L1 - 06/08/2024

Instructor: Prof. Madhusudan Manjunath

Course Structure

- Paradigm – Learning Math (✓)

  v/s

  Learning about Math
- Define Mathematics
- Study devp. of Math in diff. cultures across the world at various time periods.
- Ancient Indian Math
  - Vedas
  - Jain texts
  - Particular mathematicians (Aryabhatta, Bhaskaracharya, etc.)

- European Math
  - Ancient Greece
  - Central & Southern Europe
- Overview of Modern Math

Project

Explore Math of other civilizations across the world.

eg - Japan, China, Arabia,
(Wasan)
Kerala School of Math, etc.
(devp. of Calculus)

Refer to more than one sources
Take a scholarly approach
Groups of 2.

Approach

Emphasis will be on the philosophical grasp of the material presented; neither historical or technical proficiency.

Grading Policy

- Mid-sem – 30%
- Project – 30%
- End-sem – 40%

L2 - 09/08/2024

What is Mathematics?

सोयम देवदत्त:।

He is Devdutta

Imagine an acquaintance Devdutt whom you have met only once in IIT Bombay. Consider your thought process on meeting him 25 years later.

Earlier $\rightarrow$	Later
· Silky Hair	· Bald
· Well-built	· Frail

The physical characteristics of the person have completely changed. Yet, you recognise him as the Devdutt you had met 25 years ago.

Why?

The mathematical techique being used in the situation is Abstraction.

Abstraction

Removing the inessential features which can be contradictory

eg - Devdutt being bald after many years doesn't change his identity.

Mathematics

Study of nature via precise, symbolic, logical abstraction.

- 'Study of nature' disqualifies music.
- 'abstraction' disqualifies Physics.

L3 - 13/08/2024

- **Nature** – Physical, Inner (Mind), Inner (Study for its own sake), outer space

- **Precise** – formulation of statements having a <u>definite truth value</u> – True or False

Moreover, every term has a <u>def^n</u>.

- **Symbolic** – Representing nature via symbols & manipulating them.

- **Logical** – Systematic & rigorous application of the principles of logic.

Syllogism

$$(A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$$

eg - (Socrates $\Rightarrow$ Man) $\wedge$ (Man $\Rightarrow$ Mortal)
$\Rightarrow$ (Socrates $\Rightarrow$ Mortal)

- **Abstraction** - Capturing the essence of the situation / problem at hand.

This depends on the problem as well as its aesthetic aspects.

REMARK - 1. Abstraction is the most essential feature of Mathematics.

2. Mathematics can be considered a part of Philosophy.

L4 - 14/08/2024

Etymology

- Mathematics – Greek
  'that which is learnt'

- गणित – From $\underbrace{\text{गणः}}_{\text{grouping/classification}}$
  OR
  enumerate/count

History of Mathematics

Study of the evolution of Mathematics starting from its origin to its current state with reference to cultural, social & economic background.

Benefits of studying its History

1. Sheds light on the intersection with other human endeavours. This can nurture Mathematics.

2. Help answer questions like –

- What are the ideal cond^{n}s for the study of Math?
- How can Math contribute best to humanity?

3. Appreciate contributions to Math by cultures & civilizations across the world.

Hence, neutralize biases & stereotypes

4. Provides insight into Math through the context in which these devp. took place.

eg – Pythagoras thm & Calculus were discovered by diff. cultures.

5. Studying lives of Mathematicians, guides us in our own practice.

LS - 20/08/2024

Ancient India

Decimal System of Numeration & the Concept of Zero

<u>Text</u> - 'History of Ancient Indian Math' by Srinivasa Iyengar

1. The most fundamental contribution of Ancient India to Math.

2. Use of 9 digits and a symbol '0' (called zero) to represent a no.

3. Each digit in this representation has a <u>place value</u> and a <u>face value</u>

The no. being represented can be recovered by summing up the place values of all the digits in its representation

$$S_n \ldots . S_0 = \sum_{k=1}^{n} S_k \cdot 10^k$$

eg - $9807 = 9 \times 10^3 + 8 \times 10^2 + 0 \times 10^1 + 7 \times 10^0$

4. $\because$ this is taught at such a young age that we lose sight of its profoundity & importance.

5. Civilizations who used unary system made slower progress

eg - Greeks

- Controversy - Who should be given credit for devp. of decimal system

– Arabs OR Indians ?

- **Current consensus –**

Indians introduced decimal system.
Arabs assimilated it & conveyed it to Europe & Africa

- **Quotes from ancients texts**

  - Vedas
  - Ramayana
  - Harappa & Mohenjodaro (~3000 BCE)
  - Jain texts (500 BCE – 100 BCE)
  - Buddhist texts

1. Veda (Yajur) – Ek, Das, Shat, Sahastra, Ayut, Niyut, Prayut, Arbud, etc.
(Powers of Ten)

L6 - 22/08/2024

2. <u>Ramayana</u> - Shat Shat Sahastra

= Koti

Shat Koti Sahastra

= Shankha

(This was used to describe sizes of units in Rama's army)

So, Ramayana depicts a rather advanced civilization with astronomy, engineering and music.

- Duration of Kalayuga is estimated to be 432,000 years

3. <u>Harappa & Mohenjodaro</u> –

Evidence in the form of archaeological discoveries – pottery, coins, insciptions, etc.

4 <u>Jain texts</u> –

Mathematics in the context of religion.

5. <u>Buddhist texts</u> –

- Shunya Vada (Nagarjuna)

- Account in Lalitavistara – Buddha mentions nos. of the order $10^7 - 10^{53}$ to a mathematician Arjuna.

- **Conclusion** –

Quote from Laplace

"The idea of expressing quantities by 9 figures whereby a no. is imparted both an absolute and a posit. value is so simple that its very simplicity is the reason for our not being aware of how much admiration it deserves."

L7 - 27/08/2024

Math in the Vedas

(Sulvasutras)

$\underbrace{\text{Sulva}}_{\text{Rope/ String/Cord}}$

Rope/ String/Cord

Deals with the geometry of fire altars (havan kundas)

Ancient Indian Scriptures

- Shruti - Vedas
- Smriti - Ramayana, Mahabharata
- Purana - Shiva, Skanda
- Others - Arthashastra, Ganitashastra

- Vedas - Rig, Yajur, Sama, Atharv
- Mantras - Samhita, Brahmna, Aranyaka, Upanishad

- **Auxiliary (Vedangas)** - Shiksha, Kalpa, Vyakrana, Niyukta, Chandas, Jyotisha

S - Phonetics, intonation & chanting

K - Details of rituals

V - Sanskrit grammar

N - Etymology of words

C - Aesthetic, rythmic, metric aspect of poetry. (Prosody)

J - Astrological/Astronomical predictions

They contain a diverse compilation of knowledge ranging from hymns, chants, rituals, mythical accounts to the highest philosophical/spiritual teachings.

L8 - 29/08/2024

Sulva Sutras

(~800 – 500 BCE)

- Fire rituals were a very important part of day to day life.
- Took place in chambers
- Householders performed certain rituals everyday.

Hence, they typically maintained 3 kinds of fires at home –

Dakshin, Grihpatya, Aahavaniya

- The alters these fires resided in had to be constructed with great care so as to conform to shapes and sizes.

- **Dakshin** - Semicircle altar
  Garhipatya - Square, Circle altar
  Aahavaniya - Square

- **Unit of Measurement** -
  Length - Vyam / Vyayam
  OR
  Purusha
  (~ 96 inch)

  Area - 1 sq. Vyam

- These were the simplest kind of fires

- There were other types of rituals for fulfilling specific desires

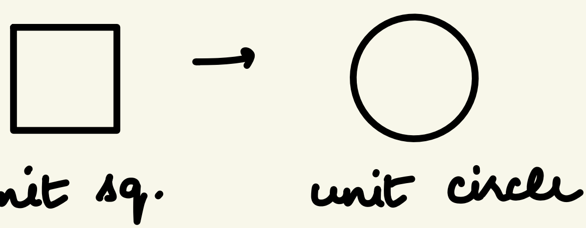
  eg - for rain, expanding kingdom, well-being of subjects, etc.

- These were supposed to be performed exactly as specified. Failing to do so might cause no results or worse, opposite results.

- These rituals required sophisticated altars – combinations of squares, rectangles, triangles, circles, etc.

