We have then

$$\lim_{(h,k)\to(0,0)}\frac{|f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)|}{\sqrt{h^2+k^2}}$$
$$= \lim_{(h,k)\to(0,0)}\frac{|hk(h^2-k^2)|}{(h^2+k^2)^{3/2}} = 0 \text{ since}$$

$$0 \leq \frac{|hk(h^2-k^2)|}{(h^2+k^2)^{3/2}} \leq \frac{|hk|}{\sqrt{h^2+k^2}}\frac{h^2+k^2}{h^2+k^2}$$
$$\leq \frac{\sqrt{h^2+k^2}\sqrt{h^2+k^2}}{\sqrt{h^2+k^2}} = \sqrt{h^2+k^2}.$$

Thus f is differentiable at $(0,0)$.

(ii) Note that, for any unit vector $v=(a,b)$ in $\mathbb{R}^2$, we have

$$D_v f(0,0) = \lim_{h\to 0}\frac{h^3a^3}{h(h^2(a^2+b^2))} = \lim_{h\to 0}\frac{a^3}{(a^2+b^2)} = \frac{a^3}{(a^2+b^2)}.$$

To consider differentiability, note that $f_x(0,0)=1,\ f_y(0,0)=0$ and

$$\lim_{(h,k)\to(0,0)}\frac{|f(h,k)-h\times 1-k\times 0|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)}\frac{\left|h^3/(h^2+k^2)-h\right|}{\sqrt{h^2+k^2}}$$
$$= \lim_{(h,k)\to(0,0)}\frac{\left|hk^2\right|}{(h^2+k^2)^{3/2}}$$

does not exist (consider, for example, $k=mh$). Hence f is not differentiable at $(0,0)$.

(iii) For any unit vector $v \in \mathbb{R}^2$, one has

$$(D_v f)(0,0) = \lim_{h\to 0} \frac{h^2(a^2+b^2)\sin\left[\frac{1}{h^2(a^2+b^2)}\right]}{h} = 0.$$

Also,

$$\lim_{(h,k)\to(0,0)} \frac{\left|(h^2+k^2)\sin\left[\frac{1}{(h^2+k^2)}\right]\right|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \sqrt{h^2+k^2}\sin\left(\frac{1}{h^2+k^2}\right) = 0$$

therefore f is differentiable at $(0,0)$.

(10) $f(0,0) = 0$, $|f(x,y)| \leq \sqrt{x^2+y^2} \implies f$ is continuous at $(0,0)$. Let v be a unit vector in $\mathbb{R}^2$. For $v = (a,b)$, with $b \neq 0$, one has

$$(D_v)\, f(0,0) = \lim_{h\to 0} \frac{1}{h}\frac{hb}{|hb|}\sqrt{h^2a^2+h^2b^2} = \frac{(a^2+b^2)b}{|b|}.$$

If $v = (a,0)$, then $(D_v f)(0,0) = 0$. Hence $(D_v f)(0,0)$ exists for every unit vector $v \in \mathbb{R}^2$. Further,

$$f_x(0,0) = 0, \;\; f_y(0,0) = 1,$$

and

$$\lim_{(h,k)\to(0,0)} \frac{|f(h,k) - 0 - h\times 0 - k\times 1|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{\left|\frac{k}{|k|}\sqrt{h^2+k^2} - k\right|}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \left|\frac{k}{|k|} - \frac{k}{\sqrt{h^2+k^2}}\right|$$

does not exist (consider, for example, $k = mh$) so that f is not differentiable at $(0,0)$.

3.7. Tutorial sheet 7

(1) $(\nabla F)(1,-1,3) = (\dfrac{\partial F}{\partial x}(1,-1,3), \dfrac{\partial F}{\partial y}(1,-1,3), \dfrac{\partial F}{\partial z}(1,-1,3)) = (0,4,6).$

The **tangent plane** to the surface $F(x,y,z) = 7$ at the point $(1,-1,3)$ is given by $0 \times (x-1) + 4 \times (y+1) + 6 \times (z-3) = 0,$ i.e., $2y + 3z = 7.$

The **normal line** to the surface $F(x,y,z) = 7$ at the point $(1,-1,3)$ is given by
$x = 1,\ 3y - 2z + 9 = 0.$

(2) $u = \dfrac{(2,2,1)}{\sqrt{2^2+2^2+1^2}} = \left(\dfrac{2}{3}, \dfrac{2}{3}, \dfrac{1}{3}\right)$
and
$(\nabla F)(2,2,1) = (3,-5,2).$
Therefore,
$(D_u F)(2,2,1) = (\nabla F)(2,2,1) \cdot u = \dfrac{6}{3} - \dfrac{10}{3} + \dfrac{2}{3} = -\dfrac{2}{3}.$

(3) Given that $\sin(x+y) + \sin(y+z) = 1$ (with $\cos(y+z) \neq 0$).
(It may be assumed that z is a sufficiently smooth function of x and y).
Differentiating w.r.t. x while keeping y fixed, we get
$\cos(x+y) + \cos(y+z)\frac{\partial z}{\partial x} = 0.$ $(*)$
Similarly, differentiating w.r.t. y while keeping x fixed, we get
$\cos(x+y) + \cos(y+z)\left(1 + \frac{\partial z}{\partial y}\right) = 0.$ $(**)$
Differentiating (*) w.r.t y we have
$-\sin(x+y) - \sin(y+z)\left(1 + \frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x} + \cos(y+z)\frac{\partial^2 z}{\partial x \partial y} = 0.$
Thus, using (*) and (**), we have

$$\begin{aligned}
&\frac{\partial^2 z}{\partial x \partial y} \\
&= \frac{1}{\cos(y+z)}\left[\sin(x+y) + \sin(y+z).\left(1 + \frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x}\right] \\
&= \frac{1}{\cos(y+z)}\left[\sin(x+y) + \sin(y+z)\left(-\frac{\cos(x+y)}{\cos(y+z)}\right)\left(-\frac{\cos(x+y)}{\cos(y+z)}\right)\right] \\
&= \frac{\sin(x+y)}{\cos(y+z)} + \tan(y+z)\frac{\cos^2(x+y)}{\cos^2(y+z)}.
\end{aligned}$$

(4) We have

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k},$$

where (noting that $k \neq 0$)

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h} = -k \text{and } f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0.$$

Therefore,

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{-k-0}{k} = -1; \text{ similarly} f_{yx}(0,0) = 1.$$

Thus

$$f_{xy}(0,0) \neq f_{yx}(0,0).$$

By directly computing f_{xy}, f_{yx} for $(x,y) \neq (0,0)$, one observes that these are not continuous at $(0,0)$.
(In the following $H_f(a,b)$ denotes the **Hessian matrix** of a sufficiently smooth function f at the point (a,b)).

(5) (i) We have

$$f_x(-1,2) = 0 \quad = \quad f_y(-1,2); H_f(-1,2) = \begin{bmatrix} 12 & 0 \\ 0 & 48 \end{bmatrix}.$$
$$D = 12 \times 48 \quad > \quad 0, f_{xx}(-1,2) = 12 > 0 \Rightarrow (-1,2) \text{ is a point of local minimum of } f.$$

(ii) We have

$$f_x(0,0) = 0 \quad = \quad f_y(0,0); H_f(0,0) = \begin{bmatrix} 6 & -2 \\ -2 & 10 \end{bmatrix}.$$
$$D = 60 - 4 \quad > \quad 0, f_{xx}(0,0) = 6 > 0 \Rightarrow (0,0) \text{ is a point of local minimum of } f.$$

(6) (i) $f_x = e^{-\frac{(x^2+y^2)}{2}} \left(2x - x^3 + xy^2\right), f_y = e^{-\frac{(x^2+y^2)}{2}} \left(-2y + y^3 - x^2 y\right).$
Critical points are $(0,0), (\pm\sqrt{2},0), (0,\pm\sqrt{2})$.

$$H_f(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \quad \Rightarrow \quad (0,0) \text{is a saddle point of } f.$$
$$H_f(\pm\sqrt{2},0) = \begin{bmatrix} -\frac{4}{e} & 0 \\ 0 & -\frac{4}{e} \end{bmatrix} \quad \Rightarrow \quad (\pm\sqrt{2},0) \text{is a point of local maximum of } f.$$
$$H_f(0,\pm\sqrt{2}) = \begin{bmatrix} \frac{4}{e} & 0 \\ 0 & \frac{4}{e} \end{bmatrix} \quad \Rightarrow \quad (0,\pm\sqrt{2}) \text{is a point of local minimum of } f.$$

(ii) $f_x = 3x^2 - 3y^2$ and $f_y = -6xy$ imply that $(0,0)$ is the only critical point of f. ∎
Now,

$$H_f(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, the standard derivative test fails.
However, $f(\pm\epsilon, 0) = \pm\epsilon^3$ for any ϵ so that $(0,0)$ is a saddle point of f. ∎

(7) From $f(x,y) = (x^2 - 4x)\cos y$ $(1 \leq x \leq 3, -\pi/4 \leq y \leq \pi/4)$, we have $f_x = (2x-4)\cos y$ and $f_y = -(x^2-4x)\sin y$.
Thus the only critical point of f is $P = (2,0)$; note that $f(P) = -4$.
Next, $g_\pm(x) \equiv f(x, \pm\frac{\pi}{4}) = \frac{(x^2-4x)}{\sqrt{2}}$ $(1 \leq x \leq 3)$ has $x = 2$ as the only critical point so that we consider $P_\pm = (2, \pm\frac{\pi}{4})$; note that $f(P_\pm) = \frac{-4}{\sqrt{2}}$.
We also need to check $g_\pm(1) = f(1, \pm\frac{\pi}{4})$ $(\equiv f(Q_\pm))$ and $g_\pm(3) = f(3, \pm\frac{\pi}{4})$ $(\equiv f(S_\pm))$; note that $f(Q_\pm) = \frac{-3}{\sqrt{2}}, f(S_\pm) = -\frac{-3}{\sqrt{2}}$.
Next, consider $h(y) \equiv f(1,y) = -3\cos y$ $(-\pi/4 \leq y \leq \pi/4)$. The only critical point of h is $y = 0$; note that $h(0) = f(1,0)$ $(\equiv f(M)) = -3$. ($h(\pm\pi/4)$ is just $f(Q_\pm)$).
Finally, consider $k(y) \equiv f(3,y) = -3\cos y$ $(-\pi/4 \leq y \leq \pi/4)$. The only critical point of k is $y = 0$; note that $k(0) = f(3,0)$ $(\equiv f(T)) = -3$. ($k(\pm\pi/4)$ is just $f(S_\pm)$).
Summarizing, we have the following table:

Points	P_+	P_-	Q_+	Q_-	S_+	S_-	T	P	M
Values	$-\frac{4}{\sqrt{2}}$	$-\frac{4}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}}$	-3	-4	-3

By inspection one finds that

$f_{\min} = -4$ is attained at $P = (2, 0)$ and
$f_{\max} = -\frac{3}{\sqrt{2}}$ at $Q_\pm = (1, \pm\pi/4)$ and at $S_\pm = (3, \pm\pi/4)$.

