LEC-1

Pre-Modern era of Physics - 4000 BC - 1900 AD

Celestial Mechanics $\longrightarrow$ Universal Law of Gravitation

$\Downarrow$

Observation of celestial bodies

* After the development of mechanics

$\downarrow$

Concept of a **field** (E&M)

(*) General relativity

$\Downarrow$

Differential Geometry

Early calendars $\rightarrow$ developed based on motion of heavenly bodies

Day $\Rightarrow$ Sunrise to Sunrise

But sunrise to sunrise time not always constant. Later refined as midday to midday $\Rightarrow$ More accurate

Year $\Rightarrow$ Time in which the stars regain the same position in the night sky

Sub-units of a year

Month
- lunar cycle
(New moon to New moon)

Week
- Why 7 days?

LEC-5

'Planets without discs' → Sun, Moon → Planets w/ discs

'Planets without discs' → Objects that move against the background of the stars

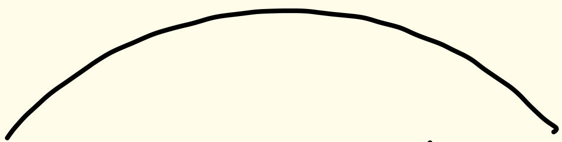
'Planets without discs' →
- Mars
- Mercury
- Jupiter
- Venus
- Saturn

Current celestial objects

→ sun & moon that move on their own spheres which also rotate wrt Earth at their own speed

→ stars fixed on celestial sphere that rotate wrt Earth

However, none of these explain retrograde motion in planets



Movement of sun & moon in night sky (against background of stars)

Movement of some planets in night sky

Retrograde motion

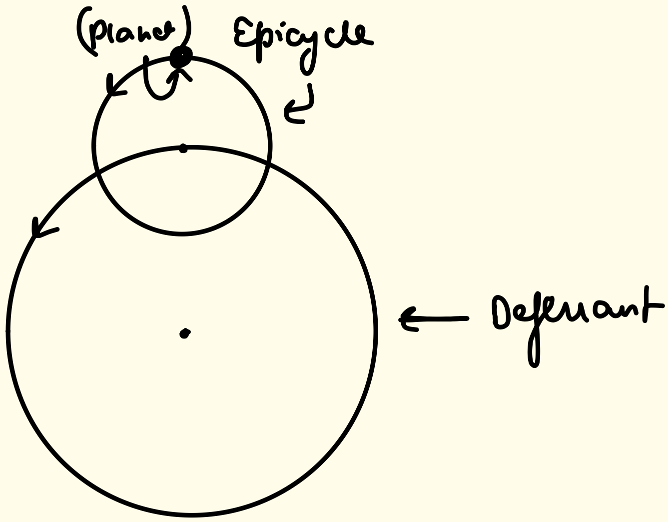
Aristotle - 'Motion in the heaven's is perfect'

How would you explain retrograde motion being confined to Aristotlian philosophy?

Appolonius & Hipparchus came up w/ an explanation

↓

Epicycles



It was believed that planetary rotation in epicycle as well as the motion of the cycle itself were constant

However, data over the next few centuries contradicted this theory → Ptolemy came up with a model

(Still Aristotlian theory)

↓

Concept of Equant

2nd century AD

There was the elephant in the room

↓

CATHOLIC CHURCH → for some reason they liked Aristotle's philosophy

And so was the story until

↓

15th century AD

Nicholas Copernicus → Heliocentric theory

Still kinda reeks of Aristotlian philosophy

↓

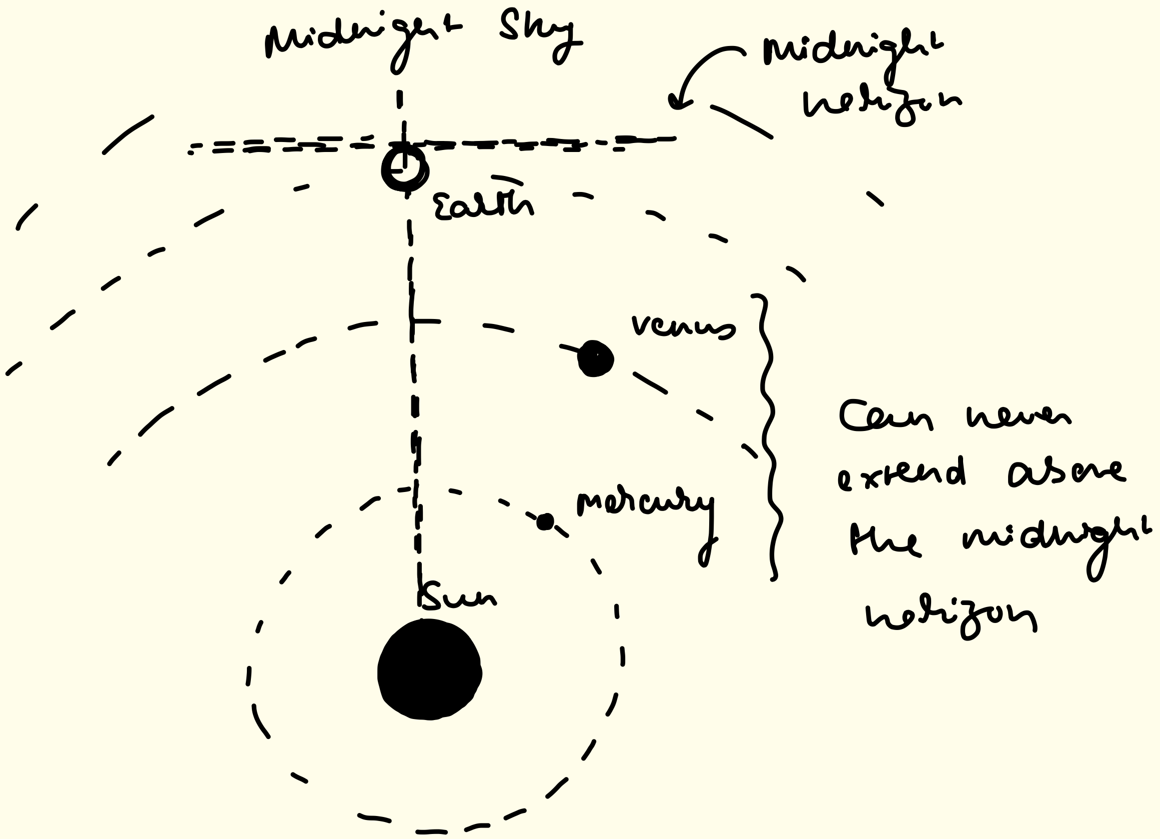
[Planets (incl. Earth excl. Moon) revolve in a circular path around the Sun w/ constant angular philosophy]

* Even observationally, we can make a distinction b/w planets

Mercury	Mars
Venus	Jupiter
	Saturn

Mercury, Venus: Cannot be seen at midnight → Not enough angular displacement from the Sun.

Mars, Jupiter, Saturn: Might be seen at midnight



Tycho Brahe → created one of the best naked eye observationaries w/ help from King of Denmark

He then pissed himself to death in a royal banquet and Johannes Kepler got hold of his observations

Kepler was the first one to bin Aristotle and any pre-conceived notions, which led him to

↓

Kepler's Laws of Planetary Motion

→ Popularised by Galileo

→ Discovery of Law of Gravitation by Newton

LEC-6

Themes of this course so far

1. Motion in Heavens is perfect

*2. Only those ideas supported by data are given the distinction of being called theories

3. There are universal laws which hold true in heaven and on Earth

Galileo (16th century)

- Italian, hence devout catholic
- Started to question Aristotle → "Heavier objects fall faster" (Allegedly)

↓

Realised through experimentation and observation that time period of a pendulum is independent of its mass

Galileo thought of as first modern scientist ↓ "Only ideas supported by data are valid theories"

Galileo also came up w/ eqns of motion by letting objects fall down inclined plane

Galileo's astronomical discoveries using a telescope
↳ Milky way
↳ 4 moons of Jupiter
↳ features of surface of the moon
→ Against Aristotle

Then dude wrote this → 'Dialogue on Two World Systems'
↳ wanted to bitchslap Aristotle once and for all.

- No equations
- Conversation b/w three people
- First pop-sci book
- Argued for heliocentrism in a manner the general public could understand

(*) The people arguing for the respective theories
→ Heliocentrism - Sagredo (Sage)
→ Geocentrism - Simplicio (Dumbass)

Church kinda didn't like that... there was a trial and he had to apologise so the higher-ups at Church didn't get their nips in a twist (and so Galileo didn't get tortured)

↓

then kept on house arrest for the rest of his life

Next dude → Huygens (17th century)

→ Analysed uniform circular motion and identified Centripetal force → (Before Newton)

→ Built on Tautochrone

↓

kind of clock

⇓

Used oscillations in a cycloidal shape to produce constant time period irrespective of amplitude

⊛ Pendulum clocks depending on simple harmonic oscillation were unreliable

→ If amplitude large, time period differs from expected time period

→ If amplitude small, difficult to practically deal with problems caused by friction

Newton

→ Childhood trauma

→ fairly brilliant student, got into Cambridge

→ College closed due to plague outbreak, Newton sent home

→ Bored Newton looked at apple falling and moon in the sky

⇓

Apple falling → apple accelerating

Moon revolving around Earth → moon accelerating

} Newton decided to work on relation b/w the two

He found

$$\frac{a_{apple}}{a_{moon}} = \frac{g}{\omega^2 R} = \left(\frac{R}{R_{Earth}}\right)^2$$

$$\rightarrow a_{apple} \propto \frac{1}{(\text{Dist. of apple from centre of Earth})^2}$$

Same case for moon

Newton's leap of imagination

$a \propto \frac{1}{d^2}$ → Universal Law of Gravitation

" There are universal laws which hold true in heaven and on Earth. "

During this time, bro also casually invented calculus

↓

Used it to find a relation b/w inverse square law and Kepler's laws

(They both, in fact, imply each other)

LEC-7

up till now, Galileo → Huygens → Newton

Revolutionised the very way people thought of Physics

Law of inertia (Newton's 1st law)

An object which does not have a force acting on it is either at rest or in a state of uniform velocity

But really he came up w/ it first → Galileo

Law of kinematics

↓

Emphasised the role of time ⇒ The concept of time as an independent variable that helps describing motion of objects → Galileo

Uniform Circular motion → Huygens

↓

Requires centripetal force

These were concepts known before Newton

Now comes the man himself → **Newton's 2nd Law**

$\bar{F} = m\bar{a}$

Describes the motion of every single moving body in the universe.

↓

Newton's 3rd Law

Every action has an equal and opposite reaction.

$$\bar{F}(\bar{r}) = m \cdot \frac{d^2\bar{r}}{dt^2}$$

$\downarrow$

"Ordinary differential equation"
in modern language.

Ever since Newton, physicists solve stuff by making differential equations $\Longrightarrow$ Newton is our Aristotle lol

Theorems regarding differential equations

→ Ordinary Diff Eqn of nth order has 'n' independent solutions

→ To obtain a unique solution, you need 'n' initial conditions.

* Independent soln → NO linear combination of the two solns will be zero for every value in their domain

"Now after praising Newton so much let us look at the cheating he did" – Prof Uma Sankar

Looking at the eqn for 2nd law,

$\bar{r}$ is only defined at a single point

$\downarrow$

CM of the body

But forces acting on surface of the body, not on CM

So what? $\bar{F}$ = Sum of forces acting on the body.

Even in law of gravity, when Newton described motion of Earth and Sun, he considered their entire mass at their CM

$\Downarrow$

Later Newton proved that for all gravitational purpose, solid sphere $\equiv$ point at CM

Moreover, another complication w/ IRL objects

$\downarrow$

Solid objects rotate $\Rightarrow$ Later solved by Euler.

Now, Newton's actual cheating

$$\bar{F}(\bar{r}) = \frac{d^2\bar{r}}{dt^2}$$

$t \rightarrow$ Time $\Rightarrow$ measured by looking at $\underline{\text{motion}}$ of celestial objects or $\underline{\text{motion}}$ of objects in an hourglass

$$\begin{tikzcd} \text{Motion} \arrow[d, bend left, "\text{depends on}"] \\ \text{Time} \arrow[u, bend left, "\text{depends on}"] \end{tikzcd}$$

Circular argument

Newton just hacked the problem to pieces $\longrightarrow$ Said there was 'Absolute time' which is independent of any motion

Now, we know $\bar{F} = m\bar{a}$ true only in "inertial frames"

$\downarrow$

what is that?

Newton simply declared that there exists 'Absolute space', and that any frame moving with constant velocity wrt that frame is hence inertial.

$$\bar{F}_{NI} = \underbrace{\bar{F}_{Pseudo}} + \bar{F}_{Real} = m\bar{a}_{NI}$$

Restoration of Newton's law for Non-inertial frames

* Newton's law of gravity

$\downarrow$

$$\bar{F}_{gr} = -\frac{GMm}{r^2}\hat{r}$$

Important discovery because of Newton's Law

$\downarrow$

Discovery of Uranus

On observation over a period of time, people realise Uranus was not moving in a Kepler orbit. So either

- Newton was wrong (GOD PLEASE NO)
- There is another massive body very close to Uranus that fucks its orbit up

$\Downarrow$

found to be Neptune

LEC 8

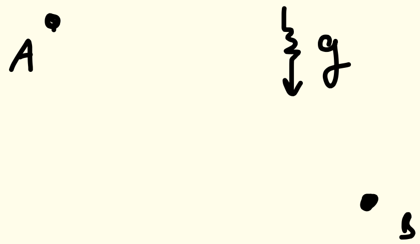
Bernoulli $\longrightarrow$ Brachistochrone problem

D'Alembert

Euler $\}\rightarrow *$

Lagrange

Hamilton

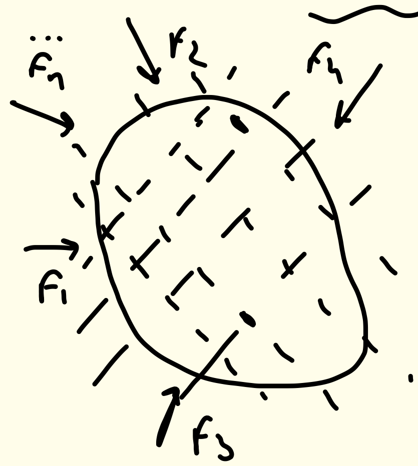


Find curve such that you could travel from A to B in the least amount of time.

