Islamic Mathematics

Aarush Bhatt & Vipin Gour

Indian Institute of Technology, Bombay

October 2024

Contents

Introduction

Baghdad Academy
Indian Heritage

Baghdad Academy

Reference : Sesiano, J. (2000). Islamic Mathematics. In: Selin, H. (eds) Mathematics Across Cultures. Science Across Cultures: The History of Non-Western Science, vol 2. Springer, Dordrecht.

In **762**, Arabs received their new capital of **Baghdad** founded in by Caliph al-Mansur after conquering vast territories in the peninsula. This was followed by the establishment of an institution in Baghdad called the 'house of wisdom' **(bayt al-hikma)**, also referred to as the **Baghdad Academy.**

This academy was similar to the one created by the Greeks a millennium before in Alexandria which had remained the foremost scientific centre until late antiquity.

The first task of the Baghdad Academy was to **collect and synthesize the scientific knowledge available at that time.**

Indian Heritage (1/2)

Reference : Sesiano, J. (2000). Islamic Mathematics. In: Selin, H. (eds) Mathematics Across Cultures. Science Across Cultures: The History of Non-Western Science, vol 2. Springer, Dordrecht.

Indian science was at this time very much alive and growing, and **direct contact occurred**, since Arabic chronicles of the time report a **visit by Indian scholars to Baghdad** during the early years of the Academy.

The first major Indian mathematical and astronomical works date from the sixth and seventh centuries. At that time, the system used to designate numbers was already the **positional** one, with ten signs including the zero for the empty place. Indian mathematicians had also developed the **arithmetical operations** adapted to that system, to the extent that numerical calculations became an essential part of arithmetical and algebraic problems dealing with the **practical needs of daily life and trade.**

Indian Heritage (2/2)

In astronomy, some technical terms and the use of geometrical models to represent the planetary movements point to a Greek influence. However, unlike the Greek use of chords in plane trigonometry, the **Indians used half-chords** and thus introduced the **sine** and **cosine.**

Since the Indian heritage arrived at an early date, **Islamic science was from the outset marked by a knowledge of Indian arithmetic and astronomy**, though in the latter case for a transitory period only.

Arithmetic

Al-Kashi’s Root Extraction Algorithm

Al-Kashi's Root Extraction Algorithm (1/4)

Reference : Aydin, N., & Hammoudi, L. (2015). Root extraction by Al-Kashi and Stevin. Archive for History of Exact Sciences, 69(3), 291–310.

- Given an integer N, its digits are divided into groups of two that Al-Kashi calls "cycles" starting from the units digit. eg - If the integer is $abcdef$ the cycles are ab, cd, and ef. (In case of a 5-digit integer, the last cycle would have contained one digit).
- The **number of cycles determines the number of digits in the integer part of the square root**. Here, since the number has 3 cycles; the integer part of square root will be formed by 3 digits.
- After creating the cycles, Al-Kashi looks for $\sqrt{N}$ one digit at a time starting from the left. First, he looks for the integer part of the root of the left most cycle (Here, ab).

Al-Kashi's Root Extraction Algorithm (2/4)

- He considers the greatest integer A whose square is less than or equal to the number formed by the cycle; $A^2 < ab$. A will be the left most digit of the integer part of $\sqrt{N}$.
- He then computes the difference $P_0 = ab - A^2$ and forms a new number P_0cd, where is the concatenation of P_0 and cd.
- Now, Al-Kashi looks for the greatest integer B s.t. $(2A \cdot 10 + B) \cdot B \leq P_0cd$. The number B is the second digit of $\sqrt{N}$
- To look at the third digit, Al-Kashi considers the difference $P_1 = P_0cd - (2A \cdot 10 + B) \cdot B$ and forms the number P_1ef.
- He looks for the largest integer C s.t. $(2AB \cdot 10 + C) \cdot C \leq P_1ef$. This digit C is the third digit of $\sqrt{N}$.