- Involved transferring the fire from one altar to another s.t the area of the latter is either equal or in fixed proportion to the former.

eg.



- Such problems led to study of basic properties of triangles, rectangles, circles, etc.

  Hence, led to topics we study in Euclidean geometry & number theory.

Reference

- Rigveda Samhita
- Taittiveya Samhita
- Taittiveya Brahmna
- Observations of Ram in the hermitage of Agatsya in Chitrakoot & Panchvati

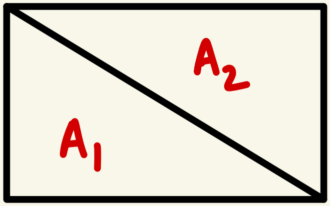
- K Jayashankar (PhD Thesis, 2007)

  'Sulva sutras : A critical study'

Geometry

Simple geometrical facts –

- Any diagonal of a rectangle divides it into 2 triangles of same area

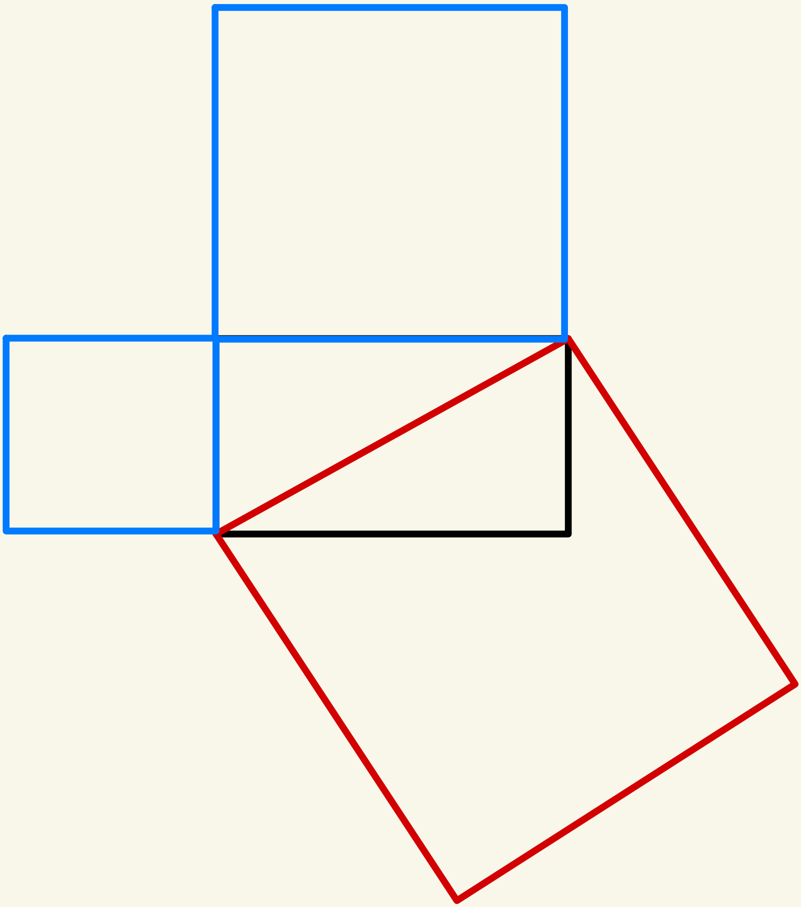


$A_1 = A_2$

- Diagonals of a rectangle have equal length & bisect each other.

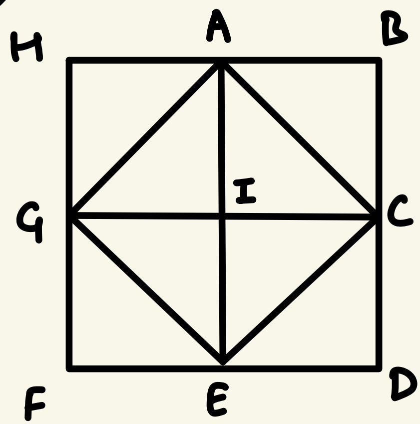
- The perpendicular from vertex to base of an isosceles triangle divides it into 2 triangles of equal area.

Boudhayana (Pythagoras Thm)



Given any rectangle, the area of sq. formed by diag. is equal to the sum of areas of sq. of similar sq. formed by the 2 adjacent sides.

Pf – (For special case)



$$[ACEG] = [ACI] + [CEI] + [EGI] + [GAI]$$

$$= [ACI] + [CEI] + [ACB] + [CED]$$

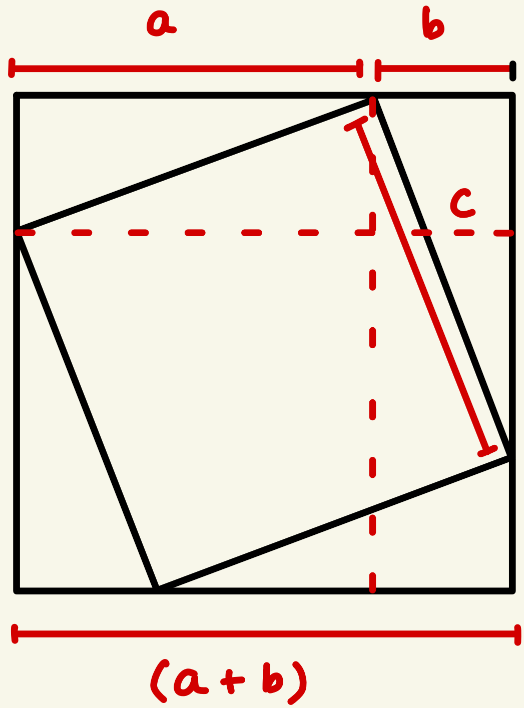
$$= [ABCI] + [CDEI]$$

$$\left(\because \begin{array}{l} [EGI] = [ACB] \\ [GAI] = [CED] \end{array} \right)$$

$\square$

L9 - 03/09/2024

Pf - (General Case)



We will compute the area of sq. with side (a+b) in 2 ways

1. ar. of sq. (side a)
   + ar. of sq. (side b)
   + 2 × ar. of rect. (sides a, b)

2. ar. of sq. (side c)
   + 4 × ar. of tr. (sides a, b)

$$(a+b)^2 = a^2 + b^2 + 2(ab)$$
$$= c^2 + 4\left(\frac{1}{2}ab\right)$$
$$\Rightarrow \quad c^2 = a^2 + b^2 \qquad \square$$

- Lifetime of Pythagoras : 572 - 501 BC

So, Baudhayana's discovery predates Pythagoras'.

- <u>Henkel</u> - (Historian)

'Pythagoras' proof savours the Indian style more than the Greek.'

NOTE – The pf. of the general case is **NOT** given in Sulvasutras.

The author believes that if they had produced a pf., it would have been along the lines of the one presented here.

L10 - 05/09/2024

'Dirghachatustrayakshyanaha rajuh pashriyavami tiryagamani ch yat prithag bhoote kurutstdubhaya karoti'

Hypotenuse   Adjacent sides

- Pythagoras Thm in number theory

Determine $(a,b,c) \in \mathbb{Z}^3_{\geq 0}$ s.t they form sides of a right-angled triangle with c as hypotenuse

$$\Rightarrow \quad a^2+b^2=c^2 \quad - \text{(E)}$$

eg - (3,4,5), (5,12,13), (7,24,25), (8,15,17), (12,35,37), etc.

NOTE - (a, b, c) satisfies E

$\Rightarrow (ka, kb, kc)$ satisfies E

$\forall\ k \in \mathbb{Z}_{\geq 0}$

Sol^{n}s s.t $GCD(a,b,c) = 1$ are called primitive sol^{n}s.

There are infinitely many such primitive sol^{n}s.

Non-trivial examples in Sautramani Altar -

$(5\sqrt{3}, 12\sqrt{3}, 13\sqrt{3})$

$(15\sqrt{2}, 36\sqrt{2}, 39\sqrt{2})$

- **General Sol^n** -

(**NOT** in Sulvasutras.