Euler derived that Newton's 2nd law can be reduced to the following form

$$\downarrow$$

$$\bar{F}_{res} = M\left(\frac{d^2 \bar{r}_{cm}}{dt^2}\right)$$



N (>>1) pieces

for ith piece ≡ point mass,

$$m_i \cdot \frac{d^2 \bar{r}_i}{dt^2} = \bar{f}_i|_{ext} + \bar{f}_i|_{int}$$

$$\therefore \sum m_i \cdot \frac{d^2 \bar{r}_i}{dt^2} = \sum \bar{f}_i|_{ext} + \underbrace{\cancel{\sum \bar{f}_i|_{int}}}_{0}$$

$$\Rightarrow \frac{d^2}{dt^2}\underbrace{\left(\sum m_i \bar{r}_i\right)}_{M \cdot \vec{r}_{cm}} = \bar{F}_{res} \Rightarrow M \cdot \frac{d^2 \bar{r}_{cm}}{dt^2} = \bar{F}_{res}$$

* Euler's thm for motion of a rigid body

$\downarrow$

Arbitrary motion involves translation of CM and rotation about CM

$$m_i \cdot \frac{d^2 \bar{r}_i}{dt^2} = \bar{f}_{i|ext} + \bar{f}_{i|int}$$

$$\therefore \bar{r}_i \times m_i \cdot \frac{d^2 \bar{r}_i}{dt^2} = \bar{r}_i \times \left(\bar{f}_{i|ext} + \bar{f}_{i|int}\right)$$

$$\bar{r}_i \times \frac{d}{dt}(m_i \bar{v}_i) = \bar{r}_i \times \left(\bar{f}_{i|ext} + \bar{f}_{i|int}\right)$$

$$\bar{v}_i = \bar{\omega}_i \times \bar{r}_i = \bar{\omega} \times \bar{r}_i$$

All particles have same $\bar{\omega}$

$$\therefore \bar{r}_i \times \frac{d}{dt}\left(m_i(\bar{\omega} \times \bar{r}_i)\right) = \bar{r}_i \times \left(\bar{f}_{i|ext} + \bar{f}_{i|int}\right)$$

$$\bar{l}_i = \bar{r}_i \times \bar{p}_i$$

$$= \bar{r}_i \times (m_i \bar{v}_i)$$

$$\therefore \frac{d\bar{l}_i}{dt} = \underbrace{\frac{d\bar{r}_i}{dt} \times \cancel{(m_i \bar{v}_i)}}_{0} + \bar{r}_i \times \frac{d}{dt}(m_i \bar{v}_i)$$

$$= \bar{r}_i \times m_i \left(\frac{d^2 \bar{r}_i}{dt^2}\right)$$

$$\frac{d\bar{L}}{dt} = \frac{d}{dt} \sum_i \bar{l_i}$$

$$= \frac{d}{dt} \sum \bar{r_i} \times (m_i \cdot \bar{\omega} \times \bar{r_i})$$

$$= \frac{d}{dt} \Big[\underbrace{(\ )}_{I} \bar{\omega} \Big] = \sum_i \bar{r_i} \times (\bar{f_i}_{ext} + \bar{f_i}_{int})$$

$$= \underbrace{\sum_i \bar{r_i} \times \bar{f_i}_{ext}} + \underbrace{\sum_i \bar{r_i} \times \bar{f_i}_{int}}_{0}$$

$$\boxed{\frac{d\bar{L}}{dt} = \bar{\tau}_{ext}}$$

All internal forces appear as actn-reactn pairs net moment of such pairs about any axis is zero

$\circledast$ $\bar{p} = m\bar{v}$

$\bar{L} = I\bar{\omega}$

Are m & I similar quantities?

They appear to be so, but not really

$\bar{p}$ & $\bar{v}$ must be in same directn

$\bar{L}$ & $\bar{\omega}$ may not be in same directn

Scalar → Number

vector → Set of $\underbrace{\text{3 numbers}}$

↳ Components (which we then treat as scalars)

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = m \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

→ Tensor

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$\boxed{\begin{aligned} (3\times1) &= \text{scalar} \cdot (3\times1) \\ (3\times1) &= (3\times3) \cdot (3\times1) \end{aligned}}$$

$\bar{A} = A_1\hat{x} + A_2\hat{y} + A_3\hat{z}$

$\bar{B} = B_1\hat{x} + B_2\hat{y} + B_3\hat{z}$

$$(\vec{A}\cdot\bar{B}) = (A_1\hat{x} + A_2\hat{y} + A_3\hat{z})\cdot(B_1\hat{x} + B_2\hat{y} + B_3\hat{z})$$

$$= \underbrace{A_1B_1\hat{x}\hat{x} + A_1B_2\hat{x}\hat{y}\ \ldots}_{\text{Tensor (of order 2)}} \Rightarrow 9 \text{ terms}$$

⊛ Young's Modulus $= \dfrac{\text{Stress}}{\text{Strain}}$

Strain $= \dfrac{\overline{\text{Force}}}{\overline{\text{Area}}}$ → Impossible to take ratio of vectors

$\Rightarrow \overline{\text{Force}} = \underbrace{(\text{Strain})}_{\text{Tensor-2}}\ \overline{\text{Area}}$

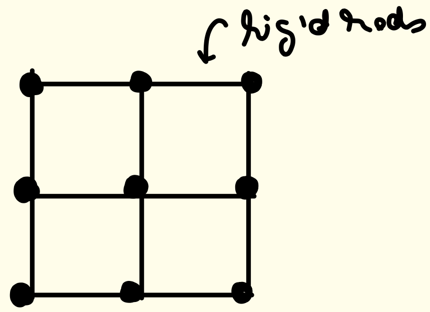
Strain $= \dfrac{\overline{\text{Change in length}}}{\overline{\text{length}}}$ $\Rightarrow$ $\overline{\text{Change in length}} = \overbrace{\text{Strain}}^{\text{Tensor-2}} \cdot \text{length}$

Similarly, $\underbrace{\text{Stress}}_{T-2} = (Y) \cdot \underbrace{\text{Strain}}_{T-2}$

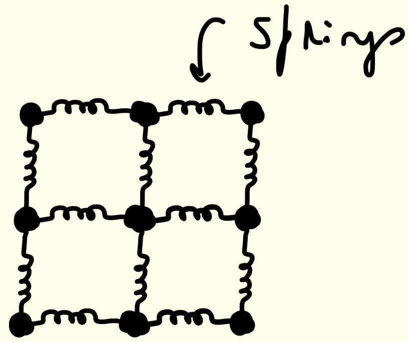
$\boxed{\text{Tensor of order 4}}$

Rigid body → No matter how it moves, the distance b/w any two mass points on a rigid body remains constant.

But all solids do deform sooner or later, so what happens when these points translate wrt each other/ the distance b/w them changes



Rigid body



Elastic body

Strain tensor → fn of position of mass point

$\Downarrow$

$g_{ij}(x)$

↓

Related to non-Euclidean Geometry

Eucl. geo → $ds^2 = [dx \; dy \; dz] \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$

Non Eucl. geo → $[dx \; dy \; dz] \cdot \underline{f(x)} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$

* Index notation

$\bar{A} = A_i \; \underbrace{(i = 1,2,3)}_{\text{implicit}}$

In Euclidean geometry, displacement is irrespective of where the point initially was

↓

In non Euclidean geometry, even the displacement is a function of the position of the point

LEC 9

$\boxed{\text{Fluid Dynamics}}$

Technical def^n of a fluid

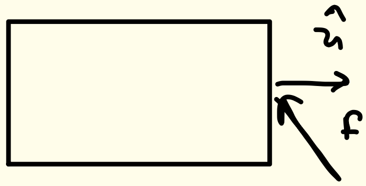
↓

Fluids can't support shear stress

$$\bar{F} = [\sigma]\,\bar{A}$$

Area having only one component

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \cdot & \cdot \\ \sigma_{21} & \cdot & \cdot \\ \sigma_{31} & \cdot & \cdot \end{bmatrix} \begin{bmatrix} A \\ 0 \\ 0 \end{bmatrix}$$



Shear stress

⇒ Component of stress due to force applied perpendicular to area

$\sigma_{11} \rightarrow f_x \Leftrightarrow A_x$

$\sigma_{21} \rightarrow f_y \Leftrightarrow A_x$

$\sigma_{31} \rightarrow f_z \Leftrightarrow A_y$

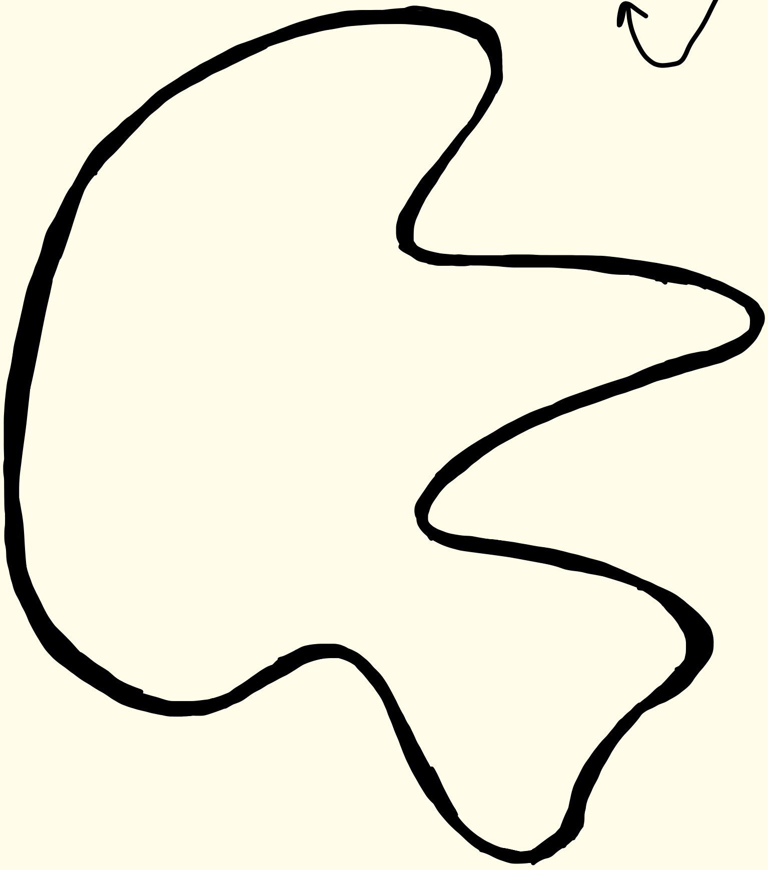
. . . and so on

Fluids

- Liquid: Do not compress under normal stress
- Gas → Compressible fluids

Formalism for fluid mechanics → Defines a macroscopic body of fluid called a continuum

No gaps in between



* Transcendental no.s

↓

No.s which cannot be obtained as the root of an algebraic eqn

No. of integers

$\equiv$

No. of fractions

$\equiv$

(No. of algebraic no.s)

Obtained as root of alg. expr

↓

Integers, fractions, surds

≥

Even irrationals

However,

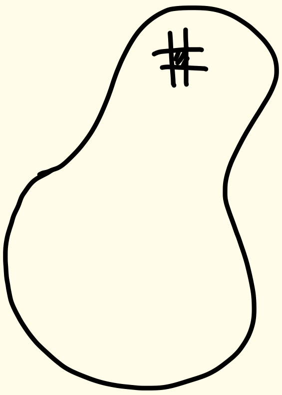
No. of transcendental no.s

>

All of these

⇓

Real no. line is filled with transcendental numbers.



Taking a volume element
$\downarrow$
$d^3 x_i$
with mass
$$dm_i = \rho(x_i)\, d^3 x_i$$

velocity pattern of fluids
$\downarrow$
$\bar{v}(x, y, z)$

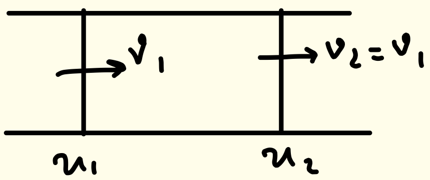
$$V = \int d^3 x$$

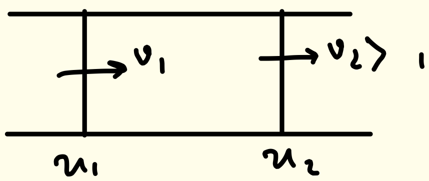
$$M = \int \rho(x)\, d^3 x$$

For simplicity's sake, let us assume there is no explicit dependence of velocity of time $\longrightarrow$ No change in flow pattern with time

$\Downarrow$

Steady State flow





Considering no leaks / loss/gain of water

Perhaps there is somehow a source of water in the pipe
$\downarrow$
Relⁿ b/w surface integral of velocity field and its source $\longrightarrow$ Any function depending on spatial coordinates

$$\underset{S}{\oint_{\substack{\text{closed}\\\text{surface}}}} \bar{V}\cdot d\bar{A} = (V_2 - V_1)A > 0 \quad \text{(So is the claim)}$$

Physics - Matter cannot be created or destroyed (c. ages ago - 1930ish)

Mass flowing out of the volume is greater than mass flowing into the volume

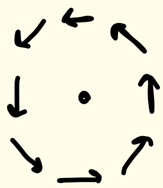
∴ our velocity flux is related somehow to strength of $\underbrace{\text{source}}$ pumping water into the canal

Physical action causing such motion

we see this in E&M

∴ Integrals of a vector field over a closed loop or surface contain crucial physical info about what produces it.