3.8. Tutorial sheet 8

(1) (i) $\int_1^e \left(\int_{\log y}^1 dx \right) dy$

(ii) $\int_{-1}^1 \left(\int_{x^2}^1 f(x,y)dy \right) dx.$

(2) (i) $\int_0^\pi \left(\int_0^y \frac{\sin(y)}{y} dx \right) dy = \int_0^\pi \sin(y)dy = 2.$

(ii) $\int_0^1 \left(\int_0^x x^2 e^{xy} dy \right) dx = \int_0^1 x(e^{x^2} - 1)dx$

$$= \frac{1}{2}(e-1) - \frac{1}{2} = \frac{1}{2}(e-2).$$

(iii) $\int_0^2 (\tan^{-1}(\pi x) - \tan^{-1}(x))dx = \int_0^2 \left(\int_x^{\pi x} \frac{dy}{1+y^2} \right) dx$

$$= \int\int_{R_1+R_2} \frac{d(x,y)}{1+y^2} = \int_0^2 \left(\int_{y/\pi}^y \frac{dx}{1+y^2} \right) dy + \int_2^{2\pi} \left(\int_{y/\pi}^2 \frac{dx}{1+y^2} \right) dy$$

$$= \int_0^2 (1 - \frac{1}{\pi}) \frac{ydy}{1+y^2} + \int_2^{2\pi} (2 - \frac{y}{\pi}) \frac{dy}{1+y^2}$$

$$= \frac{\pi - 1}{2\pi} \log(1+y^2)|_0^2 + 2\tan^{-1} y|_2^{2\pi} - \frac{1}{2\pi} \log(1+y^2)|_2^{2\pi}$$

$$= \frac{\pi-1}{2\pi} \log 5 + 2(\tan^{-1} 2\pi - \tan^{-1} 2) - \frac{1}{2\pi} \left[\log \frac{(4\pi^2+1)}{5} \right].$$

(3) $\iint_D e^{x^2} d(x,y) = \int_0^1 \left(\int_0^{2x} e^{x^2} dy \right) dx = \int_0^1 2xe^{x^2} dx = e - 1.$

(4) Put

$$x = \frac{u-v}{2}, y = \frac{u+v}{2}.$$

Then the rectangle $R = \{\pi \le u \le 3\pi,\ -\pi \le v \le \pi\}$
in the uv-plane gets mapped to D, a parallelogram in the xy-plane.
Further,

$$J = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = \frac{1}{2}$$

and then

$$\iint_D (x-y)^2 \sin^2(x+y) dxdy = \iint_R v^2 \sin^2(u) \frac{1}{2} dudv$$

$$= \frac{1}{2} \left(\int_{-\pi}^{\pi} v^2 dv \right) \left(\int_\pi^{3\pi} \sin^2(u) du \right) = \frac{1}{2} \left(2 \times \frac{\pi^3}{3} \right) (\pi) = \frac{\pi^4}{3}.$$

(5) Put

$$x = \frac{u}{v},\ \ y = uv.$$

Then the rectangle $R = \{1 \le u \le 3, 1 \le v \le 2\}$ in the uv-plane gets mapped to D
in the xy-plane.
Further,

$$J = \begin{vmatrix} 1/v & -u/v^2 \\ v & u \end{vmatrix} = \frac{2u}{v}$$

and then

$$\iint_D d(x,y) = \text{Area}(D) = \iint_R \frac{2u}{v} du dv = \left(\int_1^3 2u du\right)\left(\int_1^2 \frac{dv}{v}\right) = 8\log 2.$$

(6) (i) Setting

$$x = \rho\cos(\theta), y = \rho\sin(\theta), \quad 0 \le \rho \le r, 0 \le \theta \le 2\pi,$$

and using $J = \rho$, we get

$$\iint_{D(r)} e^{-(x^2+y^2)} d(x,y) = \int_0^{2\pi}\int_0^r e^{-\rho^2}\rho d\rho d\theta = \pi(1-e^{-r^2}).$$

Therefore, letting $r \to \infty$, we obtain the limit to be π.

(ii) By symmetry, the required limit is $\lim_{r\to\infty} \frac{\pi}{4}\left(1-e^{-r^2}\right) = \frac{\pi}{4}$.

(iii) Let

$$I(r) = \{|x| \le r, |y| \le r\}.$$

Then

$$\iint_{D(r)} e^{-(x^2+y^2)} d(x,y) \le \iint_{I(r)} e^{-(x^2+y^2)} d(x,y)$$
$$\le \iint_{D(r\sqrt{2})} e^{-(x^2+y^2) d(x,y)}.$$

Therefore, letting $r \to \infty$, we obtain the limit to be π using the Sandwich theorem.

(iv) The required integral being one-fourth of the integral in (iii) is $\frac{\pi}{4}$.

(7) By symmetry, the given volume is 8 times the volume in the positive octant. In that octant the volume lies above the region $Q = \left\{x \ge 0, \ y \ge 0, \ x^2+y^2 \le a^2\right\}$ and underneath the cylinder $x^2+z^2 = a^2$.
Therefore,

$$V = 8\int_0^a \left(\int_0^{\sqrt{a^2-x^2}} \left(\int_0^{\sqrt{a^2-x^2}} 1 dz\right) dy\right) dx = \frac{16a^3}{3}.$$

(8)

$$D = \{(x,y,z) | -1 \le x \le 1, -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}, \sqrt{x^2+y^2} \le z \le 1\}.$$

(9)

$$I = \int_0^{\sqrt{2}} \left(\int_0^{\sqrt{2-x^2}} \left(\int_{x^2+y^2}^2 x dz\right) dy\right) dx.$$

We can also write the region of integration D as

$$D = \{(x,y,z) | 0 \le z \le 2, \ 0 \le y \le \sqrt{z}, \ 0 \le x \le \sqrt{z-y^2}\}.$$

Thus

$$I = \int_0^2 \left(\int_0^{\sqrt{z}} \left(\int_0^{\sqrt{z-y^2}} x dx\right) dy\right) dz = \frac{8\sqrt{2}}{15}.$$

(10) (i) Using cylindrical coordinates, one has

$$I = \int_{-1}^1 \int_0^{2\pi} \int_0^1 (z^2 r^2) r dr d\theta dz = \pi/3.$$

(ii) Using spherical coordinates, one has

$$I = \int_0^{2\pi} \int_0^{\pi} \int_0^1 (\exp(r^3) r^2 \sin \phi dr d\phi d\theta = \frac{4\pi(e-1)}{3}.$$

3.9. Tutorial sheet 9

(1) (i) Note that
$$\frac{\partial r}{\partial x} = \frac{\partial (x^2+y^2+z^2)^{1/2}}{\partial x} = x/r.$$
Similarly, $\frac{\partial r}{\partial y} = y/r$ and $\frac{\partial r}{\partial z} = z/r$. Now,
$$\frac{\partial r^n}{\partial x} = nr^{n-1}\frac{\partial r}{\partial x} = nr^{n-2}x, \text{ etc.}$$
Hence, $\nabla(r^n) = nr^{n-2}\mathbf{r}$.

(ii) Letting $n=-1$ in (i) we have $\nabla(r^{-1}) = -r^{-3}\mathbf{r}$. Hence, $\mathbf{a}\cdot\nabla(r^{-1}) = -r^{-3}(\mathbf{a}\cdot\mathbf{r})$.

(iii) First we compute $\nabla(\mathbf{a}\cdot\nabla(r^{-1}))$:
$$\frac{\partial}{\partial x}\Big(\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}\Big) = \frac{\partial}{\partial x}\Big(\frac{a_1x+a_2y+a_3z}{r^3}\Big) = \frac{a_1}{r^3} + \mathbf{a}\cdot\mathbf{r}\frac{\partial r^{-3}}{\partial x} = \frac{a_1}{r^3} - 3xr^{-5}(\mathbf{a}\cdot\mathbf{r}).$$
Hence,
$$\mathbf{b}\cdot\nabla(\mathbf{a}\cdot\nabla(r^{-1})) = -r^{-3}(\mathbf{a}\cdot\mathbf{b}) + 3r^{-5}(\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r}).$$

(2) (i) $\nabla(fg) = \sum \mathbf{i}\frac{\partial(fg)}{\partial x} = \sum \mathbf{i}\frac{\partial f}{\partial x}g + \sum \mathbf{i} f\frac{\partial g}{\partial x} = g\nabla f + f\nabla g.$

(ii) Since $\frac{\partial f^n}{\partial x} = nf^{n-1}\frac{\partial f}{\partial x}$, hence $\nabla f^n = nf^{n-1}\nabla f$.

(iii) Since,
$$\frac{\partial}{\partial x}\left(\frac{f}{g}\right) = g^{-2}\left(g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}\right),$$
$$\nabla\left(\frac{f}{g}\right) = \sum \mathbf{i}\frac{\partial}{\partial x}\left(\frac{f}{g}\right) = g^{-2}\left(g\sum \mathbf{i}\frac{\partial f}{\partial x} - f\sum \mathbf{i}\frac{\partial g}{\partial x}\right),$$
which is the desired result.