Al-Kashi's Root Extraction Algorithm (3/4)

- At last, he computes the difference $P_2 = P_1 ef - (2AB \cdot 10 + C) \cdot C$
- If $P_2 = 0$, then the process is finished and $ABC = sqrt(N)$. Otherwise, Al-Kashi takes P_2 as the numerator of the fractional part of the square root and $(2AB \cdot 10 + C) + C + 1 = 2ABC + 1$ as the denominator. Hence, approximation to $\sqrt{N}$ is $ABC \frac{P_2}{2ABC+1}$

Al-Kashi made great use of these approximate square roots in his **approximation of π upto 16 digits.**

Al-Kashi's Root Extraction Algorithm (4/4)

He had also extended this principle to higher root extractions. For instance, in his **Key of Arithmetic** he extracted the fifth root of 44240899506197 as

$$536 + \frac{21}{414237740281}$$

Number Theory

Thabit’s Amicable Number Generator

Thabit's Amicable Number Generator (1/4)

Reference : Sesiano, J. (2008). Number Theory in Islamic Mathematics. In: Selin, H. (eds) Encyclopaedia of the History of Science, Technology, and Medicine in Non-Western Cultures. Springer, Dordrecht

Let N be a natural number and $s(N)$ denote the sum of proper divisors of N.

The Greeks called N "abundant" if $s(N) > N$, "defective" if $s(N) < N$, and "perfect" if $s(N) = N$. Euclid had demonstrated that $N = 2^{(n-1)}(2^n - 1)$ is perfect if $2^n - 1$ is prime.

Thābit ibn Qurra (836–901) noticed that $2^{(n-1)} \cdot p$, n given and p prime, is defective or abundant depending on whether $p > 2^n - 1$ or $p < 2^n - 1$.

Thabit's Amicable Number Generator (2/4)

In the same treatise, he provided for the first time a rule for finding a pair of "amicable" numbers, in which each number is equal to the sum of the proper divisors of the other: numbers N_1, N_2 such that $s(N_1) = N_2$ and $s(N_2) = N_1$. The Greeks knew of only one pair (220, 284).

His rule is: if $s = 3 \cdot 2^n - 1$, $t = 3 \cdot 2^{(n-1)} - 1$, $r = 9 \cdot 2^{2n-1} - 1$ (n $\neq$ 0, 1) are prime, then $2^n \cdot s \cdot t$ and $2^n \cdot r$ are amicable numbers.

Thabit's Amicable Number Generator (3/4)

Proof

$$
\begin{aligned}
s(2^n \cdot st) &= \left(\frac{2^{n+1}-1}{2-1}\right)(s+1)(t+1) \\
&= (2^{n+1}-1)(3 \cdot 2^{n-1})(3 \cdot 2^n) \\
&= 9 \cdot 2^{2n-1}(2^{n+1}-1) \\
&= 2^n(9 \cdot 2^n - 9 \cdot 2^{n-1}) \\
&= 2^n(9 \cdot 2^{2n-1} + 9 \cdot 2^{2n-1} - 3 \cdot 2^{n-1} - 6 \cdot 2^{n-1}) \\
&= 2^n(9 \cdot 2^{2n-1} - 3 \cdot 2^{n-1} - 3 \cdot 2^n + 1 + 9 \cdot 2^{2n-1} - 1) \\
&= 2^n[(3 \cdot 2^n - 1)(3 \cdot 2^{n-1} - 1)(9 \cdot 2^{2n-1} - 1)] \\
&= 2^n pq + 2^n r \\
s(2^n \cdot r) &= \left(\frac{2^{n+1}-1}{2-1}\right)(r+1) \\
&= (2^{n+1}-1) \cdot 9 \cdot 2^{2n-1} \\
&= 2^n pq + 2^n r
\end{aligned}
$$

$\blacksquare$

Thabit's Amicable Number Generator (4/4)

Thabit's derivation was **longer and indirect.** But, in essence, he performs the **very same computation.**

Given that the number theoretic functions and geometric series-sum formulae were not well established in his time, he had to break down the proof into that of several lemmas.

Thabit proved about **9 to 10 lemmas** in order to arrive at this result.