⊛ Vortex



$$\oint_C \bar{V}\cdot d\bar{l} = \text{Strength of vortex}$$

closed contour

Here, we abruptly end our conversation on fluid mechanics and jump into...

$\boxed{\text{Electricity and Magnetism}}$

$$\bar{F}_{12} = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{q_1 q_2 (\bar{r}_1 - \bar{r}_2)}{|\bar{r}_1 - \bar{r}_2|^3} \qquad \bar{F}_{21} = -\bar{F}_{12}$$

Force on q_1 due to q_2

Similarly, Ampere's law for force b/w closed loops, Newton said - figure out what the force is, then figure out the movement caused by such forces

↓

Both Culomb & Ampere wrote their eqns in terms of force → under Newton's influence

But Faraday's POV

⇓

If there is q_2 at $\bar{r}_2$, then Electric field $\bar{E}_1$ at $\bar{r}_1$ is given as follows

$$\bar{E}_1 = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{q_2}{|\bar{r}_1 - \bar{r}_2|^3}(\bar{r}_1 - \bar{r}_2)$$

If there happens to be a charge q_1 at $\bar{r}_1$, then force on $q_1 = q_1 \cdot \bar{E}_1(\bar{r}_1)$

LEC - 10

Concept of field introduced by Faraday

Usefulness → Allowed us to think of electrostatic forces as a charge somehow interacting with an 'aura' produced by a 'source' charge

thanks to fluid mech, we had vast amt of mathematical machinery to deal with these kinds of problems

Similarly, a magnetic field was also defined

$$d\bar{B}(\bar{r}_1) = \frac{\mu_0}{4\pi} \cdot \frac{I_2 \cdot d\bar{l}_2 \times (\bar{r}_1 - \bar{r}_2)}{|\bar{r}_1 - \bar{r}_2|^3}$$

$\downarrow$

'Aura' produced at $\bar{r}_1$ due to tiny length of dl_2 carrying current I_2 located at $\bar{r}_2$.

$$d\bar{F}_{12} = I_1 \cdot d\bar{l}_1 \times d\bar{B}(\bar{r}_1)$$

↳ This formulation of magnetic force on current elements doesn't satisfy Newton's third law ⇒ Small current elements do not exist in physical reality, current always flows in loops

$\downarrow$

Integrating this expression over the loop results in equal and opposite forces on both bodies

We know source ⇒ field

But also, field ⇒ source

"Given a field on a surface, how can we figure out the sources?"

Necessary mathematical armoury ⟶ Calculus of fields

Field = some physical quantity whose value is a function of its location.

For a fn in two variables $f(x,y)$,

$$\Delta f = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$= \underbrace{f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y)}_{} \qquad \frac{\partial f}{\partial y}\cdot\Delta y$$

$$+ \underbrace{f(x+\Delta x, y) - f(x,y)}_{} \longrightarrow \frac{\partial f}{\partial x}\cdot\Delta x$$

Now, for single variable fn $g(x)$,

$$\Delta g = \frac{dg}{dx}\Delta x$$

Similarly, we can argue there must be a similar relⁿ b/w Δf & $\Delta \bar{r}$

i.e. $\Delta f = \bar{\nabla} f \cdot \Delta \bar{r}$

del operator

$$\bar{\nabla} = \hat{x}\cdot\frac{\partial}{\partial x} + \hat{y}\cdot\frac{\partial}{\partial y}$$

Differential operator

Takes one f^n as input and gives another function as output

$$\bar{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y}$$

$$\Delta \bar{r} = \Delta x\,\hat{x} + \Delta y\,\hat{y}$$

$$\therefore \Delta f = \frac{\partial f}{\partial x}\cdot\Delta x + \frac{\partial f}{\partial y}\cdot\Delta y$$

How does del operator work on vector fields?

Considering a vector field $\bar{V} = V_1 \hat{x} + V_2 \hat{y} + V_3 \hat{z}$

$$\nabla \bar{V} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\left(V_1 \hat{x} + V_2 \hat{y} + V_3 \hat{z}\right)$$

$\rightarrow$ Tensor-2

$$\nabla . \bar{V} = \frac{\partial V_1}{\partial x}\hat{x} + \frac{\partial V_2}{\partial y}\hat{y} + \frac{\partial V_3}{\partial z}\hat{z} \rightarrow \underline{\text{Divergence}}$$

$$\nabla \times \bar{V} = \sim\sim\sim \rightarrow \underline{\text{Curl}}$$

Our jump $\rightarrow$ Divergence and curl are "sources" of a vector field

Helmholtz theorem - If divergence & curl of a vector field are known, then the vector field can be reconstructed <u>provided</u> the field vanishes as $\frac{1}{r^2}$ as $r \rightarrow \infty$ (or any higher exponent)

Electro $\underline{\text{Static}}$ field
$\hookrightarrow \bar{E}$ remains constant w/ time.

$$\underline{\oint_S \bar{E}.d\bar{A} = \frac{Q_{in}}{\varepsilon_0}} \rightarrow \frac{1}{\varepsilon_0}\int_V \rho(\bar{r})\, dV \rightarrow \int_V \left(\nabla . \bar{E} - \frac{\rho(\bar{r})}{\varepsilon_0}\right) . dV = 0$$

Gauss law

Gauss Thm $\rightarrow$ $\boxed{\oint_S \bar{E}.d\bar{A} = \int_V (\nabla . \bar{E})\, dV}$

True for $\underline{\text{any arbitrary vector field}}$

vanishes for arbitrarily small closed loop

$\Downarrow$

only if integrand vanishes

$$\therefore \boxed{\nabla \cdot \bar{E} = \frac{\rho(\bar{r})}{\varepsilon_0}}$$ ⟹ Divergence of electric field at a point ∝ charge density at that point

By similar argument, $\boxed{(\nabla \times \bar{B}) = \mu_0 \cdot \bar{J}(\bar{r})}$

(By Stokes' Thm)

LEC-11

Two thms of vector calculus

→ Gauss' Theorem - $\oint_S \bar{v} \cdot dA = \int_V \nabla \cdot \bar{v} \, dV$

→ Stokes' Theorem - $\oint_C \bar{v} \cdot d\bar{l} = \int_S (\nabla \times \bar{v}) \cdot dA$

True for any arbitrary vector field

$\nabla \cdot \bar{E} = \frac{\rho(\bar{r})}{\varepsilon_0}$ $\qquad \nabla \times \bar{E} = 0$ (for electrostatic field)

$\nabla \cdot \bar{B} = 0$ $\qquad \nabla \times \bar{B} = \mu_0 \bar{J}(\bar{r})$

Two more thms of vector calculus

→ $\nabla \times \nabla \phi(\bar{r}) = 0$

curl of gradient of any arbitrary scalar field is zero

→ $\nabla \cdot \left(\nabla \times \bar{v}(\bar{r})\right) = 0$

Divergence of curl of any arbitrary vector field is zero.

We know

$\nabla \times \bar{E} = 0$ → Def^n of potential

$\therefore \bar{E} = -\nabla . \phi(\bar{r})$

$$\nabla . \left(-\nabla . \phi(\bar{r}) \right) = \frac{\rho(\bar{r})}{\varepsilon_0}$$

$$\Rightarrow \boxed{\frac{\rho(\bar{r})}{\varepsilon_0} = -\nabla^2 \phi(\bar{r})}$$

Our beloved 2nd order differential eqn just like Newton

Also,

$\bar{\nabla} . \bar{B} = 0$

$\therefore \bar{B} = \bar{\nabla} \times \bar{A}$ → some vector field

$$\nabla \times (\nabla \times \bar{A}) = \mu_0 \bar{J}(\bar{r})$$

$$= \nabla(\cancel{\nabla . \bar{A}}) - \nabla^2 \bar{A}$$

$\int \partial$

["Trust me bro" – Prof]

Poisson's Equation ⟵ $\boxed{\mu_0 \bar{J}(\bar{r}) = -\nabla^2 \bar{A}(\bar{r})}$

How to solve Poisson's eqn → Analogue w/ Newton's 2nd law

$\dfrac{d^2 x}{dt^2} = \dfrac{F(x)}{m}$ → we need

$x(0)$, $\dot{x}(0)$ → Condition at initial pt

or

To get a unique eqn →

$x(0)$, $x(T)$
$\dot{x}(0)$, $\dot{x}(T)$ → Conditions at boundary pts

↓

Dirichlet generalised this concept over higher dimensions

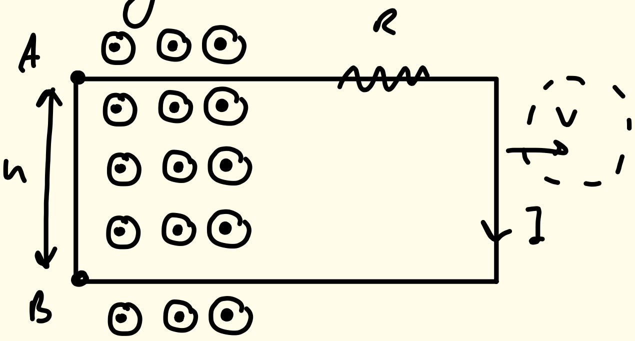
$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= -\frac{\rho(\bar{r})}{\varepsilon_0}$$

Boundary Cond^{n}s → In two dimensions, we need to specify conditions over a closed contour to get a unique soln (can extend to 3D)

LEC-12

Taking this experimental setup



when wire is moved with velocity $\bar{v}$, a current starts flowing through the wire

↓

what causes this charge flow?

Lorentz force ⟸

$$\bar{F}_{Lor} = q(\bar{v}\times\bar{B})$$

$$= q(v\hat{x}\times B\hat{z})$$

$$= -qvB\hat{y}$$

EMF (potential diff) $= \displaystyle\int_A^B \frac{\bar{F}}{q}\cdot\bar{dl} = vBh$

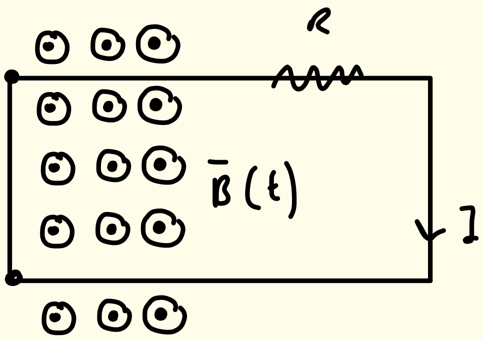
$$= \frac{dl}{dt}\cdot B\cdot h$$

$$= B\cdot\frac{dA}{dt}$$

ignoring signs

$$\boxed{\text{EMF} = -\frac{d\Phi_B}{dt}}$$ Lenz's law

However, now taking a similar but different experimental setup

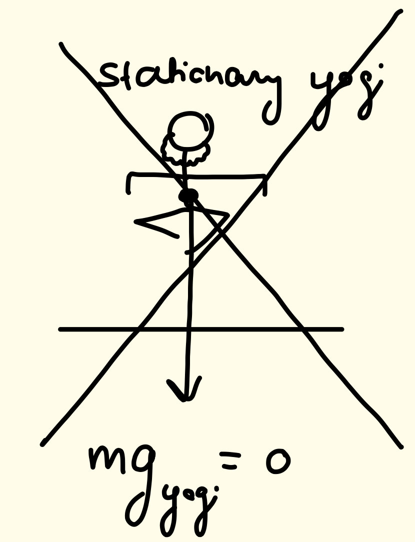


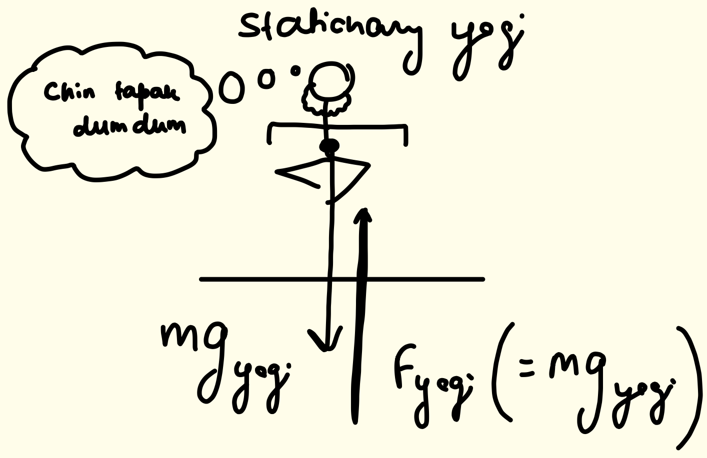
Magnetic field is taken as a function of time.

Current flows through the wire.

What is the force that causes the current to flow?

Faraday's proposal





Faraday merely concluded that the force causing charges to move inside the wire is produced <u>by the changing magnetic field</u>.