(3) (i) $\nabla\cdot(f\mathbf{v}) = \sum \frac{\partial}{\partial x}(fv_1) = \sum \frac{\partial f}{\partial x}v_1 + f\sum \frac{\partial v_1}{\partial x} = \nabla f\cdot\mathbf{v} + f\nabla\cdot\mathbf{v}.$

(ii) $\nabla\times(f\mathbf{v}) = \sum \mathbf{i}\times\frac{\partial}{\partial x}(f\mathbf{v}) = \sum \mathbf{i}\times(\frac{\partial f}{\partial x}\mathbf{v}) + \sum \mathbf{i}\times(f\frac{\partial \mathbf{v}}{\partial x})$
$= \Big(\sum \frac{\partial f}{\partial x}\mathbf{i}\Big)\times\mathbf{v} + f\sum \mathbf{i}\times\frac{\partial}{\partial x}\mathbf{v} = \nabla f\times\mathbf{v} + f(\nabla\times\mathbf{v}).$

(iii) $\nabla\times(\nabla\times\mathbf{v}) = \sum \mathbf{i}\times\frac{\partial}{\partial x}\Big(\sum \mathbf{j}\times\frac{\partial}{\partial y}\mathbf{v}\Big) = \sum\sum\Big(\mathbf{i}\times(\mathbf{j}\times\frac{\partial^2\mathbf{v}}{\partial x\partial y})\Big)$
$= \sum_{\mathbf{i}}\sum_{\mathbf{j}}\mathbf{j}\Big(\mathbf{i}\cdot\frac{\partial^2\mathbf{v}}{\partial x\partial y}\Big) - \sum\frac{\partial^2\mathbf{v}}{\partial x^2} = \sum_{\mathbf{j}}\mathbf{j}\frac{\partial}{\partial y}\sum_{\mathbf{i}}\Big(\mathbf{i}\cdot\frac{\partial}{\partial x}\mathbf{v}\Big) - \Delta\mathbf{v}$
$= \nabla(\nabla\cdot\mathbf{v}) - \Delta\mathbf{v}.$

(iv) Since
$\nabla\cdot(f\nabla g) = \sum \mathbf{i}\cdot\frac{\partial}{\partial x}\Big(\sum f\frac{\partial g}{\partial y}\mathbf{j}\Big) = \sum\frac{\partial}{\partial x}\Big(f\frac{\partial g}{\partial x}\Big) = \nabla f\cdot\nabla g + f\Delta g,$
we have
$$\nabla\cdot(f\nabla g) - \nabla\cdot(g\nabla f) = f\Delta g - g\Delta f.$$

(v) $\nabla\cdot(\nabla\times\mathbf{v}) = \sum \mathbf{i}\cdot\frac{\partial}{\partial x}\Big(\sum \mathbf{j}\times\frac{\partial}{\partial y}\mathbf{v}\Big) = \sum \mathbf{i}\cdot\Big(\sum \mathbf{j}\times\frac{\partial^2\mathbf{v}}{\partial x\partial y}\Big)$
$= \sum\sum(\mathbf{i}\times\mathbf{j})\cdot\frac{\partial^2\mathbf{v}}{\partial x\partial y} = \sum \mathbf{k}\cdot\Big(\frac{\partial^2\mathbf{v}}{\partial x\partial y} - \frac{\partial^2\mathbf{v}}{\partial y\partial x}\Big) = 0,$
by the equality of mixed partials.

(vi) $\nabla \times (\nabla f) = \sum \mathbf{i} \times \frac{\partial}{\partial x}\Big(\sum \mathbf{j}\frac{\partial f}{\partial y}\Big) = \sum\sum(\mathbf{i}\times\mathbf{j})\frac{\partial^2 f}{\partial x \partial y}$
$= \sum \mathbf{k}\Big(\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial y \partial x}\Big) = 0.$

(vii) Note that

$$g\nabla f \times f\nabla g = g\sum \mathbf{i}\frac{\partial f}{\partial x} \times f\sum \mathbf{j}\frac{\partial g}{\partial y} = \sum\sum fg(\mathbf{i}\times\mathbf{j})\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}$$

$= \sum fg\Big(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\Big)\mathbf{k}$

$= fg\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z}\end{vmatrix} = \sum fg\Big(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial z}\Big)\mathbf{i}.$

Hence,

$\nabla\cdot(g\nabla f \times f\nabla g) = \sum \mathbf{i}\cdot\frac{\partial}{\partial x}\sum \mathbf{i}fg\Big(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial z}\Big)$

$= \sum \frac{\partial}{\partial x}\Big(fg\Big(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial z}\Big)\Big)$

$= \sum f\frac{\partial g}{\partial x}\Big(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial z}\Big) + \sum g\frac{\partial f}{\partial x}\Big(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial z}\Big)$

$+\sum fg\Big(\frac{\partial^2 f}{\partial x\partial y}\frac{\partial g}{\partial z} - \frac{\partial^2 f}{\partial x\partial z}\frac{\partial g}{\partial y}\Big) + \sum fg\Big(\frac{\partial^2 g}{\partial x\partial z}\frac{\partial f}{\partial y} - \frac{\partial^2 g}{\partial y\partial x}\frac{\partial f}{\partial z}\Big).$

Each of the four sums vanishes individually.

(4) (i) Since $\frac{\partial f(r)}{\partial x} = f^{'}(r)\frac{\partial r}{\partial x} = f^{'}(r)x/r$, we have

$$\mathrm{div}(\nabla f(r)) = \sum\frac{\partial}{\partial x}\Big(f^{'}(r)\frac{x}{r}\Big)$$

$$= \sum f^{''}(r)\frac{x^2}{r^2} + \sum\frac{f^{'}(r)}{r} - \sum\frac{x^2}{r^3}f^{'}(r) = f^{''}(r) + \frac{2}{r}f^{'}(r).$$

(ii) $\mathrm{div}(r^n\mathbf{r}) = \sum\frac{\partial}{\partial x}(r^n x) = \sum(r^n + nr^{n-1}\frac{x^2}{r}) = 3r^n + nr^n = (3+n)r^n.$

(iii) Note that $\nabla\Big(\frac{r^{n+2}}{n+2}\Big) = r^n\mathbf{r}$, by exercise 1(i). So,

$$\mathrm{curl}(r^n\mathbf{r}) = \mathrm{curl}(\mathrm{grad}\Big(\frac{r^{n+2}}{n+2}\Big)) = 0,$$

by exercise 3(vi). If $n = -2$, then

$$\nabla(\log r) = r^{-2}\mathbf{r}$$

and hence,

$$\mathrm{curl}(r^{-2}\mathbf{r}) = \mathrm{curl}(\nabla\log r) = 0.$$

(iv) Using part (i) it follows that

$$\mathrm{div}(\nabla r^{-1}) = \frac{d^2}{dr^2}\Big(\frac{1}{r}\Big) + \frac{2}{r}\Big(\frac{d}{dr}(r^{-1})\Big) = 0.$$

(5) (i) $\nabla\cdot(\mathbf{u}\times\mathbf{v}) = \sum\mathbf{i}\cdot(\frac{\partial}{\partial x}(\mathbf{u}\times\mathbf{v})) = \sum\mathbf{i}\cdot(\frac{\partial\mathbf{u}}{\partial x}\times\mathbf{v}) + \sum\mathbf{i}\cdot(\mathbf{u}\times\frac{\partial\mathbf{v}}{\partial x})$
$= \sum(\mathbf{i}\times\frac{\partial\mathbf{u}}{\partial x})\cdot\mathbf{v} - \sum(\mathbf{i}\times\frac{\partial\mathbf{v}}{\partial x})\cdot\mathbf{u} = \mathbf{v}\cdot\mathrm{curl}\mathbf{u} - \mathbf{u}\cdot\mathrm{curl}\mathbf{v}.$

(**Def:** A vector-field $\mathbf{u}$ is said to be irrotational if $\nabla\times\mathbf{u} = 0$. A vector-field $\mathbf{u}$ is said to be solenoidal if $\nabla\cdot\mathbf{u} = 0$. We have now proved that if $\mathbf{u}$ and $\mathbf{v}$ are irrotational then $\mathbf{u}\times\mathbf{v}$ is solenoidal.)

(ii) $\nabla\times(\mathbf{u}\times\mathbf{v})=\sum\mathbf{i}\times\frac{\partial}{\partial x}(\mathbf{u}\times\mathbf{v})=\sum\mathbf{i}\times(\frac{\partial\mathbf{u}}{\partial x}\times\mathbf{v})+\sum\mathbf{i}\times(\mathbf{u}\times\frac{\partial\mathbf{v}}{\partial x})=\sum(\mathbf{i}\cdot\mathbf{v})\frac{\partial\mathbf{u}}{\partial x}-\sum(\mathbf{i}\cdot\frac{\partial\mathbf{u}}{\partial x})\mathbf{v}\cdot$

$=(\mathbf{v}\cdot\nabla)\mathbf{u}-(\nabla\cdot\mathbf{u})\mathbf{v}+(\nabla\cdot\mathbf{v})\mathbf{u}-(\mathbf{u}\cdot\nabla)\mathbf{v}.$

(iii) $\nabla(\mathbf{u}\cdot\mathbf{v})=\sum\mathbf{i}\frac{\partial}{\partial x}(\mathbf{u}\cdot\mathbf{v})=\sum\mathbf{i}(\frac{\partial\mathbf{u}}{\partial x}\cdot\mathbf{v})+\sum\mathbf{i}(\frac{\partial\mathbf{v}}{\partial x}\cdot\mathbf{u})$

$=\sum\Big[(\mathbf{v}\cdot\frac{\partial\mathbf{u}}{\partial x})\mathbf{i}-(\mathbf{v}\cdot\mathbf{i})\frac{\partial\mathbf{u}}{\partial x}\Big]+\sum\Big[(\mathbf{u}\cdot\frac{\partial\mathbf{v}}{\partial x})\mathbf{i}-(\mathbf{u}\cdot\mathbf{i})\frac{\partial\mathbf{v}}{\partial x}\Big]$

$$+(\mathbf{v}\cdot\nabla)\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{v}$$

$=\mathbf{v}\times\Big(\sum\mathbf{i}\times\frac{\partial}{\partial x}\mathbf{u}\Big)+\mathbf{u}\times\Big(\sum\mathbf{i}\times\frac{\partial}{\partial x}\mathbf{v}\Big)+(\mathbf{v}\cdot\nabla)\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{v}$

$=\mathbf{v}\times(\nabla\times\mathbf{u})+\mathbf{u}\times(\nabla\times\mathbf{v})+(\mathbf{v}\cdot\nabla)\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{v}.$

(6) (i) Let $\mathbf{w}=f\mathbf{u}$, where f is a scalar field and $\mathbf{u}$ is a constant vector. Then, $\mathbf{w}\cdot(\nabla\times\mathbf{w})=f\mathbf{u}\cdot(\nabla\times f\mathbf{u})=f\mathbf{u}\cdot(f\nabla\times\mathbf{u}+\nabla f\times\mathbf{u})$ (using 3(ii)) $=$ $f^2(\mathbf{u}\cdot(\nabla\times\mathbf{u}))+f\mathbf{u}\cdot\nabla f\times\mathbf{u}=0$ (using 3(v)).

(ii) Here $\mathbf{r}=(x,y,z)$. Thus, $\nabla\times\mathbf{v}=\nabla\times(\mathbf{w}\times\mathbf{r})=(\mathbf{r}\cdot\nabla)\mathbf{w}-(\mathbf{w}\cdot\nabla)\mathbf{r}+(\nabla\cdot\mathbf{r})\mathbf{w}-(\nabla\cdot\mathbf{w})\mathbf{r}$ (using 5(ii)) $=(\nabla\cdot\mathbf{r})\mathbf{w}-(\mathbf{w}\cdot\nabla)\mathbf{r}=3\mathbf{w}-\mathbf{w}=2\mathbf{w}$.