By Brahmagupta)

$$a = (m^2 - n^2)$$

$$b = 2mn$$

$$c = (m^2 + n^2), \qquad GCD(m,n) = 1$$

- **Connection to Modern Math**

Study of integer sol^ns to polynomial is an active and open discipline.

eg - **Fermat's Last Thm**

$$a^n + b^n = c^n$$

has no non-trivial integer sol^ns for $n \geq 3$

Geometric Constructions

- Only compass & straight edge available for construction

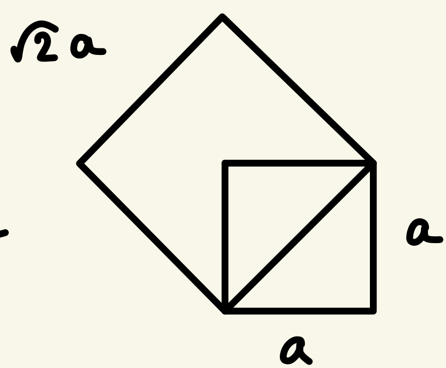
- **Area axiom** – Area of rect. with sides of lengths a & b is ab.

- **Problems**

1. Given a sq. of side a and a natural no. n, construct a sq. with area na^2

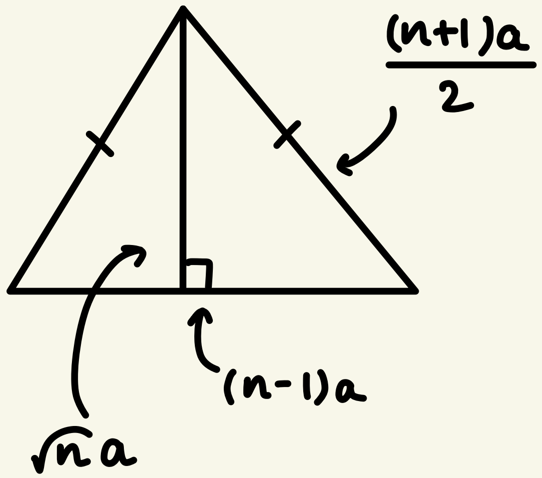
A. **1 |** **For $n = 2$**

Construct a sq. on with the diagonal of original sq. as side.



1.2 General case

Construct an isosceles triangle with base $(n-1)a$ & equal sides $(n+1)a/2$



So, the altitude will be

$$\sqrt{\frac{(n+1)^2}{4}a^2 - \frac{(n-1)^2}{4}a^2} = \sqrt{n}a$$

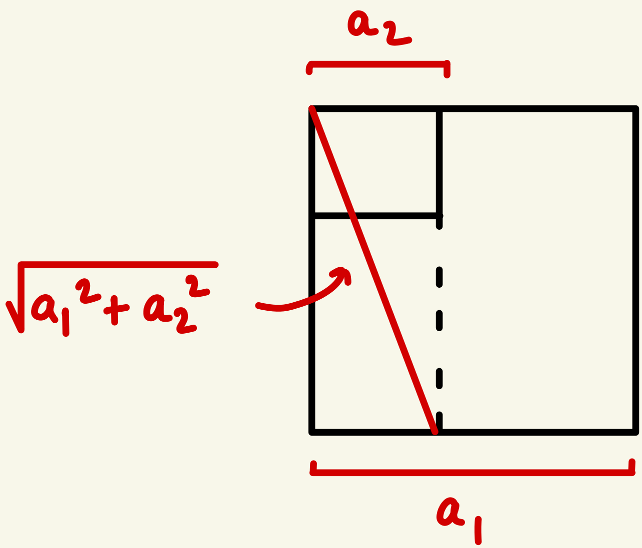
Now construct a sq. with this altitude as the side.

2\. Given 2 sq. S_1 & S_2, construct a sq. with area which is

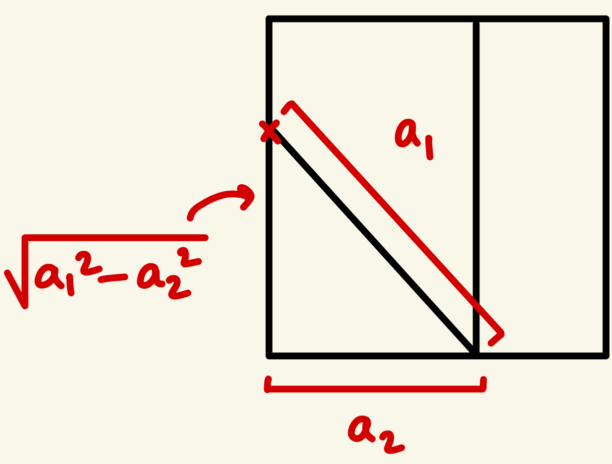
2.1 sum of areas of given sq.

2.2 diff of areas of given sq.

A. 2.1



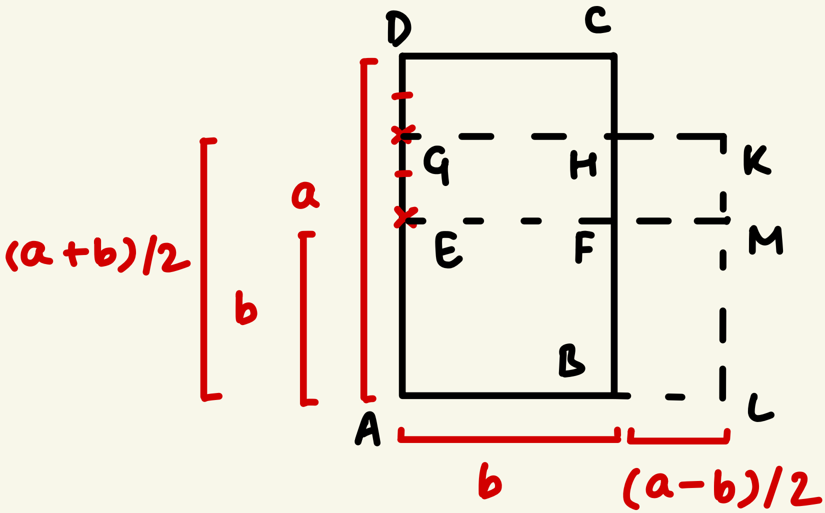
2.2



L11 - 10/09/2024

3. Given a rect., construct a sq. with the same area.

A.



Apply diff. of sq. construction of KL & BL.

L12 - 12/09/2024

4. <u>Circling a sq.</u>

<u>A.</u> From our knowledge,

$$\pi R^2 = a^2 \quad \Rightarrow \quad R = \frac{a}{\sqrt{\pi}}$$

But, if R is constructible by ruler & compass, so is π.

And we know from Galois Theory that π isn't constructible.

Hence, such an R is <u>not</u> constructible.

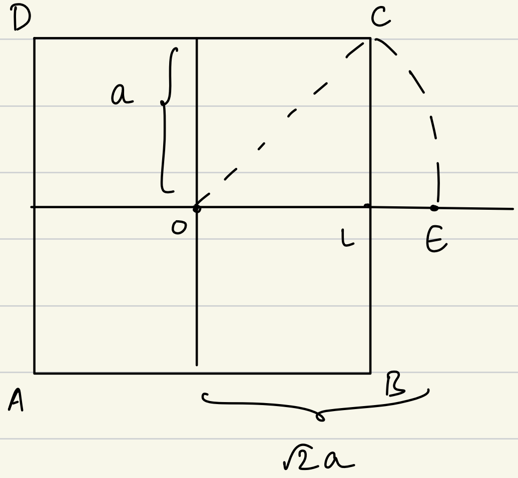
Since the problem can't be solved exactly by ruler & compass

Thus, we seek approximate sol^ns.

Note The no. π isn't defined explicitly in the Sulva Sutras.

<u>Approx. Sol^n</u> –

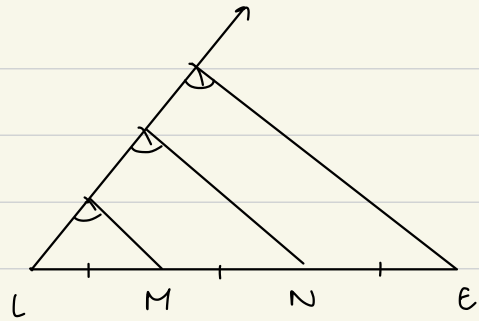
Consider a sq. of length $2a$



1. Mark E on OL s.t $OC = OE$

2. Trisect LE

3. Circle Ω with radius OM gives approximate sol^n the problem



Trisection of line segment

$$\text{ar}(\Omega) = \pi (OM)^2$$
$$= \pi (OE - ME)^2 = \pi \left(\sqrt{2}a - \tfrac{2}{3}(\sqrt{2}-1)a\right)^2$$
$$= \pi a^2 \left(\frac{\sqrt{2}+2}{3}\right)^2$$
$$\sim 1.29\, a^2$$

This gives us an approximation of π.

$$\left.\begin{array}{ll} \text{Expected ar.} = 4a^2 \\ \text{Approx. ar.} = \pi \left(\frac{\sqrt{2}+2}{3}\right)^2 \end{array}\right\} \Rightarrow \pi \left(\frac{\sqrt{2}+2}{3}\right)^2 \sim 4a^2$$

$$\Rightarrow \pi \sim 18(3-2\sqrt{2})$$
$$\sim 3.088$$

For squaring a circle one can try to reverse the constr^n.