This changing magnetic field produced an <u>induced electric field</u> that applied force on charges inside the wire

Area vector

$$\oint_C \bar{E}\cdot d\bar{l} = \int_A (\nabla\times\bar{E})\cdot d\bar{a} = -\frac{d}{dt}\left[\bar{B}\cdot\bar{A}\right]$$

$$= \int_A -\frac{\partial\bar{B}}{\partial t}\cdot d\bar{a}$$

$$\therefore \int_A \left(\nabla\times\bar{E} + \frac{\partial\bar{B}}{\partial t}\right)\cdot d\bar{a} = 0 \Rightarrow \boxed{\nabla\times\bar{E}_{ind} = -\frac{\partial B}{\partial t}}$$

Slight cheating but okay

Faraday's Law

Meanwhile, $\nabla \cdot \bar{\mathcal{E}}_{ind} = 0$

Static electric field → Non-zero divergence but zero curl

Induced electric field → Non-zero curl but zero divergence

⇓

Resembles magnetic field

$$\bar{\mathcal{E}} = \bar{\mathcal{E}}_{ind} + \bar{\mathcal{E}}_{stat}$$

$$\nabla \cdot \bar{\mathcal{E}} = \nabla \cdot \bar{\mathcal{E}}_{stat} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \bar{\mathcal{E}} = \nabla \times \bar{\mathcal{E}}_{ind} = -\frac{\partial B}{\partial t}$$

Now,

$$\nabla \times \bar{B} = \mu_0 \bar{J}$$

$$\nabla \cdot (\nabla \times \bar{B}) = \mu_0 \nabla \cdot \bar{J} = 0$$

Must necessarily be true ⇒ divergence of a curl is zero

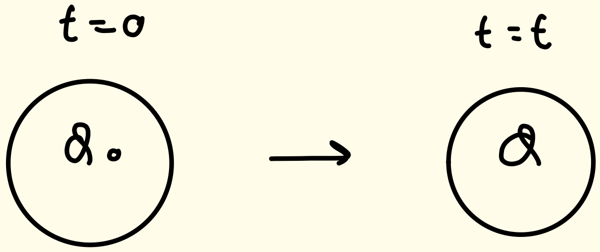
However, $\nabla \cdot \bar{J}$ is only zero when $\bar{J}$ is independent of time ⇒ Ampere's law doesn't hold for time varying $\bar{J}$.

Maxwell's modification

$$\nabla \times \bar{B} = \mu_0 (\bar{J} + \bar{J}_D)$$

"unfortunate name" Displacement current

$$\nabla \cdot \bar{J} + \nabla \cdot \bar{J}_D = 0$$



$Q \neq Q_0$

$$\therefore \frac{dQ}{dt} + \int \bar{J}\cdot d\bar{A} = 0$$

$$\frac{d}{dt}\int \rho \cdot dv + \int_V (\nabla\cdot\bar{J})\, dv = 0$$

$$\Rightarrow \int_V \Big(\underbrace{\nabla\cdot\bar{J}}_{\bar{J}_D} + \frac{d\rho}{dt}\Big) dv = 0$$

→ As it holds for arbitrarily small volumes,

$$\boxed{\nabla\cdot\bar{J}_D + \frac{d\rho}{dt} = 0}$$

$$\nabla\cdot\bar{J}_D + \frac{d}{dt}\left(\varepsilon_0 . \nabla\cdot\bar{E}\right) = 0$$

$$\Rightarrow \bar{\nabla}\cdot\left(\bar{J}_D + \varepsilon_0 \frac{d\bar{E}}{dt}\right) = 0$$

$$\therefore \bar{J}_D = \varepsilon_0 . \frac{d\bar{E}}{dt}$$

$$\therefore \boxed{\nabla\times\bar{B} = \mu_0\bar{J} + \mu_0\varepsilon_0\frac{d\bar{E}}{dt}}$$

Maxwell's manipulation

↓

We get 'same kind' of $\bar{B}$ ⇒ $\nabla\cdot\bar{B} = 0$

$\nabla\times\bar{B}$ = something

⇓

Just the value changes

Meanwhile Faraday's manipulations were far more imaginative

↳ Gave rise to a new kind of $\bar{E}$ which resembles $\bar{B}$ rather than regular static $\bar{E}$

Imagination starts new

$\downarrow$

In a space, there exist

$\bar{E}(\bar{r},t)$
$\bar{B}(\bar{r},t)$
$\left[\left.\begin{matrix}\rho=0\\ \bar{J}=0\end{matrix}\right\}\text{in immediate environment}\right]$

$\nabla\times\bar{E} = -\frac{dB}{dt}$ $\qquad$ $\nabla\times\bar{B} = \mu_0\varepsilon_0\frac{d\bar{E}}{dt}$

$\nabla\times(\nabla\times\bar{E}) = -\nabla\times\left(\frac{dB}{dt}\right)$

$= -\frac{d}{dt}(\nabla\times\bar{B})$

$= -\mu_0\varepsilon_0\frac{d^2\bar{E}}{dt^2}$

$\nabla(\cancel{\nabla\cdot\bar{E}}) - \nabla^2\bar{E}$

$\downarrow$

0
$(\rho=0)$

$$\boxed{\nabla^2\bar{E} = \mu_0\varepsilon_0\frac{d^2\bar{E}}{dt^2}}$$

Now, its one dimensional analo

$\frac{d^2f}{dx^2} = \frac{1}{v^2}\frac{d^2f}{dt^2} \rightarrow$ wave eqn in one dimension

Describes a wave w/ wave function $\bar{E}$ moving with speed $\frac{1}{\sqrt{\mu_0\varepsilon_0}} = c$

Imagination
$\downarrow$
Light is an electromagnetic wave consisting of $\bar{E}$ & $\bar{B}$ moving at speed c

Maxwell proved his leap in imagination by deriving laws of reflection and refraction from electromagnetic equations

Earlier accepted to be empirical laws

⇓

Eqns derived purely from a large number of observations

Maxwell's next leap in imagination

↓

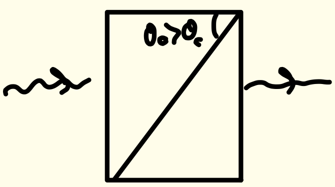
There exist other kinds of EM waves other than (visible) light having their own wavelength & frequency → Radiowaves

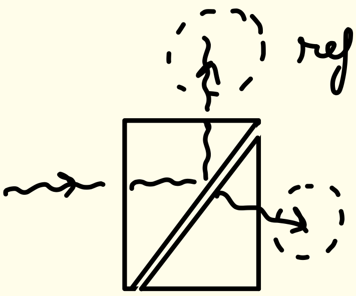
Produced by Hertz

Soon enough, microwaves were discovered

↓

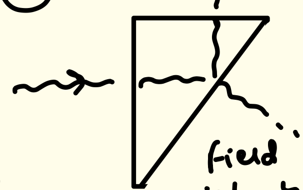
Experimented on by JC Bose → Crucial in proving Maxwell's shit





reflection

Transmission?



TIR ⇒ 100% energy reflected

field exists, but dissipates quickly → No energy propagation

wave eqn in the form of derivatives

⇓

fields must be continuous at all points

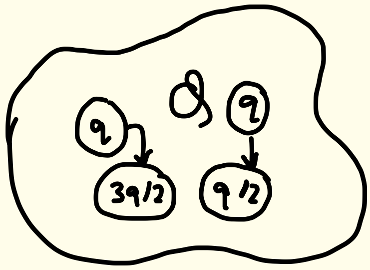
Consequence of reqn of the fields to be continuous

LEC-13

Maxwell's EM Theory

→ Ampere's law is not consistent with local charge conservation

→ formulation of displacement current



Charge conservation in large volume

However, charge conservation is also applicable in infinitesimally small volumes

↓

locality

⇓

we can trace the path of movement of a charge.

In regions of $\rho = 0$, $\bar{J} = 0$,

$\bar{E}, \bar{B}$ satisfy

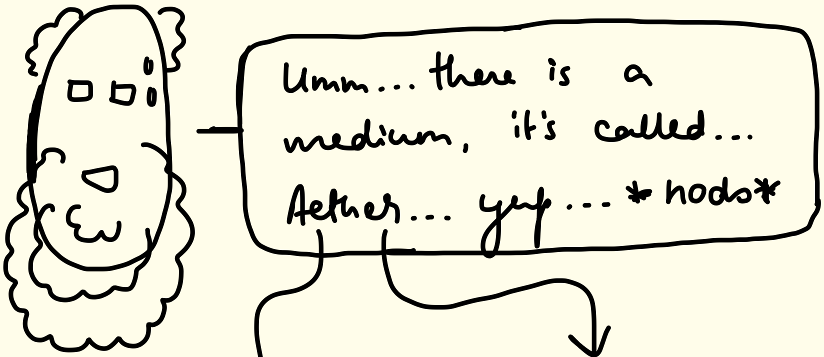
$$\underbrace{\frac{1}{c^2}\frac{d\bar{E}^2}{dt} - \nabla^2\bar{E} = 0}_{\text{wave eqn}}$$

⇒ light is an EM wave

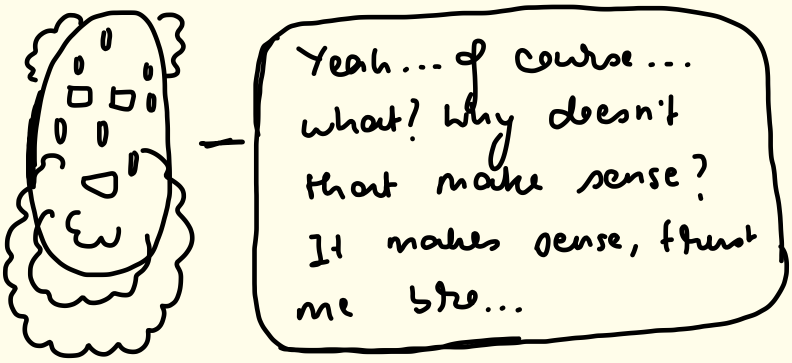
↓

Lots of people weren't comfy with this idea

Up till that point, every wave had a medium it propagated in, so what medium did light propagate through?



So tenuous that it passes through all living matter + Such strong mechanical properties that it could transmit a wave of speed of the order $10^8 ms^{-1}$?

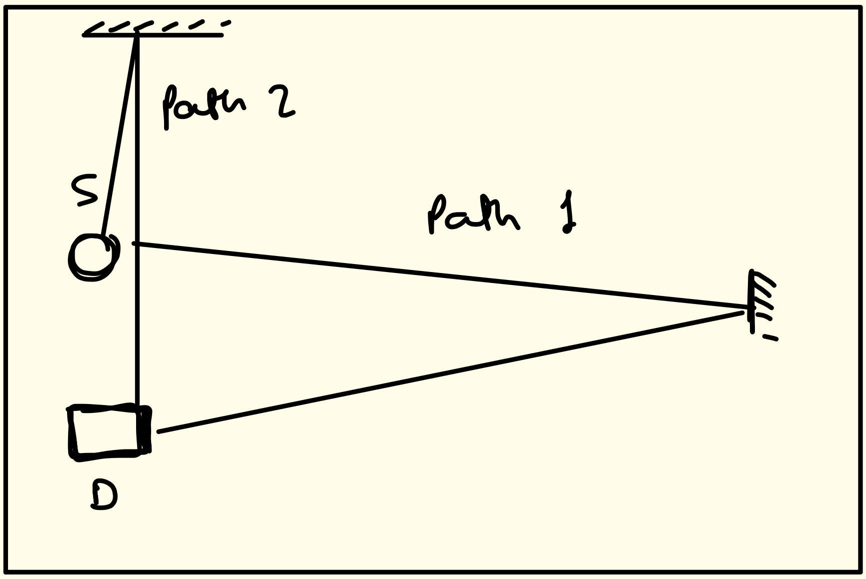


Scientists trusted the bro

Now, how to detect this Aether thing?

↓

Michelson Experiment



Path 1 $\perp$ Path 2

$$l_1 = l_2$$

$\Downarrow$

Constructive interference should occur at detector → Some people argued

↳ orient path 1 along motion of the Earth

$\Downarrow$

revolution of Earth around sun

($V_{orb} >> V_{rot}$)

Speed of travelling light in path 1 would be disturbed by movement of Earth, we wouldn't get perfectly constructive interference, there would be some fringe shift

Expected → 0.4 Error → ±0.1 Observed → **0**

Over the years, Michelson improved his experiment

Expected → 40   Error → ± 0.1   Observed → 0

So what is happening with Aether?

Shut up you're dead now

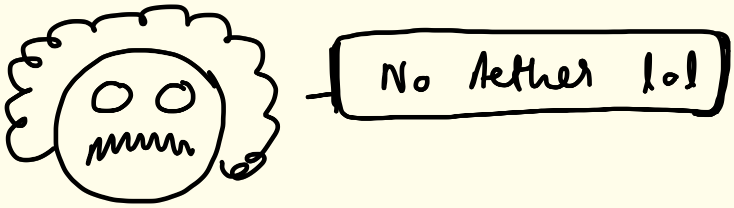
Proposed explanation - Lorentz-Fitzgerald Contraction

↓

'Lengths along direction of motion are contracted'

→ Could explain Michelson's results, but not further experiments

But then came the man, the myth, the one and only...



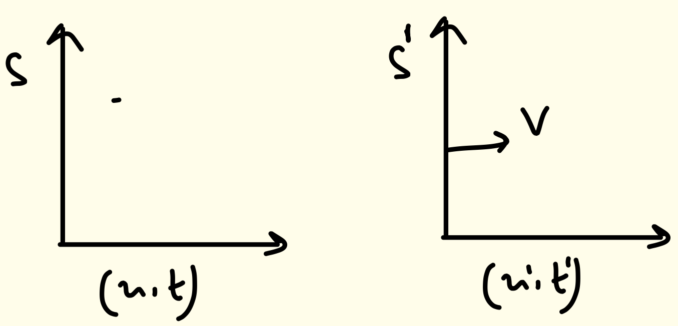
Why try to forcibly make sense of something that doesn't make sense?