(iii) Let $f=\rho^{-1}$. Then, using problem 3(ii) and 3(vi),

$$\nabla\times\mathbf{v}=\nabla\times(f\nabla p)=f(\nabla\times\nabla p)+\nabla f\times\nabla p=-f^2(\nabla\rho\times\nabla p).$$

Hence

$$\mathbf{v}\cdot(\nabla\times\mathbf{v})=-f^3\nabla p\cdot(\nabla\rho\times\nabla p)=0.$$

(7) Parameterize the curve C as $\Phi(t)=(t,t^2)$. Then, $\Phi'(t)=(1,2t)$. Thus

$$\begin{aligned}\int_C F\cdot ds &= \int_{-1}^{1}(t^2-2t^3,t^4-2t^3)\cdot(1,2t)dt\\ &= \int_{-1}^{1}(t^2-2t^3)+2t(t^4-2t^3)dt=-\frac{14}{15}.\end{aligned}$$

(8) A parametrization of the ellipse is given by

$$\gamma(\theta)=(a\cos\theta,b\sin\theta),0\le\theta\le 2\pi,$$

and

$$\gamma'(\theta)=(-a\sin\theta,b\cos\theta).$$

Thus,
$F(a\cos\theta,b\sin\theta)\cdot\gamma'(\theta)$
$=(a^2\cos^2\theta+b^2\sin^2\theta,a\cos\theta-b\sin\theta)\cdot(-a\sin\theta,b\cos\theta)$
and

$$\begin{aligned}&\int_C F(x,y)\cdot ds\\ =&\int_0^{2\pi}\big[(-a^3\cos^2\theta\sin\theta-ab^2\sin^3\theta)+(ab\cos^2\theta-b^2\sin\theta\cos\theta)\big]\,d\theta\\ =&\ \pi ab.\end{aligned}$$

(9) A parametrization of the curve is given by

$$\gamma(\theta)=(a\cos\theta,a\sin\theta),0\le\theta\le 2\pi.$$

Thus, $x(\theta) = a\cos\theta, y = a\sin\theta$, and the required integral is

$$\int_0^{2\pi} \frac{a^2(\cos\theta + \sin\theta)(-\sin\theta) + a^2(\sin\theta - \cos\theta)\cos\theta}{a^2} d\theta = \int_0^{2\pi} \frac{-a^2}{a^2} d\theta = -2\pi.$$

(10) For the curve $z = xy, x^2 + y^2 = 1$, we can use the parametrization

$$x = \cos\theta, y = \sin\theta, z = \sin\theta\cos\theta,\ 0 \le \theta \le 2\pi.$$

Thus,

$$\int ydx + zdy + xdz = \int ydx + (xy)dy + x(xdy + ydx)$$

$$= \int_0^{2\pi} [\sin\theta(-\sin\theta) + \sin\theta\cos\theta\cos\theta + \cos^2\theta\cos\theta + \sin\theta\cos\theta(-\sin\theta)]d\theta$$

$$= \int_0^{2\pi} [-\sin^2\theta + \sin\theta\cos^2\theta + \cos^3\theta - \sin^2\theta\cos\theta]d\theta$$

$$= -\int_0^{2\pi} \sin^2\theta d\theta + \int_0^{2\pi} \sin\theta\cos^2\theta d\theta$$

$$+ \int_0^{2\pi} \cos^3\theta d\theta - \int_0^{2\pi} \sin^2\theta\cos\theta d\theta$$

$$= -\pi.$$

3.10. Tutorial sheet 10

(1) Let
$$\gamma(t) = (a\cos t, a\sin t, ct).$$
Then $\gamma'(t) = (-a\sin t, a\cos t, c)$ and $\|\gamma'(t)\| = \sqrt{a^2+c^2}$. Hence
$$u(t) = \int_0^t \sqrt{a^2+c^2}dt = (\sqrt{a^2+c^2})t.$$
Thus the arc length parametrization is
$$\tilde{\gamma}(u) = \left(a\cos\left(\frac{u}{\sqrt{a^2+c^2}}\right), a\sin\left(\frac{u}{\sqrt{a^2+c^2}}\right), \frac{cu}{\sqrt{a^2+c^2}}\right).$$

(2)
$$\int_{C_1} = -\int_{-1}^{+1} \frac{x^2dx}{(1+x^2)^2} = -\int_{-\pi/4}^{+\pi/4} \sin^2\theta = -\pi/4 + \frac{1}{2},$$
$$\int_{C_2} = -\int_{-1}^{+1} \frac{dy}{(1+y^2)^2} = -\int_{-\pi/4}^{+\pi/4} \cos^2\theta d\theta = -\pi/4 - 1/2,$$
$$\int_{C_3} = \int_{-1}^{+1} \frac{-x^2dx}{(1+x^2)}^2 = -\pi/4 + \frac{1}{2},$$
$$\int_{C_4} = \int_{-1}^{+1} \frac{-dy}{(1+y^2)^2} = -\pi/4 - 1/2.$$
Hence
$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = -\pi.$$

(3) A parametrization of C is
$$\gamma(t) = (\cos t, \sin t),\ 0 \le t \le 2\pi.$$
Also,
$$\nabla(x^2 - y^2) = (2x, -2y).$$
Thus
$$\oint_C \nabla(x^2-y^2)\cdot ds = \int_0^{2\pi} (2\cos t, -2\sin t)\cdot(-\sin t, \cos t)dt$$
$$= \int_0^{2\pi} (-2\sin 2t)dt = 0.$$

(4) Parameterize C as
$$\gamma(t) = (t, t^3)$$
for $0 \le t \le 2$. Then $\gamma'(t) = (1, 3t^2)$. Since $\nabla(x^2-y^2) = (2t, -2t^3)$,
$$\int_C \nabla(x^2-y^2)\cdot ds = \int_0^2 (2t - 6t^5)dt = 4 - 64 = -60.$$

(5) The required integral is
$$= \int_{C_1} \frac{dx+dy}{|x|+|y|} + \int_{C_2} \frac{dx+dy}{|x|+|y|} + \int_{C_3} \frac{dx+dy}{|x|+|y|} + \int_{C_4} \frac{dx+dy}{|x|+|y|}.$$
Along C_1: $x+y=1$ and $|x|+|y| = x+y = 1$. Thus
$$\int_{C_1} \frac{dx+dy}{|x|+|y|} = \int_1^0 dx - \int_1^0 dx = 0.$$

Along C_2: $-x+y=1$ and $|x|+|y|=-x+y=1$. Thus

$$\int_{C_2} \frac{dx+dy}{|x|+|y|} = \int_0^{-1} dx + \int_0^{-1} dx = -2.$$

Along C_3: $x+y=-1$ and $|x|+|y|=-x-y=1$. Thus

$$\int_{C_3} \frac{dx+dy}{|x|+|y|} = \int_{-1}^{0} dx - \int_{-1}^{0} dx = 0.$$

Along C_4: $x-y=1$ and $|x|+|y|=x-y=1$. Thus

$$\int_{C_4} \frac{dx+dy}{|x|+|y|} = \int_0^{1} dx + \int_0^{1} dx = 2.$$

Hence

$$\int_{C} \frac{dx+dy}{|x|+|y|} = 2-2=0.$$

(6) The work done W is

$$\begin{aligned}\int_C F\cdot ds &= \textstyle\int_C xydx + x^6y^2dy \\ &= \textstyle\int_0^1 ax^{b+1}dx + \int_0^1 (a^2x^{2b+6})(abx^{b-1})dx \\ &= \tfrac{a}{b+2} + \tfrac{a^3b}{3b+6} \\ &= \tfrac{a}{b+2}\left(1+\tfrac{a^2b}{3}\right) \\ &= a\left(\tfrac{3+a^2b}{3(b+2)}\right).\end{aligned}$$

This will be independent of b iff $\frac{dW}{db}=0$ iff $0=\frac{(b+2)a^2-(3+a^2b)}{(b+2)^2}$ iff $a=\sqrt{\frac{3}{2}}$(as $a>$ 0).

(7) First we observe that the cylinder is given by

$$\left(x-\frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}.$$

From the equations of the sphere and the cylinder we have that, on the intersection C,

$$z^2 = a^2 - ax.$$

Noting the requirement $z \geq 0$, a parametrization of C is given by

$$x=\frac{a}{2}+\frac{a}{2}\cos\theta, \qquad y=\frac{a}{2}\sin\theta,\ z=a\sin\frac{\theta}{2}; 0\leq\theta\leq 2\pi.$$

Then

$$\begin{aligned}\int_C F\cdot ds &= \int_0^{2\pi}[\left(\frac{a^2}{4}\sin^2\theta\right)(-\frac{a}{2}\sin\theta)+(a^2\sin^2\frac{\theta}{2})(\frac{a}{2}\cos\theta)\\ &\qquad +\left(\frac{a^2}{4}+\frac{a^2}{4}\cos^2\theta+\frac{a^2}{2}\cos\theta\right)(\frac{a}{2}\cos\frac{\theta}{2})]d\theta\\ &= \int_0^{2\pi}[-\frac{a^3}{8}\sin^3\theta+\frac{a^3}{2}\sin^2\frac{\theta}{2}\cos\theta+\frac{a^3}{8}\cos\frac{\theta}{2}+\frac{a^3}{8}\cos^2\theta\cos\frac{\theta}{2}\\ &\qquad +\frac{a^3}{4}\cos\theta\cos\frac{\theta}{2}]d\theta\\ &= -\frac{\pi a^3}{4}.\end{aligned}$$

(8) $\dfrac{\partial P}{\partial y}=3x$, $\dfrac{\partial Q}{\partial x}=3x^2y$ where $(P,Q)=F$. Now

$$\begin{aligned}\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x} \quad &\text{iff}\quad 3x=3x^2y\\ &\text{iff}\quad \text{either } x=0 \text{ or } xy=1.\end{aligned}$$

Since the sets $\{(x,y)|x=0\},\{(x,y)|xy=1\}$ are not open, $F(x,y)$ is not the gradient of a scalar field on any open subset of $\mathbb{R}^2$.