Brick Constructions

Altar consists of various layers of bricks. Each layer consisted of bricks of diff. sizes.

This led to mathematical problems involving integer sol^{n}s to poly. eq^{n}s.

eg - 1. Garhipatyagni : 5 layers of 21 bricks each.

Each layer has same ar. - 1 sq. vyam

There are 2 types of bricks of length 1/m & 1/n vyam.

Let x : # type I bricks

y : # type II bricks

Constraints : $x + y = 21$

$$\frac{x}{m^2} + \frac{y}{n^2} = 1 \quad , \quad x, y \geq 0$$
$$m > n$$

$$\Rightarrow \quad \frac{x}{m^2} + \frac{(21-x)}{n^2} = 1$$

$$\Rightarrow \quad x = \frac{m^2(21-n^2)}{(m^2-n^2)}$$

$$y = \frac{n^2(21-m^2)}{(n^2-m^2)}$$

$\because \quad m > n$

$\therefore \quad$ For $x \geq 0$, $\quad n^2 < 21 \quad \Rightarrow \quad n = 1, 2, 3, 4$

By sub^n n & finding corresponding m, we get.

$$(m, n) = \underbrace{(6,3)}^{\text{Config 1}}, \underbrace{(6,4)}^{\text{Config 2}}$$

$$\Rightarrow \quad (x, y) = (16, 5), \ (9, 12)$$

Thus, we get 3 types of bricks − 1/3, 1/4, 1/6.

For stability of layers placed atop others, the following constrn is suggested -

L5 - Config 1
L4 - Config 2
L3 - Config 1
L2 - Config 2
L1 - Config 1

2. <u>Garud Chayan</u> : 5 layers

Each layer consists of 200 bricks & has area 15/2 sq. vyam

There are 4 types of bricks with area $1/a_i$ for type i

Let x_i : # bricks of type i

<u>Const^n</u> :

$$\sum x_i = 200$$
$$\sum x_i/a_i = 15/2$$

<u>Sol^n</u> : $(a_1, a_2, a_3, a_4) = (16, 25, 36, 100)$

$(x_1, x_2, x_3, x_4) = (2, 120, 36, 20) ; (12, 125, 63, 0)$

L13 - 24/09/2024

Modern perspective on Brick Problems

Let us consider the problem that came up

$$\frac{x}{m^2} + \frac{(21-x)}{n^2} = 1$$

$$\Rightarrow \quad n^2x + m^2(21-x) = m^2n^2$$

$$\Rightarrow \quad \underbrace{m^2n^2 - n^2x + m^2x - 21m^2}_{\Delta(m,n,x)} = 0$$

$$\Delta(m,n,x) = 0$$

Polynomial in 3 variables

We are interested in the integral roots of $\Delta(m,n,x)$

$$V_{\mathbb{Z}}(\Delta(m,n,x)) = \{(m,n,x) \in \mathbb{Z}^3 : \Delta(m,n,x) = 0\}$$

Hypersurface

Similarly, $V_{\mathbb{Q}}, V_{\mathbb{R}}, V_{\mathbb{C}}$ can be studied using tools from algebraic geometry - for $\mathbb{R}$ & $\mathbb{C}$
(arithmetic) - for $\mathbb{Z}$ & $\mathbb{Q}$

Conclusion

They had also studied approximations of irrationals (infinite series)

Results here interact with the first two and the sixth vol. of Euclid's elements.

Difference - Euclid's work was evidently based on the principles of logic.
This is not discernable here i.e. there is no direct account of their methods.

However, Prof. Iyengar remarks that given the complexity of their methods, they must be credited with some logic in their methods.

Also, Rishis were practising yogis. So, their subtle perception can allow 'seeing truth' directly.

Remember, sulva sutras were just adjunct texts.

This offers a fresh perspective on the topics we learn in modern times & motivated by universal well-being.

L14 – 26/09/2024

Jain Mathematics

According to Prof. Iyengar, Math was an integral part of Jainism.

- A section of their literature is called 'Ganitanuyoga' which is a system of calculations.

- Mahavir (24^{th} Tirthankar) ~ 600 BCE was well-versed in Math.

- Not much is known about Math in the original Jain texts & is a topic for search-research.

- Current knowledge is based on commentaries.

Mathematically significant Jain texts

1. Surya prajyapati ~ 500 BCE
2. Jambudweep prajyapati ~ 500 BCE
3. Sthanang sutra ~ 300 BCE
4. Uttaradhyaman sutra
5. Bhagwati sutra
6. Anuyog dwar sutra ~ 300 BCE

Authors

1. Bhadrabahu
   - Born in Magadha
   - Moved to Shravanbedgoda ~ 313 BCE
   - Commentary on Surya prajyapati
   - Astrononical work Bhadrabahu samhita

2. Umaswati
   - Born in Nyogrodhika ~ 150 BCE
   - Moved to Kusumapura
   - Known more on his work on Jain metaphysics.

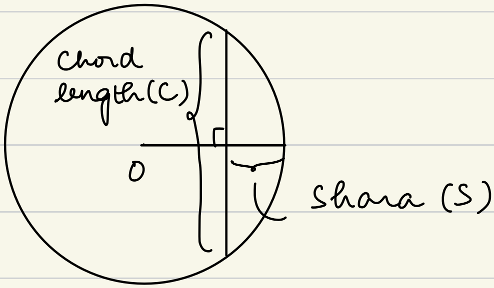
(Aryabhatta was also here ~ 476AD)

- Prof. Iyengar concludes that the Math in his work were taken from other texts from the times.

Samples from Jain Math

1. Circumference of Circle $= \sqrt{10} \times$ Diameter

2. Area of circle $= \frac{1}{4}$ Circumference $\times$ Diameter

3. 

$$C = \sqrt{4\, S(D - S)}$$

Diameter

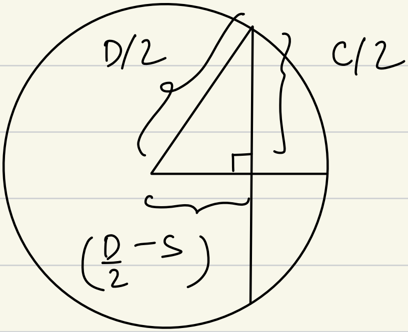
4. $S = D - \sqrt{D^2 - C^2/4}$

$$S = \frac{1}{2}\left(D - \sqrt{D^2 - C^2}\right)$$

5. Area of segment $= \sqrt{6S^2 + C^2}$

6. $D^2 = S^2 + C^2/4$

Pf (3,4,6)



$$(D/2)^2 = (C/2)^2 + (D/2 - S)^2$$

$$\Rightarrow D^2 = C^2 + (D - 2S)^2$$

$$\Rightarrow D^2 = C^2 + D^2 - 4DS + 4S^2$$

$$\Rightarrow C = \sqrt{4S(D-S)}$$

Context

- Surya prajyapati uses 2 approximations for π – 3 & $\sqrt{10}$.

- Circular model of Jambudweep (Earth)
  Diameter = 100,000 Yojana
  (1 Yojana ~ 3.5 to 15 km)

- Circumference = 316,227 Yojana

- This led them to large nos. and to infinity.

- Types of Infinities
    - Samkhya (Enumerable)
    - Shrasamkhya (Unenumerable)
    - Shrananta (Infinite)

LIS – 01/10/2024

Law of Indices – $a^1, a^2, a^{1/2}, a^{1/4}$

$$a^i \cdot a^j = a^{(i+j)}$$

(present in Anuyog dwar sutra)

Vikalpa

Permutations & Combinations
(present in Bhagwati sutra)

- 5 senses – sight, hear, smell, taste, touch

PnC was req. to form smaller groups of these 5 senses.