Concept of inertial frame → Laws of Physics should have same form in two different frames having constant relative velocity

$\Downarrow$ Unrigorous version of what Einstein stated

Another way to state Newton's 1st Law

Galilean Relativity



→ Consider inertial fr

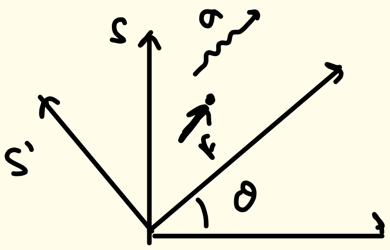
$$x' = x - vt$$

$$\bar{F} = m\bar{a}$$
$$\parallel \qquad \parallel$$
$$\bar{F}' = m\bar{a}'$$

Force wouldn't change

Acceleration is seen as the same in the two frames → $\bar{a}$ remains constant on transformation from S to S', just like $\bar{F}$

⊛ $\bar{F}$ & $\bar{a}$ are <u>invariant</u> under Galilean transformations



$x' = x\cos\theta + y\sin\theta$

$y' = x\sin\theta + y\cos\theta$

$\bar{F} = m a_x \hat{x} + m a_y \hat{y}$

$\bar{F}' = m a'_x \hat{x} + m a'_y \hat{y}$

$\bar{F} \rightarrow \bar{F}'$

$\Updownarrow$ same transformation

$a_x \leftrightarrow a'_x$

$a_y \leftrightarrow a'_y$

⊛ $\bar{F}$ & $\bar{a}$ are covariant under rotation

Now, looking at E&M eqns

$$\nabla \cdot \bar{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

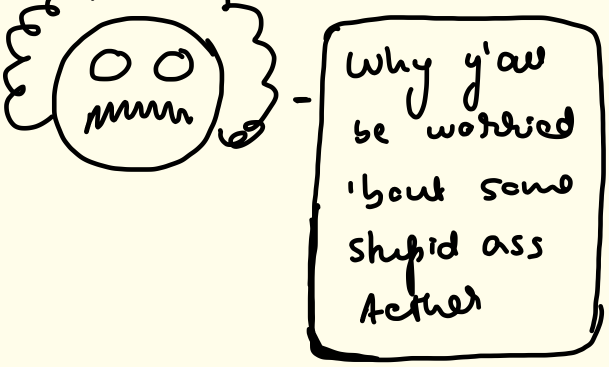
$$\nabla \cdot \bar{B} = 0 \qquad \nabla \times \bar{B} = \mu_0 \left[\bar{J} + \varepsilon_0 \frac{\partial \bar{E}}{\partial t} \right]$$

under rotations, they are covariant

However, under Galilean transformations, they are neither invariant nor covariant

$\downarrow$

Einstein pointed out this very crisis



How does Einstein solve this crisis? Stay tuned, we shall be back after a short (2 week) break

LEC-14

Theory of Relativity

Arose from conflict b/w Newton's first law and EM Theory

EM Theory $\downarrow$ Neither invariant nor covariant under Galilean transformation

Newton's first law $\downarrow$ laws of Physics should have the same form in all inertial frames of reference

Newton defined inertial frames in terms of an "absolute space" $\rightarrow$ critiqued by Mach

$\Downarrow$

* Foucault's pendulum

People solved problems by considering a frame attached to Earth as a good enough approximation for absolute space

However, Foucault argued...

Objects on surface of Earth affected by acceleration (due to Earth's rotation)

→ Centrifugal force

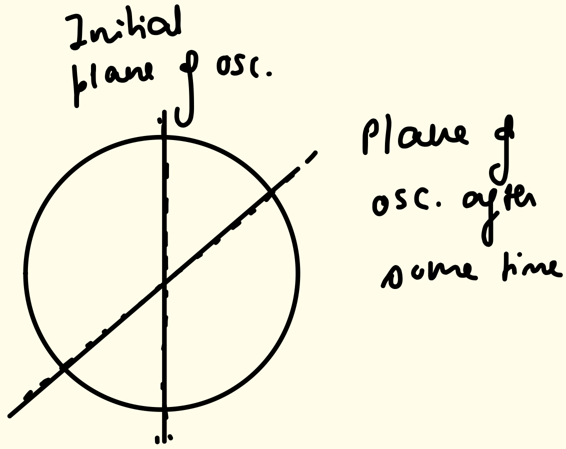
→ Coriolis force

Foucault argued we should start seeing effects of coriolis force with a large enough pendulum over a large enough time period

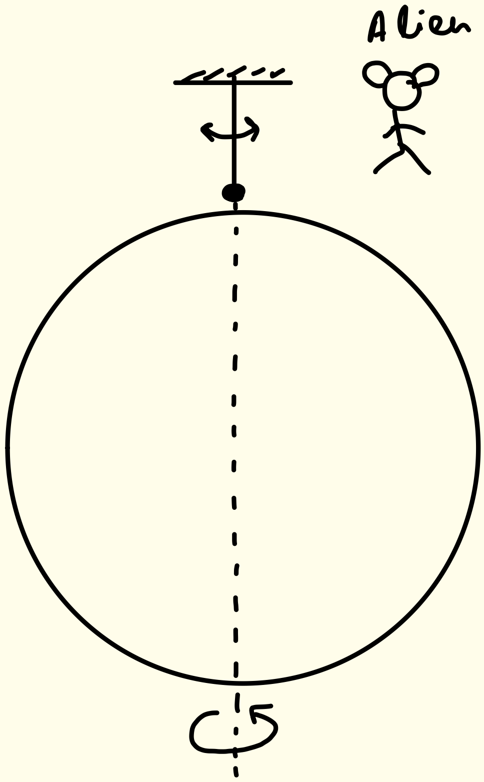
⇓

Turned out to be true

Earth is a rotating frame



Time period of revolution of plane = 24hrs × sin λ



Time period of revolution of plane of osc. = 24hrs

But an alien present in a coordinate system such that it sees the Earth's rotation observes the plane of oscillation as constant.

↳ How does the pendulum 'know' to keep its pl. of osc. constant?

Galilean transfn

$$\left.\begin{aligned} &\to x' = x - vt \\ &\to y' = y \\ &\to z' = z \\ * &\to \underline{t' = t} \end{aligned}\right\} \to$$ Can be shown that EM forces change on Galilean transformation.

<u>Woldemar Voigt</u> found a transformation such that EM eqns were invariant

$\to x' = x - vt$

$\to y' = \dfrac{y}{\gamma}$

$\boxed{\gamma = \dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}}}$

$\to z' = \dfrac{z}{\gamma}$

$\to \boxed{t' = t - \dfrac{vx}{c^2}}$ why?

Others who worked on this problem
↓
Fitzgerald
Lorentz
Poincare

Recalling the electromagnetic wave eqn,

$$\mu_0 \varepsilon_0 \frac{\partial^2 \bar{E}}{\partial t^2} = \nabla^2 . \bar{E}$$

Now, if there was an observer moving at speed c, it is logical to assume that they see a static solution → however, <u>no static soln exists for $\bar{E}$.</u>

Einstein did similar math to Voigt and came up w/ same soln → $t' \neq t$

Examining the consequences of such a revelation, Einstein published two papers on special relativity.

1st : $x' = \gamma(x - vt)$

$y' = y$

$z' = z$

$t' = \gamma\left(t - \frac{vx}{c^2}\right)$

→ Lorentz transformations ↓ name given first by Poincare

2nd : $E = mc^2$

Einstein did have two postulates, the acceptance of which allows us to derive any relativistic phenomena

→ The laws of physics must have the same form in all inertial frames (i.e. covariant)

→ Speed of light is constant in all inertial frames.

LEC-16

for $\underset{|||}{\underbrace{c \to \infty}}$, Lorentz Transf^n $\longrightarrow$ Galilean transf^n

$v <<< c$ $\rightarrow$ True for most day to day scenarios

But in Einstein's 2nd postulates, how does the speed of light suddenly pop up?

$\downarrow$

People had other points of view that tried to do away with Einstein's 2nd postulate $\longrightarrow$ Proposed replacement

$\Downarrow$

Transformation equations should form a $\underline{\underline{\text{group}}}$

Eqns linking x' to x & t' to t

$\Downarrow$

This postulate implies the existence of a universal velocity

↳ Universal velocity unmeasurably large

$\Downarrow$

Galilean transf^n

↳ universal velocity $= c$

$\Downarrow$

Lorentz transf^n

(*) However, there must always be a time delay while making a measurement

$\Rightarrow$ we get different values in the same frame

$\downarrow$

we assume there to be an infinite number of observers located at every point in spacetime, all with synchronised clocks

Confusing shit about special relativity

→ Time dilation → Pole-vaulter's paradox

→ Length contraction → Loss of Simultaneity

→ Twin Paradox

* Lorentz invariant interval

$$\underbrace{(\Delta s)^2}_{\text{symbol to denote RHS}} = \underline{c^2(\Delta t_{21})^2 - (\Delta x_{21})^2}$$

+ve if $c|\Delta t_{21}| > |\Delta x_{21}|$

↳ Not always true

Looking at Lorentz transf^n,

$x' = \gamma(x - vt)$

$t' = \gamma\left(t - \frac{vx}{c^2}\right)$

$v \rightarrow$ speed of signal b/w events 1 & 2

⇒ Any events in universe

$= \left|\frac{\Delta x_{21}}{\Delta t_{21}}\right| < c$ if $[\,] > 0$

⇒ Light pulse transmitted from one event to another

$= c$ if $[\,] = 0$

$> c$ if $[\,] < 0$

↳ Events 1 and 2 are 'disconnected'

↓

i.e. no causal connection

Causal Connection
⇓
One event occurred due to another event occuring

* If events are causally connected, $[\,]$ must be positive. Inverse not necessarily true

LEC-17

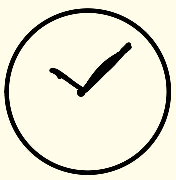
Length Contraction

L = $\underline{\text{Proper length}}$ → length of a rod wrt a frame where it appears at rest

$$\boxed{\begin{aligned} L_{\text{moving rod}} &= \frac{L}{\gamma} \\ \gamma &= \frac{1}{\sqrt{1 - v^2/c^2}} \end{aligned}}$$

Time Dilation

Time Dilation → Can be applied only in very specific circumstances



Event 1: First tick
(x_1, t_1), (x_1', t_1')

Event 2: 2nd tick
(x_1, t_2), (x_2', t_2')

$$\Delta t_{21}' = \gamma\left(\Delta t_{21} - \frac{v}{c^2}\cancelto{0}{\Delta x_{21}}\right)$$

$\Delta x_{21} = 0$ $\qquad$ $\Delta x_{21}' = x_2' - x_1'$

$\Delta t_{21} = T$ $\qquad$ $\Delta t_{21}' = T'$

$$\boxed{\therefore T' = \gamma T}$$

(*) works only when $\Delta x_{21} = 0$
i.e. the time interval b/w two events is dilated in any other frame only when the two events occur at the same spatial position in some frame

(*) whenever $[I] > 0$ b/w two events, then there exists a frame where the two events occur at the same spatial position i.e. $\Delta x_{21} = 0$. In such a frame, the period b/w the two events would be minimum

$\Rightarrow$ In any other frame,

$$\underbrace{\Delta t'_{21} = \gamma \Delta t_{21}}$$

$\gamma \rightarrow$ Always > 1

$\downarrow$

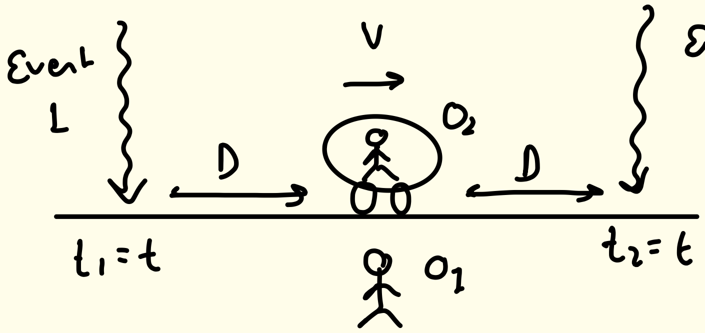
As $\Delta t'_{21}$ & Δt_{21} always have the same sign, the occurence of the two events is always in the same order in every frame of reference

Demonstration of time dilation

$\downarrow$

Penetrating component of cosmic rays

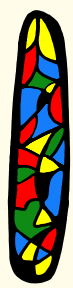
(*) It is impossible to truly say that two spatially separated events are 'simultaneous'



$\Delta x = 2D$

$\Delta t = 0$

$\therefore [I] < 0$



O_1 in stationary frame sees events occuring simultaneously

O_2 in moving frame sees Event 2 occuring first

Other observers see Event 1 first

$\rightarrow$ when $[I] < 0$, the order of events depends on the reference frame

* Pole vaulter's paradox & Twin paradox → HC Verma

LEC-18

Some further topics to be discussed

→ Non-invariance / Covariance of Newton's 2nd law on Lorentz transformations

⇓

Discussion of energy and momentum

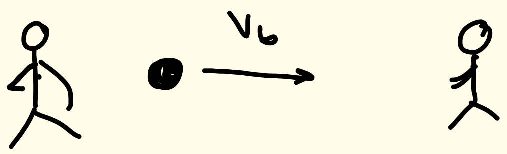
→ velocity addition

Lorentz transformations

$$\Delta x' = \gamma(\Delta x - v\Delta t)$$

$$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)$$

Imagine



wrt frame at rest,

$$v_b = \frac{\Delta x}{\Delta t}$$

wrt frame moving with speed v,

$$v_b' = \frac{\Delta x'}{\Delta t'}$$

$v_b' = \underbrace{f(v_b, v)}$ such that if $v_b = c$, $v_b' = c$

This condition gives us velocity addition function

However, looking at Lorentz transformations,
'Δx' signifies <u>direction of velocity of the object</u>

$\bar{v} = v_x\hat{x} + v_y\hat{y}$ ✗

↳ We cannot take two Lorentz transformations, one along $\hat{x}$ & one along $\hat{y}$

$\bar{v} = v_x\hat{x} + v_y\hat{y} = \sqrt{v_x^2 + v_y^2}\ \hat{v}$ → Take Lorentz transf. along $\hat{v}$

Moreover,

v_y (rather $v_\perp$ direction of Lorentz transf) $= \dfrac{\Delta y'}{\Delta t'} = \dfrac{\Delta y}{\Delta t'} \neq \dfrac{\Delta y}{\Delta t}$

All the more reason to take Lorentz transf. in direction of velocity

<u>Energy & Momentum in relativity</u>

$$\bar{p}_{\text{non rel.}} = m\bar{v} \longrightarrow \boxed{\bar{p}_{\text{rel}} = \gamma m\bar{v}} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$\boxed{E_{\text{rel}} = \gamma mc^2}$$

* Def^n of mass in rel.