(9)

$$\frac{\partial P}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}=\frac{\partial Q}{\partial x}\text{on } \mathbb{R}^2\setminus\{(0,0)\}.$$

However, $F\neq\nabla f$ for any f. Indeed, let C to be the unit circle $x^2+y^2=1$, oriented anticlockwise. Then one has

$$\begin{aligned}\oint_C F\cdot ds &= \int_0^{2\pi}(-\sin t,\cos t)\cdot(-\sin t,\cos t)dt\\ &= 2\pi\neq 0\end{aligned}$$

(10) Suppose $F=\nabla\phi$ for some ϕ.
Then

$$\frac{\partial\phi}{\partial x}=2xy+z^3\Rightarrow\phi(x,y,z)=x^2y+z^3x+f(y,z)$$

for some $f(y,z)$. Assuming f has partial derivatives, we get

$$\begin{aligned}\frac{\partial\phi}{\partial y} &= x^2+\frac{\partial f}{\partial y}=x^2\\ \text{so that } \frac{\partial f}{\partial y} &= 0\end{aligned}$$

and $f(y,z)$ depends only on z
Let $f(y,z)=g(z)$. Then $\phi(x,y,z)=x^2y+z^3x+g(z)\Rightarrow\dfrac{\partial\phi}{\partial z}=3z^2x+g'(z)=3z^2x\Rightarrow g'(z)=0$. Let us select $g(z)=0$. It can be checked that $\phi(x,y,z)=x^2y+z^3x$ satisfies $\nabla\phi=F$.
Hence

$$\oint_C F\cdot ds=0$$

for every smooth closed curve C.

(11) $F(x,y,z) = f(r)\mathbf{r} = (f(r)x, f(r)y, f(r)z)$. Since

$$r = (x^2+y^2+z^2)^{1/2},$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}.$$

If F is to be $\nabla\phi$ for some ϕ, then we must have $\phi_x = f(r)x, \phi_y = f(r)y, \phi_z = f(r)z$; that is,

$$\begin{aligned}
\phi_x = xf(r) &= \frac{x}{r}rf(r) = \frac{\partial r}{\partial x}rf(r),\\
\phi_y = yf(r) &= \frac{y}{r}rf(r) = \frac{\partial r}{\partial y}rf(r),\\
\phi_z = zf(r) &= \frac{z}{r}rf(r) = \frac{\partial r}{\partial z}rf(r).
\end{aligned}$$

Now it can be seen that $\phi(x,y,z) = \int_{t_0}^{r} tf(t)dt$, with some t_0 fixed, satisfies all the desired equations.

3.11. Tutorial sheet 11

(1) We have to show that

$$\iint_R (g_x - f_y)\, dxdy = \oint_{\partial R} (f dx + g dy).$$

(i)

$$\text{lhs } = \iint_R 4xy\, dxdy = \int_0^1 \left(\int_0^{1-x^2} 4xy\, dy \right) dx = \frac{1}{3}.$$

Observe that the boundary of R consists of three smooth curves: a segment of the x-axis, a part of the parabola $y = 1 - x^2$ and a segment of the y-axis. The integral on the rhs vanishes on both the axes. We choose the parametrization $t \mapsto (t, 1 - t^2)$, $(t \in [0, 1])$ for the part of the parabola traced in the opposite direction. This gives

$$\text{rhs } = \oint_{\partial R} (-xy^2\, dx + x^2 y\, dy)$$

$$= -\int_0^1 [-t(1-t^2)^2 + t^2(1-t^2)(-2t)]\, dt = \frac{1}{3}.$$

(ii)

$$\text{lhs } = \iint_R (e^x + 2x - 2x)\, dxdy = \int_0^1 \left(\int_0^x e^x\, dy \right) dx = 1.$$

and

$$\text{rhs } = \oint_{\partial R} [2xy\, dx + (e^x + x^2)\, dy]$$

$$= \int_0^1 (e+1) dy + \int_1^0 (3t^2 + e^t)\, dt = e + 1 - e = 1.$$

(Observe that f and dy are zero on the horizontal segment of the curve, whereas on the vertical segment $dx = 0$.)

(2) (i) Here

$$f(x, y) = y^2; g(x, y) = x.$$

Therefore, the given path integral is equal to

$$\iint_R (1 - 2y)\, dxdy = \int_0^2 \int_0^2 (1 - 2y) dy\, dx = 4 - 4 \int_0^2 dx = 4 - 8 = -4.$$

(ii) Here

$$\iint_R (1 - 2y)\, dxdy = \iint_R dxdy + \int_{-1}^1 \int_{-1}^1 (-2y) dy\, dx = 4 + 0 = 4.$$

(iii) Here

$$\iint_R (1 - 2y)\, dxdy = \iint_R dxdy + \int_{-2}^2 \left[\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} (-2y) dy \right] dx = 4\pi + 0.$$

(3) (i) $A = \frac{3\pi a^2}{2}$. (ii) $A = a^2/2$.

(4) (i) The required area is bounded by the curves

$$C_1 : r = p(\theta) = a(1 - \cos\theta),\ 0 \le \theta \le \pi/2$$

and C_2 which is a portion of the y-axis. In any case, the required area is equal to

$$\frac{1}{2}\oint_C p(\theta)^2\, d\theta.$$

Since θ is a constant along the y-axis, this integral is to

$$\frac{1}{2}\int_0^{\pi/2} p(\theta)^2\, d\theta = \frac{a^2}{8}(3\pi - 8).$$

(ii) The required area is

$$\frac{1}{2}\oint_C xdy - ydx.$$

Here the boundary curve of the interval $[0, 2\pi]$ and the cycloid above traced in the opposite direction. But the integrand is zero on the x-axis, since both y and dy vanish there. Hence the required area is

$$-\frac{a^2}{2}\int_0^{2\pi}(t - \sin t)d(1 - \cos t) - (1 - \cos t)d(t - \sin t) = 2\pi a^2.$$

(iii) Here we use the polar coordinate form as in the previous exercise:

$$A = \frac{1}{2}\oint_C p(\theta)^2 d\theta.$$

(This formula follows from Green's Theorem.)
Since θ is a constant on the two axes, this integral is equal to

$$\frac{1}{2}\int_0^{\pi/2}(1 - 2\cos\theta)^2\, d\theta = \frac{1}{2}\left(\frac{3\pi - 8}{2}\right).$$

(5) Observe that

$$xe^{-y^2}dx + (-x^2ye^{-y^2})dy = d(\frac{x^2e^{-y^2}}{2}).$$

Hence the integral of this term along a closed path vanishes. So the given integral is equal to

$$\oint_C \frac{dy}{x^2 + y^2.}$$

We compute this directly. Observe that $dy = 0$ along the two horizontal parts. But then the integral along one vertical segment cancels with that on the other since the integrands are the same and the segments are traced in the opposite direction. So the value of the required integral is equal to 0.

(6) Take $f = -y^3$ and $g = x^3$ and apply Green's theorem. We get

$$\text{rhs} = \iint_D (3x^2 + 3y^2)\, dxdy = 3I_0.$$

(7)

$$u \cdot v = (a_x - a_y)a + (b_x - b_y)b.$$

Therefore taking $f = (a^2 + b^2)/2 = g$, we see that

$$\begin{aligned} \iint_D u \cdot v\, dxdy &= \iint_D (g_x - f_y)\, dxdy \\ &= \oint_{\partial D}(f\, dx + g\, dy) \\ &= \tfrac{1}{2}\oint_{\partial D}(1 + y^2)(dx + dy) = 0. \end{aligned}$$

(The last equality follows by considering the parametrizaton $(\cos\theta, \sin\theta)$, $0 \leq \theta \leq 2\pi$). Likewise we see that

$$\iint_D u \cdot w \, dxdy = \oint_{\partial D} (ab)(dx + dy) = -\int_0^{2\pi} \sin^2\theta \, d\theta = -\pi.$$

(8) Since $\nabla^2(x^2 - y^2) = 0$, using one of Green's identities (refer to (9),(i)) one has

$$\oint_C \nabla(x^2 - y^2).n|ds| = \oint_C \frac{\partial(x^2 - y^2)}{\partial n}|ds| = \iint_R \nabla^2(x^2 - y^2)\, dxdy = 0.$$

(9) (a)

$$\nabla^2 w = 0 \text{ hence, } \oint_C \frac{\partial w}{\partial n} dS = 0$$

(b) Put $H = F - G$. Then $\operatorname{curl} H = 0$. Since D is simply connected, there exists u such that grad $u = H$. Now div $H = 0$ implies that $\nabla^2 u = 0$, i.e., u is harmonic. Finally $H \cdot n = 0$, i.e.,$\nabla u \cdot n = \frac{\partial u}{\partial n} = 0$ on the boundary implies, using (9),(ii), that u is a constant. But then $H =$ grad$u = 0$ and hence $F = G$.

(10) (i) There are two distinct cases to be considered:
Case (a): Suppose the curve does not enclose the origin. Take

$$f(x, y) = \frac{y}{x^2 + y^2}, g(x, y) = \frac{x}{x^2 + y^2}$$

and apply Green's theorem in the region R bounded by C. So the integral is equal to

$$\iint_R (g_x - f_y)\, dxdy.$$

A simple computation show that $g_x = f_y$ and hence the integral vanishes.
Case (b): Suppose the curve encloses the origin, i.e, $(0, 0) \in R$. (Now the above argument does not work!) We choose a small disc D around the origin contained in R and apply Green's theorem in the closure of $R' = R \setminus D$. As before, the double integral vanishes. But since the boundary of R' consists of C and $-\partial D$ it follows that

$$\oint_C \frac{y\, dx - x\, dy}{x^2 + y^2} = \oint_{\partial D} \frac{y\, dx - x\, dy}{x^2 + y^2}.$$

We can compute this now by using polar coordinates and see that this is equal to -2π.

(ii) Here again, we take D to be a small disc of radius ϵ around the origin and contained in the square R and apply Green's theorem in the closure of $R \setminus D$. Taking

$$f(x, y) = \frac{x^2 y}{(x^2 + y^2)^2}, g(x, y) = \frac{x^3}{(x^2 + y^2)^2},$$

we once again observe that the double integral vanishes since $g_x = f_y = \frac{(x^2+y^2)(x^4-3x^2y^2)}{(x^2+y^2)^4}$. Hence the given line integral is equal to the corresponding line integral taken over the boundary of D. This can be computed by using the parametrizaton $(\epsilon\cos\theta, \epsilon\sin\theta)$, $0 \leq \theta \leq 2\pi$. The answer is $-\pi/4$.

(iii) We have

$$\frac{\partial(\log r)}{\partial y} = \frac{y}{x^2 + y^2} \text{ and } \frac{\partial(\log r)}{\partial x} = \frac{x}{x^2 + y^2}.$$

By part (i), the required line integral is -2π.