Algebra

Indeterminate problems of second degree

Introduction

The word algebra is derived from the Arabic **al-jabr**, a term used by its founder, **Muḥammad ibn Mūsā al-Khwārizmī**, in the title of his book written in the ninth century, **al-Jabr wal-muqābalah** (The Science of Equations and Balancing)

The main aim of al-Khwārizmī's algebra was to provide the Muslim community with the necessary arithmetical knowledge essential in their daily calculation needs, such as in matters pertaining to **heritage and legacy, transaction, sharing and partnership, loss and profit, irrigation and land-acreage, and geometrical problems**. Al-Khwārizmī devoted about half of his al-Jabr wa 'l-muqābalah to such problems.

Indeterminate problems of second degree (1/4)

Reference : Ismail, M.R.B. (2008). Algebra in Islamic Mathematics. In: Selin, H. (eds) Encyclopaedia of the History of Science, Technology, and Medicine in Non-Western Cultures. Springer, Dordrecht.

In the third portion of his **Kitāb fi ‘l-al-jabr wa’l-muqābalah, Abū Kāmil** discussed indeterminate problems **(mu ‘ādalah siālah)** of the second degree. Some of these were of Greek origin and could be found in the **ṣinā ‘ah al-jabr** by Diophantus, **the translation of Arithmetica by Qusṭā ibn Lūqā.** The problems were then cited by Leonardo Fibonacci in his book Liber abaci.

Abū Kāmil concentrated on enumerating the possible solutions of simultaneous equations in his **ṭarāif al-ḥisāb.**

Indeterminate problems of second degree (2/4)

In these regions, the subject of such problems is mostly the purchase of a given number of birds with a given sum of money, whence the usual Arabic denomination of **masii'il al-tuyur** (problems of the birds) for this kind of problem.

Also given is the unit price for each kind of bird, and required is the number of birds of the ith kind. In our symbolism :

$$\sum_{i=1}^{n} x_i = k$$
$$\sum_{i=1}^{n} p_i x_i = l$$

Abu Kamil expresses his admiration for such problems because of the quantity of solutions they may admit.

Indeterminate problems of second degree (3/4)

Of his six examples is as follows : One must buy 100 birds with 100 dirhams, namely ducks (2 dirhams each), pigeons (1/2 d.), doves (1/3 d.), larks (1/4 d.), and hens (1 d.). The system is thus :

$$x_1 + x_2 + x_3 + x_4 + x_5 = 100$$
$$2x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 + \frac{1}{4}x_4 + x_5 = 100$$

Abu Kamil begins by eliminating x_5 from both the equations :

$$100 - (x_1 + x_2 + x_3 + x_4) = 100 - (2x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 + \frac{1}{4}x_4)$$
$$\Rightarrow x_1 = \frac{1}{2}x_2 + \frac{2}{3}x_3 + \frac{3}{4}x_4$$

Since $x_5 > 0$, we must have

$$x_1 + x_2 + x_3 + x_4 = \frac{3}{2}x_2 + \frac{5}{3}x_3 + \frac{7}{4}x_4 < 100$$

Indeterminate problems of second degree (4/4)

From the relation between x_1 and rest of the variables, he inferred that x_1 will be integral if :

1. $3 \mid x_3$
2. $2 \nmid x_2, 2 \mid x_4$ and $4 \nmid x_4$
3. $2 \mid x_2$ and $4 \mid x_4$

Abu Kamil systematically listed the **2676** (of the 2678) solutions omitting the 2 solutions corresponding to $x_3 = 54$. Although such problems are common in later Islamic mathematics, no mathematician seems to have gone so far in the **enumeration of all possible solutions.**

The analysis marks the birth of the field of **linear algebra.**

Conclusion

Conclusion

Even though the Arabs had influences from both the Greek and Indian mathematics, **it was the latter that shaped their approach to mathematics**. As instantiated before, the motivation for them to do mathematics was not rooted in the abstract notion of mathematics which the Greeks had.

Instead, it was the **practical needs of daily life and trade** like heritage and legacy, transaction, sharing and partnership, loss and profit, irrigation and land-acreage, which **guided them in their practice.**

In a nutshell, in the pursuit of knowledge, the Arab mathematicians had **followed the footsteps of the great Indian mathematicians.**

Thank You