Rest mass → m_0

~~'Relativistic mass' → $m = \gamma m_0$~~ We do not talk abt this

For $v \to 0$, $\bar{p}_{rel} \to 0$

$E_{rel} \to mc^2$] $\to$ A particle with mass has an intrinsic energy

(*) E_{rel} & 3 components of $\bar{p}_{rel}$ form a four vector $\to$ has a Lorentz invariance interval

$$E^2 - p^2c^2 = \gamma^2 m^2 c^4 - \gamma^2 m^2 v^2 c^2$$

$$= \frac{c^2}{c^2 - v^2} \cdot m^2 c^2 (c^2 - v^2)$$

$$= m^2 c^4$$

$$\therefore \quad E = \sqrt{m^2c^4 + p^2c^2}$$

$$= mc^2 \left(1 + \frac{p^2}{(mc)^2}\right)^{1/2}$$

$$= mc^2 \left(1 + \frac{p^2}{2(mc)^2}\right) \quad \left[\text{in lim of } p \ll mc\right]$$

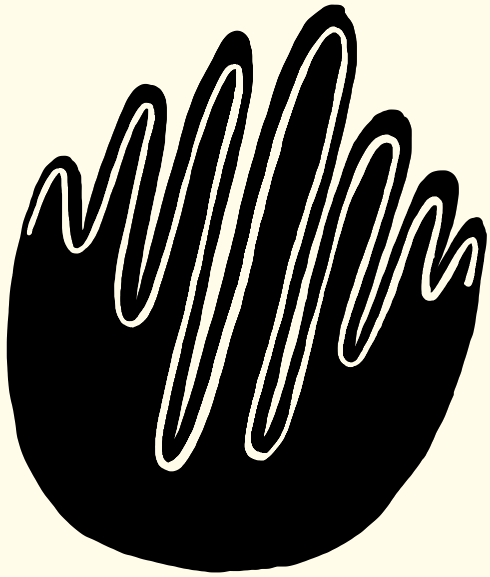
$$E = \underbrace{mc^2 + \frac{p^2}{2m}}_{\text{Energy content of a body in non relativistic limit}}$$

$\frac{p^2}{2m} \to KE$

Newton's 2nd law $\to$ $\bar{F} = \frac{d\bar{p}}{dt}$

(*) $$\frac{|\bar{p}|c}{E} = \frac{|\bar{v}|}{c}$$

General Relativity - A glimpse



Not much, just some non-Euclidean geometry type shit.

LEC-19

Ancient mathematics — natural no.

operations — -ve no.s bad — solution of quadratic eqn — we don't talk about roots of -ves

Skip to ~1500 years later ⇒ Now the drama starts

Scipione del Ferro → figured out how to solve a reduced cubic

⇓

$x^3 + px + q = 0$

$p, q \in \mathbb{Z}$

↓

Didn't publicise his solution

⇓

Monetary reasons

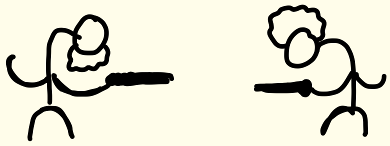
→ Only told one guy - his son-in-law Antonio Fiore

↓

Made a bunch of money winning duels over this

⇓

Eventually beat by a guy Nicolo Tartaglia → figured it out over the course of the duel



Mathematical duels

Being able to solve a problem no one could solve was a bazooka up your sleeve

Eventually Tartaglia told this solution to Girolamo Cardano on the condition he keep it secret

↓

Cardano used the result & further generalised the problem to solve → $x^3+ax^2+bx+c=0$

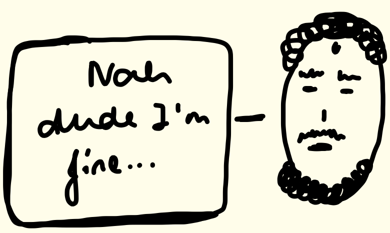
However, Cardano figured some important mathematical results in the meanwhile which he wanted to publish, but was unable to do so as it used Tartaglia's result.

Then he realised → If Fiore posed the problem he must knew the solution

↓

Talked to him, and figured it was okay to publish Ferro's result instead of Tartaglia's result

Tartaglia was pissed... ←



okay now drama over back to math

Cardano's advancement → Proved that equations involving square roots of -ve no.s were solvable

↓

His algorithm for solving cubic equations

1. Integers ⇒ Start

⋮

m. $\sqrt{-()}$ ⇒ Some intermediate

⋮

n. Rational ⇒ end

Earlier people would just give up whenever they came across the square root of a -ve no.

↓

Cardano proved these no.s could be worked with

Eventually quartics were solved

~1800 it was proved by Abel that it was impossible to find the solution to a general quintic eqn

Galois also proved the same for quintic using diff argument that could also be extended for arbitrarily higher order equations

↓

Introduced the concept of a Group

Ended up dead at 23 because he fucked around and found out
(+ government conspiracy)
(+ duels [non-mathematical])

Group

→ A set

→ May have finite / infinite no. of elements

→ Must satisfy a set of properties

- $a, b \in$ group $G \Rightarrow \underbrace{a \times b}_{\text{"Group multiplication"}} \in G$

"Group multiplication"

(but not really multiplication, it could be any operation where we get a third element from combination of two elements)

- $\exists\, \underbrace{e}_{\text{identity}} \in G$ s.t. $a \times e = e \times a \;\; \forall a \in G$

- $\exists\, \underbrace{a^{-1}}_{\text{inverse}} \in G$ s.t. $a \times a^{-1} = a^{-1} \times a = e \;\; \forall a \in G$

- Associativity → $a \times (b \times c) = (a \times b) \times c$

* $a \times b$ need not be $b \times a$
If $a \times b = b \times a \;\; \forall a, b \in G$,
then $G \rightarrow$ Abelian Group

$C_1 \rightarrow$ group w/ 1 element

$C_2 \rightarrow$

$C_3 \rightarrow$

A group w/ zero elements is not possible. But groups w/ 1, 2, 3 elements possible –

$C_1 \;\; (1) \rightarrow x - 1 = 0$

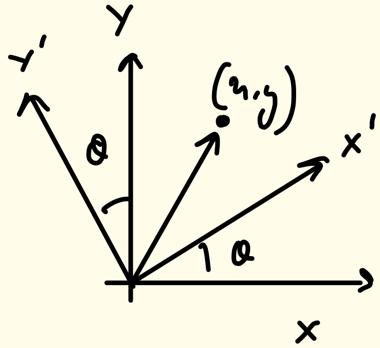
$C_2 \;\; (1, -1) \rightarrow x^2 - 1 = 0$

$C_3 \;\; (1, \omega, \omega^2) \rightarrow x^3 - 1 = 0$

cyclic groups of order n

LEC-20

Continuous parameter groups
$\downarrow$
Set of $n \times n$ matrices with $\det \neq 0$ $\rightarrow$ called $GL(n, \mathbb{R})$
(General linear matrix of n dimensions on the set of real nos)



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \underbrace{R(\theta)}_{\text{Defines a rotational matrix to transform a vector from one coordinate system to another.}} \begin{bmatrix} x \\ y \end{bmatrix}$$

Similarly,

$$[x' \ y'] = [x, y] \, R^t(\theta)$$

$$\underbrace{[x' \ y'] \begin{bmatrix} x' \\ y' \end{bmatrix}}_{\text{'length' of vector}} = [x, y] \, R^t(\theta) \, R(\theta) \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= [x, y] \begin{bmatrix} x \\ y \end{bmatrix} \hookrightarrow R^t(\theta) . R(\theta) = I$$
$$= R(\theta) R^t(\theta)$$

Consider set matrices R s.t.

$\underbrace{R^t R = R R^t = I}$ $\rightarrow$ set of R matrices forms a group
$\hookrightarrow$ orthogonality condition

* $\det(R) = +1$ or -1

↳ R s.t. $\det(R) = +1 \Rightarrow$ is also a group → group of rotations

↳ R s.t. $\det(R) = -1 \Rightarrow$ NOT A GROUP

↓

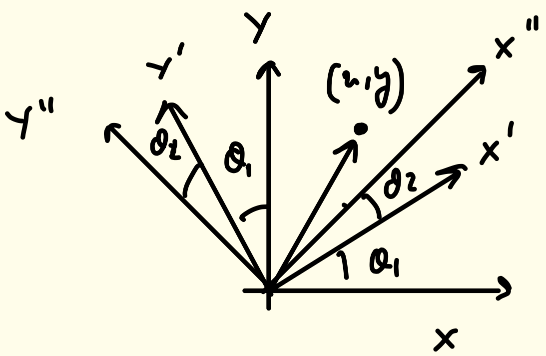
group of matrices having det $= -1$ cannot have identity

∴ Rotation matrices are those which satisfy orthogonality equation and have determinant $= 1$

⇓

$$\boxed{\begin{array}{c}\text{i.e.}\\ RR^t = R^tR = I\\ \text{AND}\\ \det(R) = 1\end{array}} \rightarrow \text{Rotational matrix}$$

Applying a second rotation to the given axis



$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \underbrace{R(\theta_2)R(\theta_1)}_{R(\theta_1+\theta_2)} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$R(\theta_1+\theta_2) = R(\theta_1)R(\theta_2)$$

↓

Rotations in 2D commute

* Grp → $\underbrace{SO(2)}$

Special orthogonal of 2 dimensions

$\downarrow$ Special: det $= 1$

$\downarrow$ orthogonal: $RR^t = R^tR = I$

* $R = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow$ 4 nos

However, rotational matrices need only one parameter i.e. the angle of rotation

∴ How does one reduce 4 points of variation to 1

$\Downarrow$

Apply three constraints

$$RR^t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For arbitrary vector product, we would need 4 constraints to solve this eqn

However, we must remember that RR^t is a symmetric → 3 independent constraints

Counting the constraints,
we have 2 diagonal constraints
and 1 off-diagonal constraint

Remember, so far, purely mathematical arguments, no physics has actually started

Now let's move a dimension up

Taking set of 3×3 matrices satisfying constraints of rotational matrices

$RR^t = I \rightarrow R = 3\times3$ matrix

$$= \begin{bmatrix} \boxed{1} & 0 & 0 \\ \boxed{0} & \boxed{1} & 0 \\ \boxed{0} & \boxed{0} & \boxed{1} \end{bmatrix}$$

Using argument of symmetricity, we now have <u>6 independent constraints</u>

$$9 \text{ numbers} \xrightarrow{6 \text{ ind. const.}} \text{3 points of variation}$$

x, y, z

Rotation in different planes

↓

$xy, yz, zx \rightarrow \underline{3C_2}$ points of variation

Similarly, in 4 dimensions

↓

$^4C_2 = 6$ points of variation

∴ In three dimensions, we have 3 independent rotations → $\theta_x, \theta_y, \theta_z$ (Rotation abt each axis)

$$R(\theta_x, \theta_y, \theta_z) = R(\theta_x, 0, 0)\, R(0, \theta_y, 0)\, R(0, 0, \theta_z)$$

$\downarrow$

$SO(3)$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{bmatrix} \begin{bmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(*) Rotations in 3d DO NOT COMMUTE $\rightarrow$ Non-Abelian Group

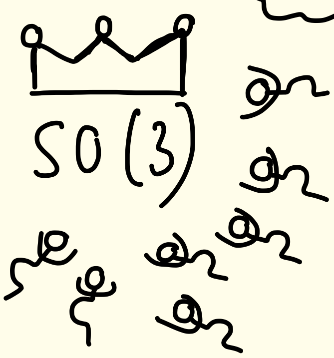
$\downarrow$

As well as higher dimensions

$$R(\theta_x, 0, 0)\, R(0, \theta_y, 0) - R(0, \theta_y, 0)\, R(\theta_x, 0, 0) \neq 0$$

The value of this difference follows a certain structure that rules the entire landscape of physics

All hail SO(3)



Worship SO(3) with your heart and soul

SO(3) is the truth of life and the universe

Probability

Gaussian distribution stuff

LEC-21

Quantum Quantum Quantum

One mole of ideal gas at temperature T
↓
One mole of He at room temp. (300K) ⟹ $U = \frac{3}{2}RT$
$\sim 4000\,J$

Taken to a height of 100m

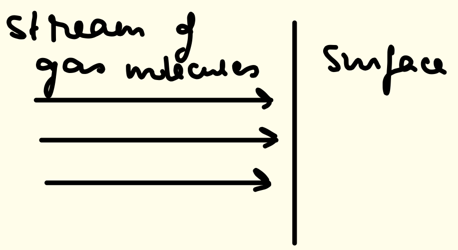
Mass of 1 mol He = 4g

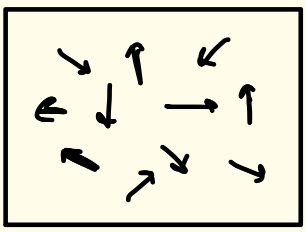
Potential energy = mgh
= 4J

Total energy = 4004J, out of which we can utilise only 4J as work

Internal energy → due to KE of gas molecules

We can't get any energy out of it as motion of gas molecules is <u>completely random</u>





Net momentum of system = 0
Large no. of randomly moving particles → their cummulative impact cancels out

the system as a whole does work on the surface ← All gas molecules have a certain momentum, which changes uniformly on collision with a surface

Entropy

$\Delta Q = T\Delta S$

Helmholtz Free Energy

$H = U - TS$

$= 0$ for a stationary gaseous system

(*) Entropy is crucial in our conception of temperature

$\downarrow$

we need internal energy as well as entropy

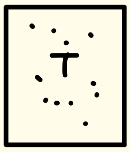
$\downarrow$

we can only define temperature of a system when there is thermodynamic equilibrium

$\Downarrow$

Maximum possible value of entropy $\rightarrow H = 0$

(*)