3.12. Tutorial sheet 12

(1) (i) On S,

$$\begin{aligned} z &= \frac{1}{2}(4+y-x) = h(x,y) \text{ so that} \\ \Phi(x,y) &= (x,y,\frac{1}{2}(4+y-x)),(x,y)\in\mathbb{R}^2 \end{aligned}$$

can be chosen as one parametrization. The normal vector is

$$\Phi_x \times \Phi_y = (\frac{1}{2}, -\frac{1}{2}, 1).$$

(ii) For $S: y^2+z^2=a^2$, a parametrization is

$$\Phi(u,v) = (u, a\sin v, a\cos v), u\in\mathbb{R}, 0\le v\le 2\pi.$$

The normal vector is

$$\Phi_u \times \Phi_v = (0, a\sin v, a\ \cos v).$$

(iii) If $\mathbf{e} = \dfrac{(1,1,1)}{\sqrt{3}}$, then $\mathbf{e}$ is a unit vector along the axis of the cylinder. Consider the planar cross-section of the cylinder through the origin O. This is a circle C of radius 1. Fix a point P on C. Then $\overrightarrow{OP}$ is a unit vector, say $\mathbf{u}$. Let $\mathbf{v} = \mathbf{e}\times\mathbf{u}$. Then a point on the cylinder is parameterizable as

$$\Phi(\theta,t) = \cos\theta\mathbf{u} + \sin\theta\mathbf{v} + t\mathbf{e}, 0\le\theta\le 2\pi, t\in\mathbb{R}.$$

The normal vector is

$$\Phi_\theta \times \Phi_t = \cos\theta\mathbf{u} + \ \sin\theta\mathbf{v}.$$

(2) (a) The area SA of the surface S with projection R on the xy-plane is given by

$$SA = \iint_R \sec\gamma dxdy$$

where γ is the acute angle between n and $(0,0,1)$ at a generic point on the surface. Thus, if this angle is the same at every point on S, we have

$$SA = \sec\gamma \iint_R dxdy = \sec\gamma SA_{xy},$$

where SA_{xy} is the area of R. Hence,

$$SA_{xy} = SA\ \cos\gamma.$$

(b) By (a) above, one has (for appropriate α, β and γ)

$$\begin{aligned} S_1 &= S\cos\alpha, \\ S_2 &= S\cos\beta, \\ S_3 &= S\cos\gamma. \end{aligned}$$

Thus $S_1^2+S_2^2+S_3^2 = S^2(\cos^2\alpha+\cos^2\beta+\cos^2\gamma) = S^2$ in view of the fact that $\cos\alpha,\ \cos\beta,\ \cos\gamma$ are the direction cosines of n.

(3) There are two pieces of the surface - one below and one above the xy-plane, both having the same area. Let S be the upper piece. Then one has

$$\text{Area}(S) = \iint_T \sqrt{1 + z_x^2 + z_y^2} dxdy,$$

where T is the disc

$$\{x^2 + y^2 \leq ay\} \ = \ \left\{x^2 + (y - \frac{a}{2})^2 \leq (\frac{a}{2})^2\right\},$$

and $z = \sqrt{a^2 - x^2 - y^2}$. Since

$$z_x = -\frac{x}{z} \text{ and} z_y = -\frac{y}{z},$$

it follows that

$$\text{Area}(S) = \iint_T \frac{adxdy}{z} = \iint_T \frac{adxdy}{\sqrt{a^2 - x^2 - y^2}}.$$

Now T is described in polar coordinates by

$$x = r\cos\theta, \ y = r\sin\theta; \ 0 \leq \theta \leq \pi, \ 0 \leq r \leq a\sin\theta.$$

Therefore,

$$\begin{aligned}\text{Area}(S) &= \int_0^\pi \left(\int_0^{a\sin\theta} \frac{ardr}{\sqrt{a^2 - r^2}}\right) d\theta \\ &= a\int_0^\pi [-\sqrt{a^2 - r^2}]|_0^{a\sin\theta} d\theta \\ &= a\int_0^\pi (-a|\cos\theta| + a)d\theta = (\pi - 2)a^2.\end{aligned}$$

Thus the required area is $2(\pi - 2)a^2$.

(4) (i) A point (x, y, z) on the surface satisfies $z = x^2 + y^2$. (The surface is thus a portion of a paraboloid of revolution). The given portion lies between the planes $z = 0$ and $z = 16$. $u = c$ gives a horizontal circular section, while $v = c$ gives a profile curve which is the portion of a half parabola.

(ii) $\Phi_u \times \Phi_v = (-2u^2\cos v, -2u^2\sin v, u)$.

(iii) $S = \int_{v=0}^{2\pi}\int_{u=0}^{4} |\Phi_u \times \Phi_v| dudv = 2\pi\int_0^4 u\sqrt{4u^2 + 1}du = \frac{\pi}{6}(65\sqrt{65} - 1)$. Therefore, $n = 6$.

(5) The area of the paraboloid $x^2 + z^2 = 2ay$ between $y = 0$ and $y = a$ is given by

$$S = \iint_T \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2} dxdz$$

where T is the region $\{z^2 + x^2 \leq 2a^2\}$ in the zx-plane. Hence,

$$\begin{aligned}S &= \iint_T \sqrt{1 + \frac{x^2}{a^2} + \frac{z^2}{a^2}} dxdz \\ &= \int_0^{2\pi}\int_0^{a\sqrt{2}} \sqrt{1 + \frac{r^2}{a^2}}\, rdrd\theta \\ &= \frac{2\pi}{3}(3\sqrt{3} - 1)a^2.\end{aligned}$$

(6) We choose the coordinate system in such a way that the center of the sphere is located at the origin and the central axis of the cylinder coincides with the z-axis. We consider the case when one plane is cutting the sphere at height h above the xy-plane and the other plane is cutting the sphere at depth k below the xy-plane. (Other cases can be treated similarly). We compute the surface areas S_1 and S_2 of the 'upper' and 'lower' caps of the sphere and subtract their sum from $4\pi a^2$. We are expected to get the result to be $2\pi a(h+k)$.
Note that the plane cutting the sphere at height h above the xy-plane intersects the sphere in the circle $x^2+y^2=a^2-h^2$. A parametrization for the upper cap of the sphere is thus given by $\Phi(x,y)=(x,y,\sqrt{a^2-x^2-y^2})$ with $(x,y)\in D=\{(x,y): x^2+y^2\le a^2-h^2\}$. We have then

$$S_1=\iint_D\sqrt{1+(\frac{-x}{\sqrt{a^2-x^2-y^2}})^2+(\frac{-y}{\sqrt{a^2-x^2-y^2}})^2}\;dxdy$$

$$\iint_D\frac{a}{\sqrt{a^2-x^2-y^2}}\;dxdy=\int_0^{2\pi}\int_0^{\sqrt{a^2-h^2}}\frac{a}{\sqrt{a^2-r^2}}\;rdrd\theta$$

$=2\pi a(-h+a)$.
Similarly, $S_2=2\pi a(-k+a)$; and then $4\pi a^2-(S_1+S_2)=2\pi a(h+k)$, as desired.

(7) (i) Note that $\Phi_u\times\Phi_v=-2(1,1,1)$ has negative z-component. Thus,

$$F\cdot n|dS|=-F\cdot(\Phi_u\times\Phi_v)dudv=2(x+y+z)dudv=2dudv,$$

one has

$$\iint_S F\cdot n|dS|=2\text{Area}(S^*),$$

where S^* is the parametrizing region in the uv-plane. As the components of Φ are affine-linear in u and v, S^* is also a triangle whose vertices are pre-images of the vertices of S. Now the vertices of S^* are $(0,0)$, $(\frac{1}{2},\frac{1}{2})$, and $(\frac{1}{2},-\frac{1}{2})$ so that the area of S^* is $\frac{1}{4}$, and hence $\iint_S F\cdot n|dS|=\frac{1}{2}$.

(ii) The surface S satisfies $z=1-x-y\ge 0,\ x\ge 0,\ y\ge 0$. Thus,

$$F\cdot n|dS|=(x,y,z)\cdot(-z_x,-z_y,1)dxdy=(x+y+z)dxdy=dxdy$$

and $S_1^*=\{x+y\le 1, x\ge 0, y\ge 0\}$ as the parametrizing region, one has

$$\iint_S F\cdot n\,|dS|=\iint_{S_1^*}dxdy=\text{Area}(S_1^*)=\frac{1}{2}.$$

(8) A parametrization of S is

$$\Phi(u,v)=(a\sin v\cos u, a\sin v\sin u, a\cos v), 0\le u\le 2\pi,\ 0\le v\le\pi$$

and

$$\Phi_u\times\Phi_v=a\sin v\ \Phi(u,v)$$

is the outward normal. The integrand is

$$F\cdot(\Phi_u\times\Phi_v)=a^4\sin^3 v\cos v(1+\cos^2 u).$$

Thus the required integral is

$$\int_{u=0}^{2\pi}\int_{v=0}^{\pi}a^4\sin^3 v\cos v(1+\cos^2 u)dudv$$

$$= a^4 \left(\int_0^{\pi} \sin^3 v \cos v \ dv \right) \left(\int_0^{2\pi} (1 + \cos^2 u) du \right) = 0.$$

(9) The hemisphere satisfies

$$z = \sqrt{1 - x^2 - y^2}, \ x^2 + y^2 \leq 1.$$

Using

$$(-z_x, -z_y, 1) = (\frac{x}{z}, \frac{y}{z}, 1)$$

and

$$F \cdot n|dS| = (x, -2x - y, z) \cdot (\frac{x}{z}, \frac{y}{z}, 1) dxdy = \frac{(1 - 2xy - 2y^2)}{z} dxdy$$

and $T = \{(x, y) : x^2 + y^2 \leq 1\}$ as the parametrizing region, one has

$$\iint_S F \cdot n|dS| = \iint_T \frac{(1 - 2xy - 2y^2)}{\sqrt{1 - x^2 - y^2}} dxdy$$

$$= \int_0^{2\pi} \int_0^1 \frac{(1 - r^2 \sin 2\theta - 2r^2 \sin^2 \theta) r dr d\theta}{\sqrt{1 - r^2}} = \frac{2\pi}{3}.$$

(10) The flux through the base T is

$$\iint_T F \cdot (0, 0, -1) dxdy = 0,$$

as $F \cdot (0, 0, -1) = -z = 0$ along T. The total flux is therefore the same as in the previous problem, namely, $\frac{2\pi}{3}$.