- Selection out of a given no. of men & women

- Formulae found in Bhagwati Sutra

$$\binom{n}{1} = n, \quad \binom{n}{2} = \frac{n(n-1)}{2 \cdot 1}, \quad \binom{n}{3} = \frac{n(n-1)(n-2)}{3 \cdot 2 \cdot 1}$$

$$^nP_r = n! / (n-r)!$$

- Anuyog dwar sutra & commentary Haemchandra (1089 AD) mention that ${}^nP_r$ was known.

Thus, credit must be given to Jains for the systematic devp. of this topic

Pingala (~ 300 BC)

His compilation Chandas sutra contains Pascal's Δ.

Conclusion

Unlike Vedas, mathematics seems to be a central part of religion for Jains.

Jain math included geometry motivated by astronomy & PnC.

Indian Mathematicians

Knowledge of history of Indian mathematics is sparse.

Very little is known about the Vedic rishis who composed Sulva sutras as well as Jain mathematicians particularly before ~ 499 AD

L16 – 03/10/2024

Aryabhatta

There are atleast 2 mathematicians of the same name.

1. Aryabhatta of Kusumapura (~ 499 AD)
2. Aryabhatta who wrote Aryasiddhanta or Maharyasiddhanta (~ 950 AD)
   (astronomical treatise)

There is some evidence regarding the existence of an Aryabhatta prior to Aryabhatta of Kusumapura

- Al-Biruni (Persian historian) ~ 1050 AD mentions 2 Aryabhatta - one of Kusumapura & the other who is elder.

- Also, he wasn't aware of the author of Aryasiddhanta.

- But, his work is known to contain errors.

- His references to the 2 Aryabhatta are both from the same book - Aryabhatiya leading to confusion.

- Aryabhatta of Aryasiddhanta mentions a quote in his work along the lines of :

'A long time has elapsed since the elderly Aryabhatta propounded his theory and hence it contains specific errors. Hence, I resay in my own terms'

But, Aryasiddhanta and Aryabhatiya are different in flavour.
Hence, the 'Aryabhatta' here does <u>not</u> seem the one of Kusumapura.

- Brahmagupta (~ 628 AD) fiercely critisizes Aryabhatiya in the beginning of his text & later speaks of reverence of Aryabhatta.

Aryabhatta of Kusumapura

- Born in Kerala (476 AD)
- Wrote Aryabhatiya (499 AD)
- Aryabhatiya is a relatively small text with 108 shlokas. (in Aryavritta chandas).

- 3 parts : Ganita, Kalakriya, Goda. (sphere)

- Topics :
  1. Enumeration (via alphabets)
  2. Cube roots
  3. Summations
  4. Approx. of π
  5. Trigonometry

6. Linear Diophantine Eq^{n}s
(integer sol^{n}s to $ax + by = c$, $(a, b, c) \in \mathbb{Z}^3$
his method is called 'Kuttaka')

After the discovery of Jain texts & the Bhakshali manuscript, some of these cannot be attributed to Aryabhatta.

But, 4, 5, 6 still stand to his credit

1. Enumeration
In Devanagari script,

25 Vargiya consonants – Ka, Kha, Ga ...
9 Avargiya consonants – Ya, Ra, La ...
10 vowels – A, Aa, E, Ee ...

Assign VC – 1 to 25
AC – 30, 40, 50 ..
V – 10^0, 10^1, 10^2 ...

eg — Khya — $\underbrace{\text{Kha}}_{2} + \underbrace{\text{Ya}}_{30} = 32$

Khyu — $(\underbrace{\text{Kha}}_{2} + \underbrace{\text{Ya}}_{30}) \cdot \underbrace{\text{u}}_{10^5} = 32 \cdot 10^5$

Ghoo — $(\underbrace{\text{Gha}}_{4}) \cdot \underbrace{\text{oo}}_{10^6} = 4 \cdot 10^6$

2. Summation

— $\sum n^2 = \frac{1}{6} n(n+1)(2n+1)$

— $\sum n^3 = \left(\frac{n(n+1)}{2}\right)^2$

— $1 + (1+2) + (1+2+3) + \dots$

$= \sum \sum n = \frac{1}{6} n(n+1)(n+2)$

4. Approx. to π — 3.1416

(given in the form that circumference of circle with diameter 20000 is 62832)

He also mentions that this is just an approx. & that π is maybe irrational.

L17 – 08/10/2024

Ancient Greek Mathematics

<u>Reference</u> — Stilwell : Mathematics & its History

Rational pts on a Circle

This is related to enumerating Pythagorean triples.

$$a^2 + b^2 = c^2 \quad , \quad a, b, c \in \mathbb{Z}_{\geq 0}$$
$$GCD(a, b, c) = 1$$

General sol^n —
$$a = p^2 - q^2$$
$$b = 2pq$$
$$c = p^2 + q^2$$

But, there's a geometric approach to the same problem.

for non-trivial solⁿs, $c \neq 0$.

$$\Rightarrow \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

Hence, consider the eqⁿ : $x^2 + y^2 = 1$

$$S = \{(x, y) : x^2 + y^2 = 1\}$$

unit circle

L18 - 10/10/2024

Pythagoras Thm

- Arithmetic & Number Theory
- Geometry & Distance
- Infinity in Greek Math

Chord - Tangent Method

1. Start with a simple sol^n (seed)

2. Construct new sol^ns in terms of the seed.

So, start with sol^n $P(0,1)$.
Notice, L passing through P (except when tangent) intersects the circle at exactly 2 pts. (the other being (x_0, y_0))

L: $(y-1) = mx$ (for finite slope m)

$x = 0$

1. So, $y_0 - 1 = mx_0$

$$x_0^2 + y_0^2 = 1 \Rightarrow x_0^2 + (mx_0+1)^2 = 1$$

$$\Rightarrow (1+m^2)x_0^2 + 2mx_0 = 0$$

$$\Rightarrow x_0 = 0, \frac{-2m}{1+m^2}$$

$$y_0 = 1, \frac{1-m^2}{1+m^2}$$

Now, $m \in \mathbb{Q} \Rightarrow (x_0, y_0) \in \mathbb{Q}^2$

Conversely, if $(x_0, y_0) \in \mathbb{Q}^2$, the slope of line joining $(0,1)$ & (x_0, y_0) $m \in \mathbb{Q}$

2. $L: x = 0$ intersects the circle at $(0,1)$ & $(0,-1)$

Hence, the set of all rational pts. $C(\mathbb{Q})$ on the circle C is

$$\left\{ \left(\frac{-2m}{1+m^2}, \frac{1-m^2}{1+m^2} \right) \mid m \in \mathbb{Q} \right\} \cup \{(0,-1)\}$$

To enumerate all primitive Pythagorean triples,

put $m = \frac{p}{q}$, $GCD(p, q) = 1$

$$\Rightarrow \left(\frac{-2m}{1+m^2}\right)^2 + \left(\frac{1-m^2}{1+m^2}\right)^2 = 1$$

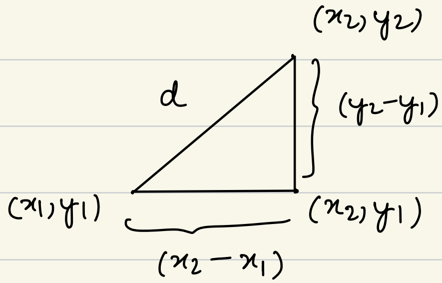
$$\Rightarrow \left(\frac{2pq}{p^2+q^2}\right)^2 + \left(\frac{p^2-q^2}{p^2+q^2}\right)^2 = 1$$

$$\Rightarrow \underbrace{(2pq)}_{a}^2 + \underbrace{(p^2-q^2)}_{b}^2 = \underbrace{(p^2+q^2)}_{c}^2$$

Geometry

- The notion of distance in coordinate geometry is derived from the Pythagoras Thm

$$d\left((x_1, y_1), (x_2, y_2)\right) = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$



L19 - 15/10/2024

Infinity

The irrationality of $\sqrt{2}$

Thm The eq^n $x^2 - 2y^2 = 0$ does not have non-trivial sol^{n}s over the integers

i.e $\nexists \, (x_0, y_0) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ s.t $\underbrace{x_0^2 - 2y_0^2 = 0}_{E}$

The proof introduces 2 useful & powerful methods.