$\frac{1}{2} m v^2 \times A = \frac{3}{2} kT$

$\Rightarrow \langle v^2 \rangle = \frac{3kT}{m}$

$\approx$

for the system

$v \in (0, \infty)$

Probability distribution function

$$\boxed{P(v)\,dv = (\ )\, e^{-\frac{\frac{1}{2}mv^2}{kT}}\, dv} \rightarrow \text{Probability of a particle to have speed in the range } v \text{ to } v+dv$$

Boltzmann factor $\equiv e^{-\frac{E}{kT}}$ $\rightarrow$ Generalised to energy in any form

Blackbody Radiation

↓

Perfectly absorbs & emits light of all frequencies

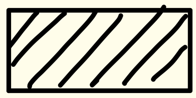
$$\boxed{\begin{array}{l} \text{* Stefan's Law} \\ \text{Power emitted} = \sigma T^4 \end{array}}$$

↓

Σ Energy emitted over all frequencies

Emmissivity $= A f(\nu, T)$

* Wien's Law



Black object immersed in an environment of radiation, in thermal eqbm w/ radiation at temp. T

$$\boxed{f(\nu, T) \sim (\ )\,\nu^3 e^{-\beta\nu}}$$

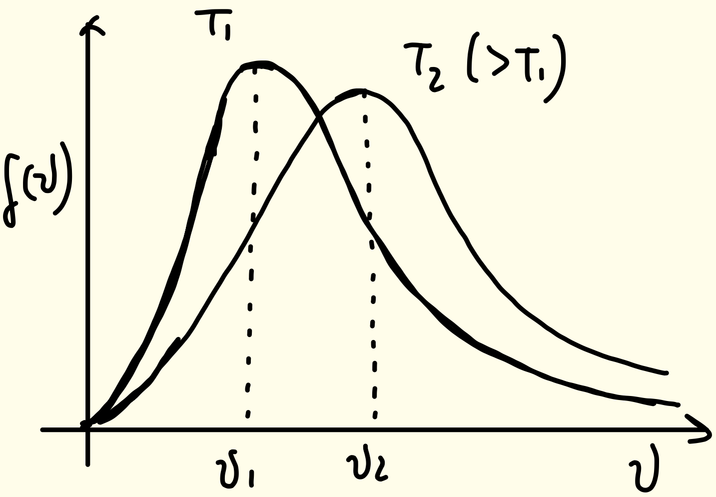
$e^{-\frac{E}{kT}} \longleftrightarrow$ It appears that Wien maybe tried to imply a relation between energy & freq.

↓

Went against ideas of classical electromagnetism

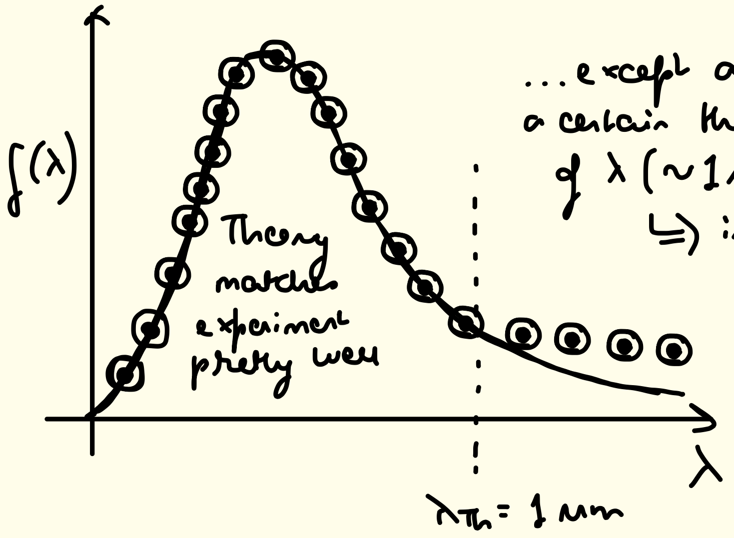
$\Downarrow$

Energy related to amplitude of electromagnetic wave



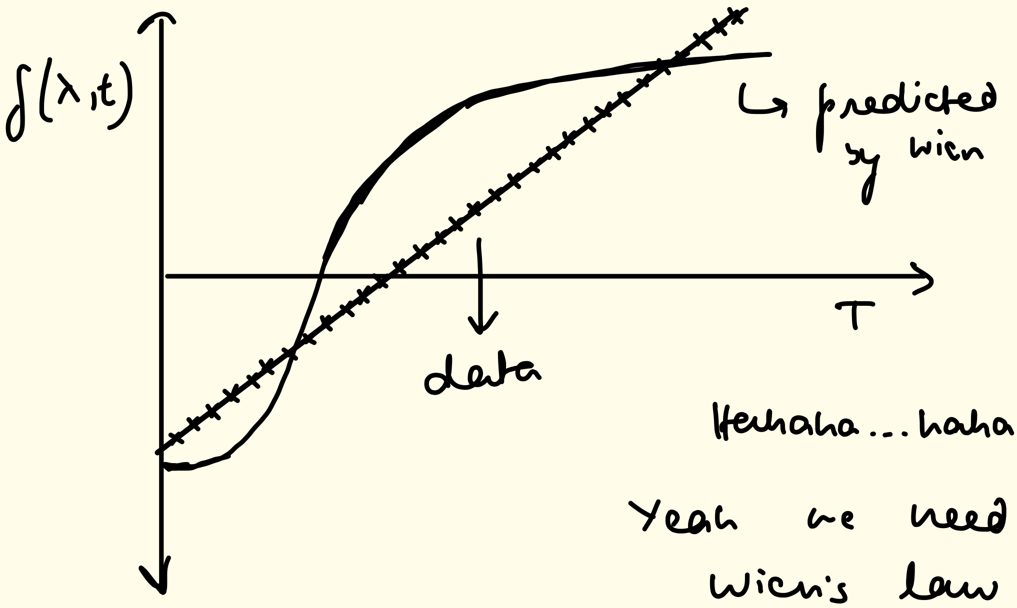
Wien's displacement law $\longrightarrow$ $\boxed{\lambda_{max} T = \text{constant}}$

Comparing theory with experiment,



...except after a certain threshold of λ (~ 1 nm)
↳ infrared range

Moreover, at far infrared frequencies, fixing λ and merely looking at variation w/ temperature



Hehaha... haha... ha...

Yeah we need to modify Wien's law

<u>Max Planck</u> came up with a solution

$$\boxed{\rho(\nu,t) = (\;)\,\nu^3 \frac{1}{e^{\frac{h\nu}{kT}} - 1} \quad \vdots \quad \begin{array}{l} \nu\uparrow \Rightarrow \text{Wien's original law} \\ \nu\downarrow \Rightarrow \text{linear eqn} \end{array}}$$

How did Planck justify this change without disturbing laws of statistics

↓

Replace $\int$ with Σ

[Declared that energy exists in quantised bundles of min. unit → $h\nu$]

Einstein went further in his explanation of the photoelectric effect

↓

[Energy only exists in 'packets' of $h\nu$, unit size is never $2h\nu$, $3h\nu$ or so on.] → In order to maximise entropy

Energy packets = photons

LEC-22

Why do we need Quantum Theory?

(a) Explain blackbody radiation

(b) Atoms radiate at discrete frequencies

(c) Photoelectric effect

Mathematical backbone arose from (a) & (c), but (b) is perhaps the most significant.

↓

Hydrogen spectra → Dark lines in absorption spectrum
Bright lines in emm. sp.

Rydberg formula → $\boxed{E_{nm} = (-13.6\,eV)\left(\frac{1}{n^2} - \frac{1}{m^2}\right) \quad m > n}$

↓

As energy levels are quantised, so is frequency

Niels Bohr attempted to explain the absorption & emmission spectra

↓

first stated that classical rules merely don't apply at atomic scales

* Bohr's postulates

1. Angular momentum of electrons is quantised

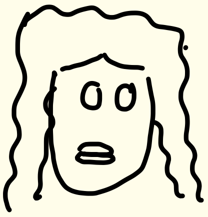
$$\boxed{L = n\hbar}$$

2. Only orbits satisfying above rule are allowed

Considering electrostatic force to provide centripetal force, the eqn for Energy we get → $E_n = \frac{-k}{n^2}$, $k = 13.6\,eV$

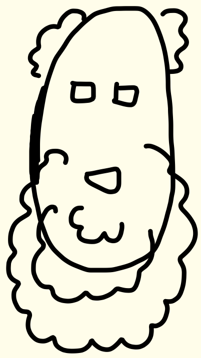
Matched eqns known at the time

Bohr ended up being totally wrong, but he introduced the idea of quantisation into atomic models



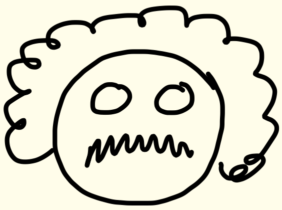
Light doth be corpuscular

Watch me double slit experiment all over your theory



→ Yup, plus look at Electromagnetism woooooow

Light do be pretty wavey



He's a quantum
She's a quantum
Light's a quantum
Your mom's a quantum

Einstein said → Light is a particle in that it appear as a bundle of energy

→ light is a wave in that it has frequency & wavelength

Now a young chap called de Broglie looked at electrons and thought "Hmmm..."

de Broglie said

→ ~~light~~ Electron is a particle in that it appears as a bunch of energy ↱ ≡ mass

→ ~~light~~ Electron is a wave in that it has frequency ? Hmmm...

↓

found a relation for energy of an electron in terms of its momentum, which could be related to its ν & λ

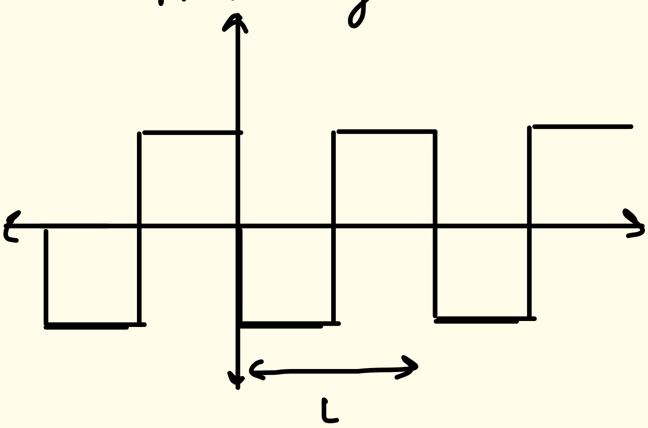
←

Eventually other wave properties of electrons were discovered

A mathematical solution to the problem was already available ⇒ Fourier Analysis

1. Fourier Series

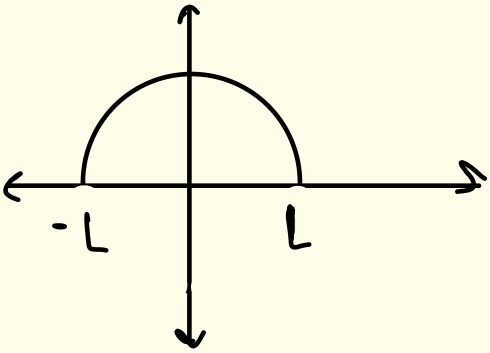
Infinite series over an infinite range



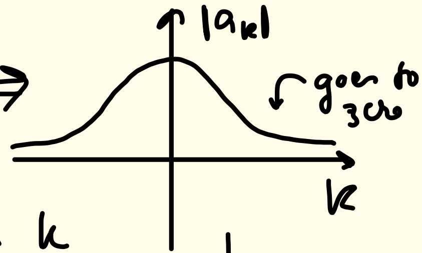
$$= \underbrace{\sum_{n=0}^{\infty}\left[A_n \cos\left(\frac{2\pi n x}{L}\right) + B_n \sin\left(\frac{2\pi n x}{L}\right)\right]}_{\text{Adding discrete harmonics}}$$

$$f(x+L) = f(x)$$

2. Fourier Integral



$$= \int_{-\infty}^{\infty} dk\, \underbrace{a_k}_{\text{function of } k}\, \overbrace{e^{ikx}}^{\cos kx \,+\, i \sin kx}$$



$\int$ → Continuous elements added

↓

Most important thm of Fourier analysis

⟸

Taking $f(x)$ → has nonzero values in a limited range

Corresponding to it we have a function a_k
↓
nonzero values in limited range → falls to zero faster

Say $f(x) : (-L, L)$ → Range of f in x X

$a_k : (-k_0, k_0)$ → Range of a in k

$\parallel$

Number of order 1 ⟹ Single digit number

Representing an electron by a wave function

$\downarrow$ Superposition of infinite no. of waves with wave numbers in range (k_{min}, k_{max})

$\Downarrow$ Uncertainty $k_{max} - k_{min} = \Delta k$

wave function $\Downarrow$ Uncertainty Δx

$\boxed{\Delta x . \Delta k \to \text{Order } 1}$

LEC-23

Major developments in quantum theory

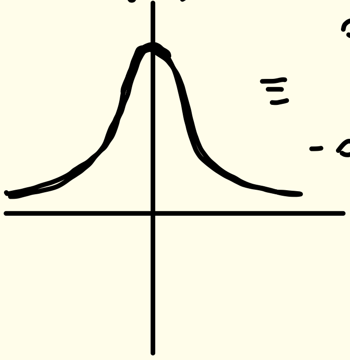
1. Energy is quantised (Planck & Einstein)
   - $\downarrow$ Radiation
2. Angular momentum is quantised (Bohr)

   $\Downarrow$ Energy of electrons is quantised

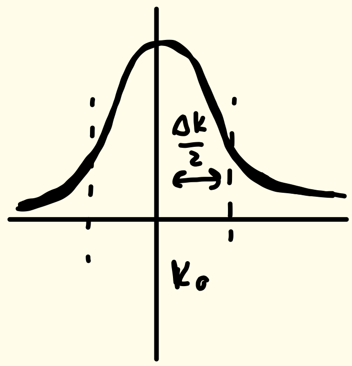
* Stable Bohr orbits $\Rightarrow$ fit an integer no. of de Broglie wavelengths

de Broglie's insight $\Rightarrow$ you need a wave eqn to represent a 'point' particle

Wave fn $\rightarrow f(\bar{r}) \equiv f(x)$ in one dimension



$$\equiv \int_{-\infty}^{\infty} a(k)\, e^{ikx}\, dk$$



Restricting our choice of wave no.s (k)

$$\downarrow$$

$$k \in \left(k_0 - \frac{\Delta k}{2},\ k_0 + \frac{\Delta k}{2}\right)$$

Schrödinger

→ Thorough knowledge of differential equations

→ de Broglie ⇒ particles show wave properties

Schrödinger ⇒ Every wave should have a wave eqn → Always partial differential equations

$\Psi(\bar{r}, t) \rightarrow \Psi(x, t)$
in one D

Considering a free particle

$E = \dfrac{p^2}{2m}$ (classical mechanics)

(de Broglie → $p = \frac{h}{\lambda} = \hbar k$)

$$\underset{\text{(Planck)}}{\hbar\omega} = \frac{\hbar^2 k^2}{2m}$$

$\therefore$ The wave eqn of this particle –

$$\psi(x,t) = (\ )\, e^{\underbrace{i(kx-\omega t)}}$$

$$\hbar\omega = \frac{\hbar^2 k^2}{2m}$$

$$\Downarrow$$

$$\underbrace{\hbar\omega\,\psi(x,t)}_{i\frac{\partial\psi}{\partial t}} = \overbrace{\frac{\hbar^2 k^2}{2m}}^{-\frac{\partial^2\psi}{\partial x^2}} \cdot \psi(x,t)$$

$$\Rightarrow \boxed{i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\cdot\frac{\partial^2\psi(x,t)}{\partial x^2}} \longrightarrow \boxed{i\hbar\frac{\partial\psi}{\partial t} = H\psi}$$

Schrödinger's wave eqn.