3.13. Tutorial sheet 13

(1) We have

$$\begin{aligned} W &= \{(x,y,z) | y^2+z^2 \le x^2, 0 \le x \le 4\} \text{ and} \\ \partial W &= S_1 \bigcup S_2, \text{ where} \\ S_1 : \Phi(x,\theta) &= (x, x\cos\theta, x\sin\theta), 0 \le \theta \le 2\pi, \\ & \quad S_2 : x = 4, y^2+z^2 \le 16. \end{aligned}$$

Along S_1, $\Phi_x \times \Phi_\theta = (x, -x\cos\theta, -x\sin\theta) = (x, -y, -z)$ so that the outward normal is $-\Phi_x \times \Phi_\theta = (-x, y, z)$. Thus

$$\begin{aligned} \iint_{S_1} F \cdot n |dS| &= \int_0^4 \int_0^{2\pi} (-x^2y^2 + y^2z^2 + z^2x^2) dx d\theta \\ &= \int_0^4 \int_0^{2\pi} x^4(-\cos^2\theta + \cos^2\theta\sin^2\theta + \sin^2\theta) dx d\theta \\ &= \left(\int_0^4 x^4 dx\right)\left(\int_0^{2\pi} (-\cos^2\theta + \cos^2\theta\sin^2\theta + \sin^2\theta)\right) \\ &= \frac{4^5}{5}\,\frac{\pi}{4} = 4^4\frac{\pi}{5}. \end{aligned}$$

Also, along S_2, the outward normal (to $x = 4 \equiv f(y,z)$) is $(1,0,0)$. Thus

$$\begin{aligned} \iint_{S_2} F \cdot n |dS| &= \iint_{S_2} 4y^2 |dS| = \iint_{y^2+z^2 \le 16} 4y^2 dy dz \\ &= 4\int_0^{2\pi} \int_0^4 r^3 \cos^2\theta dr d\theta = 4^4\pi. \end{aligned}$$

Now,

$$\begin{aligned} \iiint_W \text{div } F \; d(x,y,z) &= \iiint_{\substack{y^2+z^2 \le x^2 \\ 0 \le x \le 4}} (x^2+y^2+z^2) d(x,y,z) \\ &= \int_0^4 \left(\int_0^x \left(\int_0^{2\pi} (x^2+r^2) r dr d\theta\right) dx\right) \\ &= 2\pi \int_0^4 \left(\int_0^x (x^2 r + r^3) dr\right) dx \\ &= 2\pi \int_0^4 \frac{3x^4}{4} dx = 4^4\frac{6\pi}{5}. \end{aligned}$$

Since $4^4\dfrac{6\pi}{5} = 4^4\dfrac{\pi}{5} + 4^4\pi$, the divergence theorem is verified.

(2) We have $\text{div}F = (y + z + x)$ and

$$\begin{aligned} I &= \iiint_W (x+y+z)d(x,y,z) = \\ &= \iiint_W xd(x,y,z) + \iiint_W yd(x,y,z) + \iiint_W zd(x,y,z) \\ &= \int_0^c \int_0^{b(1-\frac{z}{c})} \int_0^{a(1-\frac{y}{b}-\frac{z}{c})} xdxdydz + (\cdots) + (\cdots) \\ &= \frac{a^2bc}{24} + \frac{ab^2c}{24} + \frac{abc^2}{24} \\ &= \frac{abc}{24}(a+b+c). \end{aligned}$$

Now,

$$\iint_S F \cdot n|dS| = \iint_{S_1} F \cdot n|dS| + \iint_{S_2} F \cdot n|dS| + \iint_{S_3} F \cdot n|dS| + \iint_{S_4} F \cdot n|dS|$$

where

$$\begin{aligned} &S_1: \quad z = 0; \quad \frac{x}{a} + \frac{y}{b} \le 1, \quad x, y \ge 0 \text{ and} \\ &S_2: \quad y = 0; \quad \frac{x}{a} + \frac{z}{c} \le 1, \quad x, z \ge 0 \text{ and} \\ &S_3: \quad x = 0; \quad \frac{z}{c} + \frac{y}{b} \le 1, \quad y, z \ge 0 \text{ and} \\ &S_4: \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, \qquad x, y, z \ge 0. \end{aligned}$$

Also,

along S_1, $n = (0,0,-1)$ $\Rightarrow F \cdot n = -xz = 0$ (as $z = 0$ on S_1);
along S_2, $n = (0,-1,0)$ $\Rightarrow F \cdot n = -yz = 0$ (as $y = 0$ on S_2);
along S_3, $n = (-1,0,0)$ $\Rightarrow F \cdot n = -xy = 0$ (as $x = 0$ on S_3).

Along S_4, the outward normal (to $z = c(1 - \frac{x}{a} - \frac{y}{b}) \equiv f(x,y)$) is $(\frac{c}{a}, \frac{c}{b}, 1)$ so that

$$\begin{aligned} \iint_{S_4} F \cdot n|dS| &= \iint_{\frac{x}{a}+\frac{y}{b}\le 1; x,y\ge 0} \Big(\frac{cxy}{a} + \frac{cyz}{b} + zx\Big) d(x,y) \\ &= \int_0^a \int_0^{b(1-\frac{x}{a})} \frac{cxy}{a} dxdy + (\cdots) + (\cdots) \\ &= \frac{ab^2c}{24} + \frac{abc^2}{24} + \frac{a^2bc}{24} \\ &= \frac{abc}{24}(a+b+c). \end{aligned}$$

(3) Consider $F = uv(1,0,0)$. By the divergence theorem, one has

$$\begin{aligned} \iint_{\partial W} uvn_x|dS| &= \iiint_W \frac{\partial}{\partial x}(uv)d(x,y,z) \\ &= \iiint_W u\frac{\partial v}{\partial x}d(x,y,z) + \iiint_W v\frac{\partial u}{\partial x}d(x,y,z) \end{aligned}$$

Hence

$$\iiint_W u\frac{\partial v}{\partial x}d(x,y,z) = \iint_{\partial W}(uvn_x)|dS| - \iiint_W v\frac{\partial u}{\partial x}d(x,y,z).$$

(4) Since $\nabla\cdot(\phi\nabla\phi) = \|\nabla\phi\|^2 + \phi\nabla^2\phi$, we have $\nabla^2\phi = 10\phi - 4\phi$, i.e., $\nabla^2\phi = 6$. Thus

$$\begin{aligned}\iint_S \frac{\partial\phi}{\partial n}|dS| &= \iint_S \mathrm{grad}\phi\cdot n|dS| \\ &= \iiint_W \mathrm{div}(\mathrm{grad}\phi)d(x,y,z) = 6\iiint_W d(x,y,z) \\ &= 6\text{ (Volume of the sphere)} = 6\times\frac{4\pi}{3} = 8\pi.\end{aligned}$$

(5) Let $F = (x,0,0)$. Using the divergence theorem, we get

$$V = \iiint_W d(x,y,z) = \iint_S xn_x|dS|$$

Similarly, letting $F = (0,y,0)$, we get

$$V = \iint_S yn_y|dS|$$

and letting $F = (0,0,z)$, we get

$$V = \iint_S zn_z|dS|.$$

(6) Let W denote the solid cube. Consider

$$\begin{aligned} I &= \iint_S x^2dy\wedge dz + y^2dz\wedge dx + z^2dx\wedge dy \\ &= \iint_S (F\cdot n)|dS|,\end{aligned}$$

where $F = (x^2, y^2, z^2)$. We have then, using the divergence theorem,

$$I = \iiint_W (\mathrm{div}\ F)d(x,y,z) = \int_0^1\int_0^1\int_0^1 2(x+y+z)dxdydz = 3.$$

(7) The required integral is $I = \iint_{\partial W}(F\cdot n)|dS|$, where $F = (yz, zx, xy)$. Since $\mathrm{div}(F) = 0$, one has

$$I = \iiint_W (\mathrm{div}F)d(x,y,z) = 0.$$

(8) By the divergence theorem, one has

$$\iint_{S\bigcup S_1}(\nabla\times F)\cdot n|dS| = \iiint_W \nabla\cdot(\nabla\times F)d(x,y,z) = 0,$$

where S_1 is the disc $x^2+y^2+z^2 = 1$, $z = 1/2$. Thus

$$\iint_S(\nabla\times F)\cdot n|dS| = \iint_{S_1}(\nabla\times F)\cdot n|dS|,$$

where n in the rhs integral is the vector $(0,0,1)$; so rhs $= \iint_{S_1}(\nabla\times F)\cdot(0,0,1)|dS|$. But the coefficient of $(0,0,1)$ in $\mathrm{curl}(F)$ is 0. Hence, $\mathrm{curl}(F)\cdot(0,0,1) = 0$. Thus

$$\iint_{S_1}(\nabla\times F)\cdot(0,0,1)|dS| = 0.$$

This gives

$$\iint_S (\nabla \times F) \cdot n|dS| = 0.$$

(9) Note that

$$p = n \cdot (x,y,z) = \frac{\left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right)}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}} \cdot (x,y,z) = \frac{1}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}.$$

(a) Let $F = (x,y,z)$. Then

$$\begin{aligned} &\iint_S F \cdot n \ |dS| \\ = \ &\iint_S p \ |dS| = \iiint_W \text{div} F \ d(x,y,z) = 3 \iiint_W d(x,y,z) = 3\left(\frac{4\pi}{3}abc\right) = 4\pi abc. \end{aligned}$$

(b) Let $F = \frac{1}{p^2}(x,y,z)$. Then

$$\begin{aligned} &\iint_S \frac{1}{p} \ |dS| \\ = \ &\iint_S F \cdot n \ |dS| = \iiint_W \text{div} F \ d(x,y,z) \\ = \ &5 \iiint_W \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right) d(x,y,z). \end{aligned}$$

Let $x = ar\sin\varphi\cos\theta$, $y = br\sin\varphi\sin\theta$, $z = cr\cos\varphi$. Then

$$\begin{aligned} &\iint_S \frac{1}{p} \ |dS| \\ = \ &5abc \int_0^{\pi} \int_0^{2\pi} \int_0^1 r^4 \left(\frac{\sin^3\varphi\cos^2\theta}{a^2} + \frac{\sin^3\varphi\sin^2\theta}{b^2} + \frac{\sin\varphi\cos^2\varphi}{c^2}\right) dr d\theta d\varphi \\ = \ &abc \int_0^{\pi} \int_0^{2\pi} \left(\frac{\sin^3\varphi\cos^2\theta}{a^2} + \frac{\sin^3\varphi\sin^2\theta}{b^2} + \frac{\sin\varphi\cos^2\varphi}{c^2}\right) d\theta d\varphi \\ = \ &\pi abc \int_0^{\pi} \left(\frac{\sin^3\varphi}{a^2} + \frac{\sin^3\varphi}{b^2} + \frac{2\sin\varphi\cos^2\varphi}{c^2}\right) d\varphi \\ = \ &\frac{4}{3}\pi abc\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = \frac{4\pi}{3abc}(b^2c^2 + c^2a^2 + a^2b^2). \end{aligned}$$

(We used the fact that the Jacobian of (x,y,z) with respect to (r,φ,θ) is $abcr^2\sin\varphi$). $\blacksquare$

Aliter: If $\psi = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$, then

$$\begin{aligned} \iint_S \frac{1}{p} \ |dS| &= \iint_S \left(\frac{n}{(x,y,z) \cdot n}\right) \cdot n \ |dS| = \iint_S \frac{\nabla\psi}{(x,y,z) \cdot \nabla\psi} \cdot n \ |dS| \\ &= \iint_S \frac{\nabla\psi}{2} \cdot n \ |dS| \\ &= \iiint_W \frac{\nabla^2\psi}{2} \ d(x,y,z) = \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \iiint_W d(x,y,z) \\ &= \frac{4}{3}\pi abc\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right). \end{aligned}$$

(10) For a simple closed (and sufficiently smooth) plane curve $C : \gamma(u) = (x(u), y(u))$, parametrized by the arc length u, the outward unit normal at $x(u)$ is $n = (y'(u), -x'(u))$. Let D be the region enclosed by the curve C.