- Proof by contradiction
- Method of infinite descent

Key steps

1. Define the notion of $\underbrace{\text{size}}_{\text{non-(-ve) int.}}$ of a sol^n

2. For any $k \in \mathbb{Z}_{\geq 0}$, the no. of int. sol^{n}s of E with size $\leq$ k is finite

3. Suppose E has a non-t'vl int. sol^n. Then, construct a non-t'vl int. sol^n of strictly smaller size.

Details

1. $$N((x_0, y_0)) = x_0^2 + y_0^2$$

2. Given k, s.t $x_0^2 + y_0^2 \leq k$

   then $x_0^2 \leq k$ & $y_0^2 \leq k$

So, $$S_k = \{ (x,y) \in \mathbb{Z}^2 : x^2 + y^2 \leq k \}$$

$$S_k \subseteq \{ (x,y) \in \mathbb{Z}^2 : \max\{|x|, |y|\} \leq \sqrt{k} \}$$

But, the set has size $\leq (2\sqrt{k})^2 = \underbrace{4k}_{\text{finite}}$

3. Suppose (x_0, y_0) is a non-t'vl sol^n to E

Then, $x_0^2 - 2y_0^2 = 0$

$\Rightarrow x_0^2 = 2y_0^2$

$\therefore x_0^2$ is even

$\therefore x_0$ is even i.e $x_0 = 2z_0$

$(2z_0)^2 - 2y_0^2 = 0$

$\Rightarrow 4z_0^2 - 2y_0^2 = 0$

$\Rightarrow y_0^2 - 2z_0^2 = 0$

$\therefore (y_0, z_0)$ is a sol^n of E

But, $N((y_0, z_0)) = y_0^2 + z_0^2$

$< y_0^2 + x_0^2 = N((x_0, y_0))$

L20 - 17/10/2024

Suppose E has a non-t'vl int. sol^n (x_0, y_0)

Let $s = N((x_0, y_0)) = x_0^2 + y_0^2$

$\because$ The set of int. pts. with size $\leq s$ is finite

$\therefore$ The set of non-t'vl int. sol^{n}s with size atmost s is also finite.

So, $\exists$ a pt. with min. size.
Let (a_0, b_0) be such a pt.

But, as shown prev., $(b_0, a_0/2)$ is also a non-t'vl int. sol^n.

However, $N((b_0, a_0/2)) < N((a_0, b_0))$ which is a contd^n.

Proof based on Aristotle's Priori Analytics

If $\exists$ a non-t'vl int. sol^n, WLOG, we can assume it is primitive.

i.e $\gcd(x_0, y_0) = 1$

$$x_0^2 = 2y_0^2$$

x_0^2 is even $\Rightarrow$ x_0 is even

So, $x_0 = 2k_0$ for some k_0

$$\Rightarrow (2k_0)^2 = 2y_0^2$$
$$\Rightarrow 2k_0^2 = y_0^2$$

y_0^2 is even $\Rightarrow$ y_0 is even which is a contd^n

The result shocked the ancient Greeks.
They did not accept $\sqrt{2}$ as a no.

So, they tried to approximate $\sqrt{2}$ in terms of rational nos.
(Diophantine approximations)

Pythagoras

- Very little is known for certain.
- None of the documents from his times have seemingly survived.
- Rely on stories passed on through generations
- Birth: Samos, Greek Island near Turkey ~ 580 BC
- Travelled to Miletus to learn Math from Thales ~ 624 – 547 BC (founder of Greek Math)
- Also travelled to Egypt & Babylon & picked up more Math.

- Settled in Croton ~ 540 BC (a Greek colony in Italy)

- Here, he founded a school, now called the Pythagoreans.

- <u>Philosophy</u>: 'All is number'
To bring all human endeavour including science, religion, philosophy in the realm of Math.

- The word 'Mathematics' is attributed to this school.

- <u>Strict conduct of conduct</u>:
Secrecy, vegetarianism, taboo against eating beans, sought numerical laws governing orbits of planets

- <u>Highlights</u> : Explanation of musical harmony of whole-no. ratios.

- <u>Death</u> : 497 BC

<u>Archaeological sites</u> · Pythagoreion , Samos

Greek Geometry

1. The method of deduction (Proof)

The method involves demonstrating the validity of a statement using prev. established statements using principles of logic.

These established statements are called postulates or axioms – 'Self-evident' or 'intuitive' statements

Hence, all statements trace back to the axioms.

L21 - 22/10/2024

Greek geo. is attributed to Thales & was devp. by Euclid in his Elements

Euclid's postulates

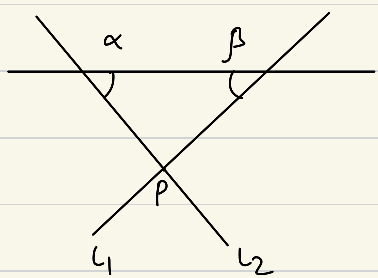
1. Two distint pts. determine a (unique) straight line.

2. A line segment can be infinitely extended to a (unique) line.

3. Given any radius & a pt., there is a (unique) circle with pt. as center & radius as the given one.

4. All right angles are equal to each other.

5. Suppose a line intersects two other lines s.t the sum of interior angles α & β ($\leq \pi/2$) with the



two lines is $\alpha + \beta < \pi$

Then, l_1 & l_2 can be extended to intersect at a pt. P.

Note, the notions of a line & circle are not defined & relies on intuition.

Limitations of Euclid's Elements

- Euclid does not (as per the text), stick only to the axioms, but also some other statements are taken as self-evident.

- This axiom based approach is awkward to deal with for higher degree curves like cubics.

The questions related to foundations of axioms were dealt by Hilbert.

The question of whether 5th postulate was indep. of the first 4 was open for hundreds of years
Now, it is known to be indep., giving rise to non-Euclidean geometry.

So, Euclidean geometry has largely been replaced by Coordinate geometry.

L22 - 24/10/2024

In contemporary Math, proof are used to

- verify that a statement is true
- explain 'why' a statement is true.

'Deep' thms usually have multiple proofs.

eg – quadratic reciprocity

See – 'Proofs from the Book' by Ziegler (coined by Erdos) attributed to Theatetus

Regular Polyhedra (Platonic Solids)

		V	E	F
Tetrahedron	-	(4,	6,	4)
Cube	-	(8,	12,	6)
Octahedron	-	(6,	12,	8)
Dodecahedron	-	(20,	30,	12)
Icosahedron	-	(12,	30,	20)

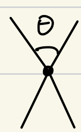
Regular solid - A solid $\subseteq \mathbb{R}^3$ is called regular if all the faces are congruent to each other & are regular polygons and the no. of faces incident on each vertex is the same.

Why only these ?

- Fix a vertex of the solid S.

- $N \cdot \theta < 2\pi$

(no. of faces incident on each vertex) $N \geq 3$



- Fix the no. of sides of regular polygon M

1. $M = 3 \Rightarrow \theta = \pi/3$

$$N \cdot \theta < 2\pi \Rightarrow N \cdot (\pi/3) < 2\pi$$
$$\Rightarrow N < 6$$

$N = 1 \to \times$
$2 \to \times$
$3 \to$ Tetrahedron
$4 \to$ Octahedron
$5 \to$ Icosahedron

2. $M = 4 \Rightarrow \theta = \pi/2$

$$N \cdot \theta < 2\pi \Rightarrow N \cdot (\pi/2) < 2\pi$$
$$\Rightarrow N < 4$$

$N = 1 \to \times$
$2 \to \times$
$3 \to$ Cube

3. $M = 5 \Rightarrow \theta = 3\pi/5$

$N \cdot \theta < 2\pi \Rightarrow N \cdot (3\pi/5) < 2\pi$

$\Rightarrow N < 10/2$

$N = 1 \longrightarrow \times$

$2 \longrightarrow \times$

$3 \longrightarrow$ Dodecahedron

4. $M = 6$ onwards $\Rightarrow \theta = \frac{(s-2)}{s}\pi$

$N \cdot \theta < 2\pi \Rightarrow N \cdot \frac{(s-2)}{s}\pi < 2\pi$

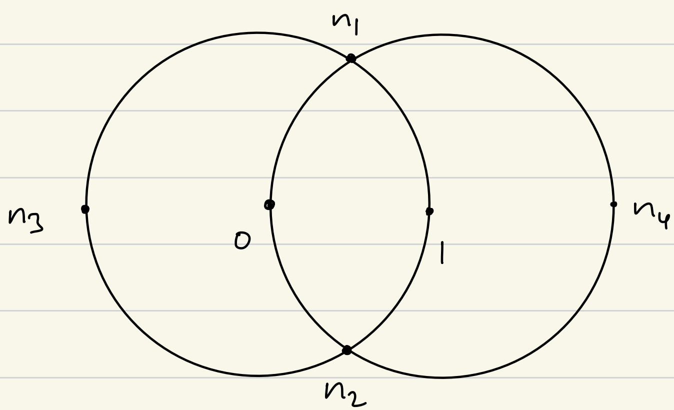
$\Rightarrow N < \frac{2s}{(s-2)}$

$\forall\, s \geq 6, \quad N \in (2, 3)$

Euler's formula - $V - E + F = 2$

Ruler Compass Construction

<u>Recall</u> : constructible nos.