Let $F = (Q, -P)$ be a continuously differentiable vector field in a region including $C \bigcup D$. Then

$$\operatorname{div}(F) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \text{ and } F \cdot n = Qy' + Px'.$$

Thus one has

$$\oint_C (F \cdot n)\ |ds| = \oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) d(x, y)$$

$$= \iint_D \operatorname{div}(F)\ d(x, y).$$

3.14. Tutorial sheet 14

(1) The cone $z = \sqrt{x^2+y^2}$ is parametrized as $\Phi(x,y) = (x, y, \sqrt{x^2+y^2})$. One has then $n|dS| = (-\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1)dxdy$. Further, $\text{curl}F = (0,0,2)$.

(a) If S is the surface lying on the cone $z = \sqrt{x^2+y^2}$ and bounded by the intersection C of the hemisphere $x^2 + (y-a)^2 + z^2 = a^2, z \geq 0$ with the cone, then the projection R of S onto the xy-plane is given by $x^2 + (y - a/2)^2 \leq a^2/4$.

Thus one has

$$\iint_S \text{curl}F \cdot n|dS| = \iint 2\ dxdy = 2\iint_R dxdy = \pi a^2/2.$$

With the choice of the normal n to S as indicated above, the induced orientation on C is counterclockwise (when viewed from high above). The projection of C onto the xy-plane can then be described by $(\frac{a}{2}\cos\theta, \frac{a}{2} + \frac{a}{2}\sin\theta)$ $(0 \leq \theta \leq 2\pi)$.

Thus,

$$\begin{aligned}\oint_C F \cdot ds &= \oint_C (x-y)dx + (x+z)dy + (y+z)dz \\ &= \oint_C (xdy - ydx) + d(yz) + \frac{1}{2}d(x^2+z^2) \\ &= \oint_C (xdy - ydx) = 2\pi\frac{a^2}{4} = \pi a^2/2.\end{aligned}$$

Stokes' Theorem now stands verified.

(b) If S is the surface lying on the cone $z = \sqrt{x^2+y^2}$ and bounded by the intersection C of the cylinder $x^2 + (y-a)^2 = a^2, z \geq 0$ with the cone, then the projection R of S onto the xy-plane is given by $x^2 + (y-a)^2 \leq a^2$.

Thus one has

$$\iint_S \text{curl}F \cdot n|dS| = \iint 2\ dxdy = 2\iint_R dxdy = 2\pi a^2.$$

With the choice of the normal n to S as indicated above, the induced orientation on C is counterclockwise (when viewed from high above). The projection of C onto the xy-plane can then be described by $(a\cos\theta, a + \sin\theta)$ $(0 \leq \theta \leq 2\pi)$.

Thus,

$$\begin{aligned}\oint_C F \cdot ds &= \oint_C (x-y)dx + (x+z)dy + (y+z)dz \\ &= \oint_C (xdy - ydx) + d(yz) + \frac{1}{2}d(x^2+y^2) \\ &= \oint_C (xdy - ydx) = 2\pi a^2.\end{aligned}$$

Stokes' Theorem now stands verified.

(2) For $F = (yz, xz, xy)$,

$$\text{curl }(F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} = (x - x, y - y, z - z) = 0.$$

Thus the required line integral is

$$\iint_S \text{curl}(F) \cdot n|dS| = 0.$$

(3) By Stokes' theorem, we have

$$\iint_S \text{curl } (F) \cdot n|dS| = \oint_{C_1} F \cdot d\mathbf{s} + \oint_{C_2} F \cdot d\mathbf{s}$$

where C_1 is the circle $x^2 + y^2 = 4, z = -3$ with the counterclockwise orientation when viewed from high above, and C_2 is the circle $x^2 + y^2 = 4, z = 0$ with the opposite orientation. Now,

$$\text{curl } F = (-3zy^2 - 3xz^2)(1,0,0) + (z^3 - 1)(0,0,1) \text{ and}$$

$$F \cdot ds = ydx + xz^3 dy - zy^3 dz.$$

$$\text{Along } C_2, \quad z = 0 \quad ; \quad F \cdot ds = ydx \text{ and } \oint_{C_2} ydx = -\int_0^{2\pi} (-4\sin^2\theta| \ d\theta = 4\pi.$$

$$\text{Along } C_1, \quad z = -3 \quad ; \quad dz = 0; \ F \cdot ds = ydx - 27xdy = d(xy) - 28xdy \text{ and}$$

$$\oint_{C_1} d(xy) - \oint_{C_1} 28xdy = -28\int_0^{2\pi} 4\cos^2\theta d\theta = -112\pi.$$

Hence

$$\iint_S \text{curl } (F) \cdot n|dS| = -108\pi.$$

(4) Note that, to apply Stokes' Theorem, one would have to work inside $U = \mathbb{R}^3 \setminus z-$axis, as F would not make sense at a point on the z-axis. But there is no surface in $U = \mathbb{R}^3 \setminus z-$axis whose boundary is C. Hence Stokes' theorem cannot be applied.

Using the parametrization $(\cos\theta, -\sin\theta)$, $(0 \le \theta \le 2\pi)$ one has

$$\oint_C F \cdot ds = \oint_C \frac{-ydx + xdy}{x^2 + y^2} = -\int_0^{2\pi} d\theta = -2\pi.$$

(5) Note that

$$\begin{aligned} F &= (y^2 - z^2, z^2 - x^2, x^2 - y^2), \\ \text{curl } F &= (-2y - 2z, -2z - 2x, -2x - 2y), \\ \text{and } n &= \frac{(1,1,1)}{\sqrt{3}}. \end{aligned}$$

Thus, along the surface S which is part of the plane $x + y + z = \frac{3a}{2}$ and which is bounded by C, one has

$$\begin{aligned} \text{curl} F \cdot n &= -\frac{2}{\sqrt{3}}(y + z + z + x + x + y) \\ &= -\frac{4}{\sqrt{3}}(x + y + z) = -\frac{4}{\sqrt{3}}\frac{3a}{2}. \end{aligned}$$

Hence

$$\iint_S \text{curl } F \cdot n|dS| = -2\sqrt{3}a \iint_S |dS| = (-2\sqrt{3}a)(\text{Area of } S).$$

The surface S is a regular hexagon with vertices $(a/2, 0, a)$, $(a, 0, a/2)$, $(a, a/2, 0)$, $(a/2, a, 0)$, $(0, a, a/2)$, $(0, a/2, a)$. Hence its area is

$$3\frac{\sqrt{3}}{2}(\text{length of side})^2 = \frac{3\sqrt{3}}{2}\frac{a^2}{2}.$$

Stokes' theorem then yields that

$$\oint_C (y^2 - z^2)dx + (z^2 - x^2)dy + (x^2 - y^2)dz = -\frac{9a^3}{2}.$$

(6) We have

$$\begin{aligned} F &= (y, z, x), \\ \text{curl } F &= -(1, 1, 1). \end{aligned}$$

Parametrize the surface lying on the hyperbolic paraboloid $z = xy/b$ and bounded by the curve C as $(x, y, \frac{xy}{b})$ $(x^2 + y^2 \le a^2)$ so that $n|dS| = (-\frac{y}{b}, -\frac{x}{b}, 1)dxdy$ and

$$\begin{aligned} \iint_S \text{curl } F \cdot n|dS| &= \frac{1}{b}\iint_{x^2+y^2 \le a^2} (y + x - b)dxdy \\ &= \frac{1}{b}\int_0^{2\pi}\int_0^a (r\sin\theta + r\cos\theta - b)rdrd\theta = -\pi a^2 \end{aligned}$$

(7) Letting $\mathbf{r} = (x, y, z)$ and using S to denote the planar area enclosed by C, one has

$$\begin{aligned} I &:= \frac{1}{2}\oint_C a(ydz - zdy) + b(zdx - xdz) + c(xdy - ydx) \\ &= \frac{1}{2}\oint_C n \times \mathbf{r} \cdot ds \\ &= \frac{1}{2}\iint_S \nabla \times (n \times \mathbf{r}) \cdot n|dS| \\ &= \frac{1}{2}\iint_S 2n \cdot n|dS| = \iint_S |dS| = \text{Area}(S). \end{aligned}$$

If C is parametrized as $\mathbf{u}\cos t + \mathbf{v}\sin t$ $(0 \le t \le 2\pi)$, then

$$\begin{aligned} \text{Area}(S) &= \frac{1}{2}\oint_C n \times \mathbf{r} \cdot ds \\ &= \frac{1}{2}\int_0^{2\pi} n \times (\mathbf{u}\cos t + \mathbf{v}\sin t) \cdot (-\mathbf{u}\sin t + \mathbf{v}\cos t)dt \\ &= \frac{1}{2}\int_0^{2\pi} n \cdot (\mathbf{u}\cos t + \mathbf{v}\sin t) \times (-\mathbf{u}\sin t + \mathbf{v}\cos t)dt \\ &= \frac{1}{2}\int_0^{2\pi} n \cdot \mathbf{u} \times \mathbf{v}\ dt \end{aligned}$$

so that, letting $n = \frac{\mathbf{u}\times\mathbf{v}}{||\mathbf{u}\times\mathbf{v}||}$, we get

$$\text{Area}(S) = \frac{1}{2}\int_0^{2\pi} ||\mathbf{u} \times \mathbf{v}||\ dt = \pi||\mathbf{u} \times \mathbf{v}||.$$