Construct nos. n_1, n_2, n_3, n_4 from $0, 1$.

Inductively repeat to define the set of constructible nos.

<u>Remark</u> – The set of constructible nos. forms a field.

The Greeks tried constructing $\sqrt[3]{2}$, π as well as trisecting the angle.

They couldn't do it & eventually accepted the impossibility & a sol^n by more advanced methods.

<u>Sol^n to</u> –

$\sqrt[3]{2}$ & trisecting angle – Wantzel, Galois (1837)

π – Lindemann

<u>Open</u> – Which n-sided regular polygons are constructible?

L23 — 29/10/2024

Conic Sections

Std. cone : $x^2 + y^2 = z^2$

Parameters – Axis of revolution
Angle of cone (θ)

When a plane inclined at diff. angles to the axis intersect the cone, conic sections are obtained

Conic Section	$\angle$ b/w axis & normal of plane
Circle	0
Ellipse	$< \theta$
Parabola	$= \theta$
Hyperbola	$> \theta$

Attributed to Menaechmus ~ 300 BC
(contemporary of Alexander)

- <u>Doubling the cube using conic sections</u> –
  Intersection of the parabola $y = \frac{1}{2}x^2$
  with the hyperbola $xy = 1$

- Kepler later built on this in his theory of elliptical orbits of planets.
  Newton derived it from his gravitation law.

Euclid

- Less is known about Euclid than Pythagoras

- Taught in Alexandria, Egypt ~ 300BC

- Told Ptolemy I : 'There is no royal road to geometry'

- Student : 'Gain from Math ?'
  Euclid : Gives him a coin

- Known for 'Elements' ~ 12 vols.

- Not all of it was original
  The following was already known :-
  1. Elementary ppts. of lines & circles
  2. Irrationality (Eudoxus ~ 400 - 347 BC)
  3. Theory of regular polyhedra (Theatetus ~ 413 - 309 BC)

- Stands out for organisation, dessemination of Math.
  Core of Math education in West and at the heart of Western culture for > 2000 years.

- Influenced : Lincoln, Russell

Greek Number Theory

Comparison b/w Geometry & Number Theory

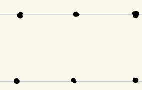
- Geometry allows for a systematic theory compared to Number Theory.

- NT has many open problems with unknown theoretical framework.

- Both are almost as old with NT being slightly older.

- Recently, connections b/w them have emerged.

Prime Nos.

- Rect. nos. — eg. 6 

  Primes : Non-rect. nos.

- Prime no. — A natural no. $n = pq$ s.t atleast one of p or q is 1.

- Divisibility — m divides n if $\exists\, k \in \mathbb{Z}$ s.t $n = km$

Not^n: $m \mid n$

Here, m is called a divisor of n

P: Every int. has a prime divisor.

Euclid's Thm - There are infinitely many primes.

Pf - Assume finitely many primes $\{p_1, \dots, p_n\}$

Consider $M = \prod_{i=1}^{n} p_i + 1$

Note $p_i \nmid M$, $1 \leq i \leq n$

which is a contd^n to prev. pp^n $\square$

L24 - 01/11/2024

Euclidian Algorithm (Book VII)

- may be known earlier
  credits to Euclid for its presentation
  & applications to Number Theory

Recall, GCD of nat. nos m, n
is the largest nat. no. s.t it divides
both m & n.

- Input: A pair of non-(-ve) int. (a_0, b_0)

Set $i = 0$

If $a_i = 0$, output b_i and if $b_i = 0$,
output a_i

Else, set

$$a_{i+1} = \max(a_i, b_i) - \min(a_i, b_i)$$
$$b_{i+1} = \min(a_i, b_i)$$

- <u>Key</u> : If $a \geq b$, then

$$GCD(a, b) = GCD(a-b, b)$$

<u>Consequences</u> (given in the elements)

<u>1</u> $\exists$ int. x_0, y_0 s.t $GCD(a,b) = ax_0 + by_0$

- <u>Key</u> : a_i & b_i are int. comb. of a, b
  By ind^n, all a_is & b_is are int. combs.
  $\because$ After finite steps, one of a_i, b_i becomes GCD.
  $\therefore$ GCD is also an int. comb. of a & b.

Furthermore, if $GCD(a,b) \mid d$,
$\exists$ int. x, y s.t $d = ax + by$
where $x = \dfrac{dx_0}{GCD(a,b)}$, $y = \dfrac{dy_0}{GCD(a,b)}$

2. If a prime p divides ab, then $p \mid a$ or $p \mid b$.

Pf – WLOG suppose $p \nmid a$

Then, $GCD(a, p) = 1$

So, $\exists\, x_0, y_0 \in \mathbb{Z}$ s.t $1 = ax_0 + py_0$

$$\Rightarrow b = abx_0 + pby_0$$

$\because p \mid ab \Rightarrow p \mid abx_0 + pby_0$

$\therefore p \mid b$

$\square$

3. Fundamental Theorem of Arithmetic

Any (+ve) int. $n\ (\geq 2)$ can be expressed as a product of primes $n = p_1 \cdots p_k$ & the seq. $(p_1, \cdots, p_k)$ is unique upto rearrangement.

- <u>Key</u>: If n is prime, then the hypothesis is true

Else, $\exists\ a, b \neq 1$ s.t $n = a \cdot b$

$\because$ a & b can be written as a prod. of primes

$\therefore$ n can also be written as a prod. of primes

Hence, existence of prime factorization follows by ind^n.

Suppose there are nos. $(n \geq 2)$ having prime factorizations which are <u>not</u> rearrangements of each other.

Consider the smallest such no.

$$n = p_1 \cdots p_k = q_1 \cdots q_r$$

This implies $\{p_1, \ldots, p_k\}$ & $\{q_1, \ldots, q_r\}$ are disjoint.

But, $q_i \mid q_1 \cdots q_r \implies q_i \mid p_1 \cdots p_k$

$\therefore$ $q_i = p_j$ for some $1 \leq j \leq k$

which is a contd^n

Pell's Eq^n

$$x^2 - Ny^2 = 1 \quad , \qquad N - \text{non-perfect square}$$

Most well studied after $x^2 + y^2 = 1$

- <u>Pythagoras</u> $(N = 2)$

Suppose (x_n, y_n) is a sol^n.

i.e $x_n^2 - 2y_n^2 = 1$.

Then
$$x_{n+1} = (x_n + 2y_n)$$
$$y_{n+1} = (x_n + y_n)$$

$$x_{n+1}^2 - 2y_{n+1}^2 = (x_n + 2y_n)^2 - 2(x_n + y_n)^2$$
$$= 2y_n^2 - x_n^2 = -1$$

Hence, (x_{n+2}, y_{n+2}) will be a sol^n.

$(x_0, y_0) = (1, 0)$

- <u>Cattle problem of Archimedes</u>

$$x^2 - 472424\, y^2 = 1$$

The smallest non-trivial solⁿ has 206545 digits

(<u>See</u> : HW Lenstra Jr – Solving the Pell's Eqⁿ)

Comparison b/w Greek & Indian Math

Greek		Indian
	Motivation	
– Intrinsic		– Vedas, rituals, astronomy, poetry
	Proofs	
– Heavy emphasis		– Not much emphasis
	Aim	
– Explain all of nature with Math		– Specific applications

L25 - 05/11/2024

Calculus

- Marks a shift in the focus from geometry to algebra ~ 16 th century

- Allowed for systematic treatment of areas, volumes, tangents, etc.

- These were considered by the Greeks. Their method is called the 'method of exhaustion'.

Key: Approximating shapes by simpler ones

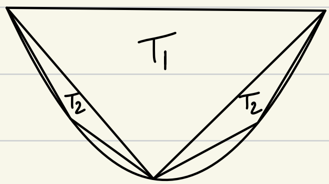
eg - Using regular n-sided polygon to approx circle.

- This method was tedious & hence, Calculus developed as a system of shortcuts.

— <u>Types of Problems</u>

- Area, Volume — Integral Calculus
- Tangent — Differential Calculus

— <u>Archimedes</u>



Method of Exhaustion
to calculate area of parabolic segment.

— More generally, the Greeks tried finding area under $y = x^k$, $k \in \mathbb{Z}_{\geq 0}$

This leads to the sum
$$1^k + 2^k + \dots + n^k$$
which the Arabs calculated for $k = 1, 2, 3, 4$
(956 — 1039 AD)

— Cavalieri (1635 AD) conjectured

$$\int_0^a x^k \, dx = \frac{a^{k+1}}{k+1}$$

Later, Fermat & Descartes established it for integral k $(k \neq -1)$

— Archimedes tried to calculate the tangents to pts in a spiral $r = \theta$

Fermat (1629)

- one of the founders of Calculus

- Introduced limits

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

for polynomials f (one & two variables)
Used this to calculate maxima, minima, tangents ~ 1625 (published 1679)

- This involved a 'sleight of hand' with an infinitesimal element.

i.e. introduce e, divide by e, simplify, and omit e as if it were 0.

eg - $$\frac{(x+e)^2 - x^2}{e} = \frac{2xe + e^2}{e} = 2x + e$$
$$\sim 2x$$

This confused Philosophers.

- Followed by Descartes in his book 'La Géometrie'

eg – $P(x, y) = 0$

$x = x(t), \quad y = y(t)$

$$\frac{dy}{dx} = -\frac{P_x}{P_y} \quad \text{where} \quad P_x = \partial P/\partial x, \quad P_y = \partial P/\partial y$$

Newton

Most imp. discovery ~ 1665–1666

Studied works of Descartes, Wallis, Viete

Works : De Analysi, De Methodis
~ 1669 ~ 1671

- Contributions to differentiation are 'minor' except Chain Rule

- Misleading to consider him the founder of calculus, unless one sees it as algebra of infinite series

Key : Manipulation of infinite series

Diff & Int carried out term by term.

eg - $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots$

$$\frac{d}{dx}(\sin(x)) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots$$

$$= \cos(x)$$

- De Methodis

'Since the operations for computing with nos. & variables are so similar ... amazed that no one (except Mercator) recognized that the doctrine recently established for decimal nos. can also be carried to variables'

L26 - 07/11/2024

His approach was based on infinite series. term by term diff. & int. & a method for inverting the power series.

Using this, he constructed Taylor series expansions for $\log(1+x)$, $\sin(x)$, $\cos(x)$, $\sin^{-1}(x)$, e^x.

Examples

1. $\log(1+x) = \int_0^x \frac{dt}{1+t} = \int_0^x 1 - t + t^2 - t^3 \dots dt$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} \dots$$

(found by Mercator & Kerala School)

2. $\tan^{-1}(x) = \int_0^x \frac{dt}{1+t^2} = \int_0^x 1 - t^2 + t^4 - t^6 \dots dt$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} \dots$$

3. Inverting / 'Extracting the Root'

$$y = \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \;\; ..$$

We wish to express x as follows

$$x = b_0 + b_1 y + b_2 y^2 + \ldots \qquad \text{(Ansatz)}$$

Substituting x in the previous expansion,

$$y = (b_0 + b_1 y \cdots) - \frac{1}{2}(b_0 + b_1 y \ldots)^2 + \frac{1}{3}(b_0 + b_1 y \ldots)^3 \ldots$$

By comparing coeffs. of powers of y,

$$y^0 : \quad 0 = b_0 - \frac{b_0^2}{2} + \frac{b_0^3}{3} \ldots = \log(1+b_0)$$

(if $|b_0| < 1$)

$\Rightarrow \quad b_0 = 0$

$y:\quad 1 = b_1 \quad\Rightarrow\quad b_1 = 1$

$y^2:\quad 0 = b_2 - b_1^2/2 \quad\Rightarrow\quad b_2 = 1/2$

$y^3:\quad 0 = b_3 - b_2 b_1 + b_1^3/3 \quad\Rightarrow\quad b_3 = 1/6$

So,
$$x = y + \frac{y^2}{2} + \frac{y^3}{6} + \dots$$
$$= \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} \dots$$
$$= e^y - 1$$

4. Binomial series

$$(1+a)^p = 1 + pa + \frac{p(p-1)}{2} a^2 + \dots$$

where $\quad p \in \mathbb{R}_{\geq 0}, \quad a \in \mathbb{R}$

Substitute $a = -t^2$, $p = -1/2$

we get,

$$\frac{1}{\sqrt{1-t^2}} = 1 + \frac{t^2}{2} + \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)t^4}{2} \dots$$

$$\sin^{-1}(x) = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt = x + \frac{x^3}{2\cdot 3} + \frac{1}{2}\cdot\frac{3}{4}x^5 \dots$$

Inverting, we get the expansion of $\sin(x)$.

Leibniz

- Newton's 2 papers submitted to the Royal Society Proceedings were rejected in the 1670s

- In the meantime, a German mathematician philosopher, diplomat published 'Nova Methodus' laying out the foundations of calculus. ~ 1684

- In this paper, he lays down the sum, product & the quotient rule and introduced the notation dy/dx

To him, dy/dx was an actual quotient of infinitesimals

- Following this, in his 'De Geometrica' introduced '$\int$' & proved ~ 1686
the fundamental theorem of calculus

This was known to Newton & his teacher Barrow in a different form

For Leibniz $\int f(x)$ was a sum of terms representing infinitesimal rectangles of height $f(x)$ & width dx

He showed $\dfrac{d}{dx}\int f(x)\, dx = f(x)$

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Characteristics of Leibniz's Math

1. His strength was identification of concepts as against their technical devp.

eg – Notations : d/dx & $\int$

2. Introduced the word 'function'
algebraic v/s transcendental

3. Preferred closed form rather than infinite series.

From his perspective, $\int f(x)\,dx$ involved finding a fx^n F (antiderivative) s.t $F' = f$

– The search for such closed forms lead to a 'wild goose chase' but, integration of rational fx^ns lead to factoring of polynomials.

- Integration of $\int \frac{dx}{\sqrt{1-x^4}}$ lead to the theory of elliptic curves

Newton

- Born : 1642 in Woolsthrope, Lincolnshire, England
- Tough early years
- Initial interest in Mechanics such as windmills, later academics
- Entered Trinity College, Cambridge ~ 1661 as a 'sizar'
  (students who earn their keep by serving wealthier students)

- Early studies : Aristotle, Descartes
- By 1664, he prepared notes 'Questiones Quadam Philosophicae'.
  Mechanics, Optics & philosophy of vision

- 1665 : Plague in England
  Newton returned to Woolsthrope & was absorbed in research

- 1664 to 1666 : Most creative period for Newton his first papers appeared
  (De Analysi, De Methodis)

- 1669: Lucasian Professor of Mathematics

- 1687: Principia Mathematica
  Theory of gravitation, elliptic loci of orbits

Leibniz

- Born: 1646, Leipzig, Germany
- Both parents were academics
- Access to father's library
- At 15, University of Leipzig & doctorate from Aldorf in Law 1666.
- 1663: Visited Jena, Germany
  studied Euclid
- 1672 – 1676: Crucial in Math
  - Pascal's triangle ~ 1666
  - Met Huygens

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

- 1673: $\pi/4 = 1 - 1/3 + 1/5 \ldots$

$$1/2 \log(2) = \frac{1}{2 \cdot 4} + \frac{1}{6 \cdot 8} + \ldots$$

- De Arta Combinatoria ~ 1666
  - Systemically deduce all true statements
  - Identified by Leibniz & later developed by Hilbert, Gödel ..

- Again, did <u>not</u> delve 'deep' into these but identified these concepts.