How we usually write it

H → Hamiltonian

Special case when potential only depends on spatial coords, not on velocity or time

↓

$$H = PE + KE$$

$$= \frac{|\bar{p}|^2}{2m} + V(\bar{r})$$

$$\boxed{p \longleftrightarrow -i\hbar\frac{\partial}{\partial x}}$$

$\Downarrow$

Momentum identified by differential operator

In general coordinates,

$$\bar{p} \longleftrightarrow -i\hbar\nabla$$

$$\bar{p}\cdot\bar{p} = p^2 \longleftrightarrow -\hbar^2\nabla^2$$

General Schrödinger eqn

$\Downarrow$

$$\boxed{i\hbar \frac{\partial \Psi}{\partial t} = \left[\frac{-\hbar^2}{2m} \cdot \nabla^2 + V(\vec{r}) \right] \Psi}$$

Time dependent Schrodinger Equation

(*) Suppose $V(r) = \frac{1}{2} m \omega^2 r^2$

$= \frac{1}{2} k r^2$

$$\left. \begin{aligned} E_n &= n \cancel{h} \nu \\ &= n \cancel{\hbar} \omega \end{aligned} \right\} \text{Planck}$$

$$E_n = \left(n + \frac{3}{2}\right) \hbar \omega$$

related to angular momentum quantisation

$\frac{3}{2} \hbar \omega$ = zero point energy

Schrödinger solved the problem for

$$V(\vec{r}) = -\frac{e^2}{r}$$

$\Downarrow$

Got same value for Bohr's energy, but didn't get angular momentum quantisation

$\downarrow$

Got something more intricate.

Schrodinger said that angular momentum quantisation not only dependent on n (principal quantum no.), but on two more numbers

$\downarrow$

l, m

Time independent Schrodinger eqn.

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = \frac{-\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V(x) \cdot \Psi$$

(*) Uniqueness thm

$\downarrow$

For a diff eqn with certain initial conditions, there is a unique soln to the eqn.

Assuming $\psi(x,t) = T(t).\psi(x)$

$$i\hbar\, \psi \frac{dT}{dt} = \left(\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x).\psi\right) T(t)$$

$$\underbrace{i\hbar\, \frac{1}{T}.\frac{dT}{dt}}_{\text{only time related terms}} = \underbrace{\frac{1}{\psi}\left(\frac{-\hbar^2}{2m}.\frac{d^2\psi}{dx^2} + V(x)\psi\right)}_{\text{Only } x \text{ related terms}}$$

$\Downarrow$

$G(t) = F(x)$ $\longrightarrow$ $G(t) = F(x) =$ some const. E

for all x & all t

$\Downarrow$

only possible solution

$$\therefore \frac{dT}{dt} = \frac{-i}{\hbar}.TE \Rightarrow \boxed{T(t) = T(0)\, e^{\frac{-iEt}{\hbar}}}$$

(similar to radioactive decay law)

Moreover,

$$\boxed{\left[\frac{-\hbar^2}{2m}.\frac{d^2}{dx^2} + V(x)\right]\psi(x) = E\psi(x)}$$

$H\psi = E\psi$ $\longrightarrow$ Energy of particle w/ wave fn ψ

⊛ Eigenvalues & Eigenvectors

$M \rightarrow n \times n$ matrix

$\bar{v} \rightarrow n \times 1$ column matrix

$M\bar{v} = \bar{w} \rightarrow n \times 1$ $\Rightarrow$ $\bar{v}$ is called an eigenvector of M with eigenvalue λ if

$$M\bar{v} = \lambda \bar{v}$$

'Matrix M multiplied by vector $\bar{v}$' ✗

$\Downarrow$

'Matrix M <u>acts upon</u>' vector $\bar{v}$

Schrödinger's eqn is essentially an eigenvalue problem for differential eqns instead of matrices

$$H\Psi = E\Psi$$

$\Psi(x)$ is the energy eigen fn of the operator H with energy eigen value E.

* Sturm-Liouville Theory

$\downarrow$

what kind of differential operators have the solution of eigenvalue form.

$$Df = \lambda f$$

Take certain free parameters $\rightarrow$ we get solns for certain <u>integer</u> values of the free parameter $\rightarrow$ <u>Energy eigenvalues are quantised</u>

LEC-24

Schrödinger eqn

$\downarrow$

$$i\hbar \frac{d\Psi}{dt}(x,t) = H\Psi(x,t) \Rightarrow$$ Time dependent Schrödinger eqn (TDSE)

Time independent Schrödinger eqn (TISE)

$\downarrow$

$$\underbrace{H}_{\text{differential operator}}\Psi(x) = \underbrace{E}_{\text{Number}}\Psi(x) \Rightarrow$$ Eigen value eqn

For differential operators there is a set of functions such that the operators act on these functions in a way so as to reproduce the function in a scaled form

Energy quantisation arises as $\Psi(\vec{r})$ is not well behaved unless E takes well-defined discrete values

$\therefore$ Quantisation arises not due to Schrödinger's equation, but as a property of differential equations

Example of quantisation in classical physics

$\downarrow$

Vibration modes of a stretched string

Defining a differential operator $D = \frac{d^2}{dx^2}$

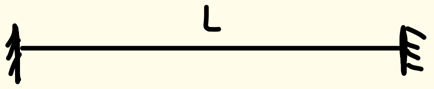
$Df = (\;)f \rightarrow f = A\cos kx + B\sin kx$

k can take any value

$Df = (-k^2)f$

$\downarrow$

Imposing conditions on f



$f = 0$ at $x = 0$ $\qquad$ $f = 0$ at $x = L$

$\Downarrow$

We get well-behaved waveforms only for certain discrete values of k

$\Downarrow$

wave number is quantised

This is the same problem we solve in quantum mechanics

But Schrodinger couldn't clarify the physical meaning behind $\Psi \rightarrow$ just a mathematical entity used to arrive at energy quantisation

Max Born proposed $\rightarrow$ Probability interpretation

$|\Psi(x)^2|\,dx$ is probability of finding the particle in dx neighbourhood of x

$$\Downarrow$$

$|\Psi(\bar{r})^2|\,dV$ = Probability of finding particle in dV neighbourhood of $\bar{r}$.

$$\underset{\text{universe}}{\int} |\Psi(\bar{r})^2|\,dV = 1$$ $\rightarrow$ for Ψ to be well behaved,

1. $\Psi(\bar{r})$ has to be finite everywhere

2. $\int |\Psi(\bar{r})^2|\,dV \rightarrow$ finite for any volume

$$\Downarrow$$

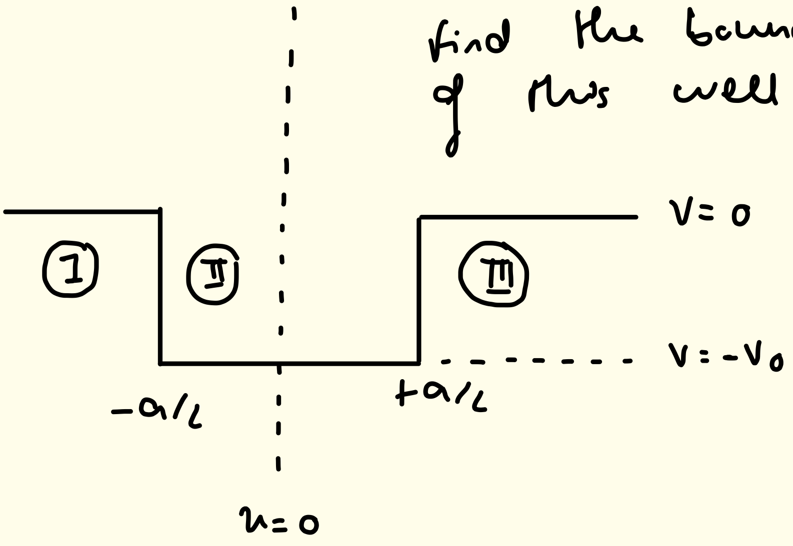
$$\Psi(\bar{r}) \rightarrow 0 \text{ as } r \rightarrow \infty$$

3. $\dfrac{d^2\Psi}{dx^2}$ must exist

$$\Downarrow$$

Ψ & $\dfrac{d\Psi}{dx}$ must be continuous

finite potential well



find the bound states of this well

Three regions I, II, III each with their own form of wave function of the particle ψ_I, ψ_{II}, ψ_{III}

In a classical system,

lower bound = $-V_0$
upper bound = 0

$$\left.\begin{matrix} \text{lower bound} = -V_0 \\ \text{upper bound} = 0 \end{matrix}\right\} \rightarrow -V_0 < E < 0$$

E may have values in a continuous range from $-V_0$ to 0

However, in a quantum mechanical system,

$-V_0 < E < 0 \rightarrow$ still true,

but there are a finite number of valid bound states

$\Rightarrow$ form of wave function in region I

$$\text{I}: -\frac{\hbar^2}{2m} \cdot \frac{d^2\psi_I}{dx^2} = E\psi_I = -|E|\psi_I$$

$$\Rightarrow \frac{d^2\psi_I}{dx^2} = \frac{2m|E|}{\hbar^2}\psi_I = \kappa^2\psi_I \longrightarrow \psi_I = Ae^{\kappa x} + \underset{0}{\cancel{B}}e^{-\kappa x}$$

As $x \rightarrow -\infty$
$\Downarrow$
$\psi_I \rightarrow 0$

Similarly, $\psi_{III} = F e^{-\alpha x} + \underset{0}{\cancel{G}} e^{\alpha x}$

II : $-\dfrac{\hbar^2}{2m} \cdot \dfrac{d^2\psi_{II}}{dx^2} + (-V_0)\psi_{II} = -|E|\,\psi_{II}$

$$\therefore \frac{d^2\psi_{II}}{dx^2} = -\frac{2m}{\hbar^2}\left(V_0 - |E|\right)\psi_{II} = -k^2 \cdot \psi_{II}$$

$$\therefore \psi_{III} = C\cos kx + \underset{0}{\cancel{D}}\sin kx$$

ψ & $\dfrac{d\psi}{dx}$ both continuous at $x = \pi/2$

$\downarrow$

$C\cos k\frac{a}{2} = A e^{-\alpha a/2}$

$Ck\sin k\frac{a}{2} = \alpha A e^{-\alpha a/2}$

$\Downarrow$

$\tan k\frac{a}{2} = \dfrac{\alpha}{k}$

$\Rightarrow \boxed{\dfrac{ka}{2}\tan\dfrac{ka}{2} = \dfrac{\alpha a}{2}}$

Theorem → If potential has symmetry, then the lowest energy state shares the same symmetry

$\Downarrow$

As ψ is symmetric about y axis in regions beyond $\pm a/2$, so should be the case in the potential well.

$\Downarrow$

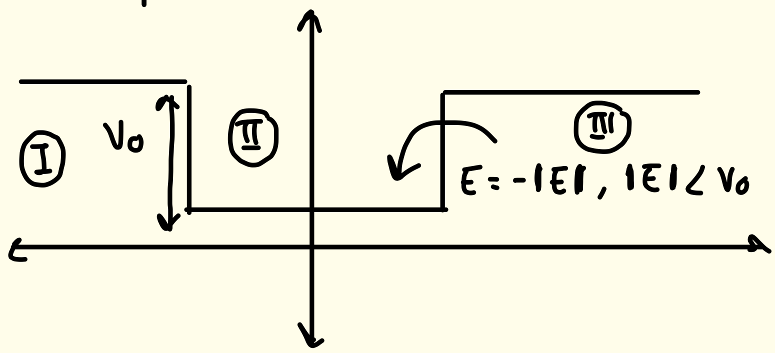
ψ must be an even function, & sin is odd fn

$$k^2 + \alpha^2 = \frac{2m}{\hbar^2}\left(V_0 - |E|\right) + \frac{2m|E|}{\hbar^2} \Rightarrow \boxed{\left(\frac{ka}{2}\right)^2 + \left(\frac{\alpha a}{2}\right)^2 = \frac{mV_0a^2}{2\hbar^2}}$$

Enforcing b'dary conditions gives us quantisation of energy

LEC-25

Finite potential well

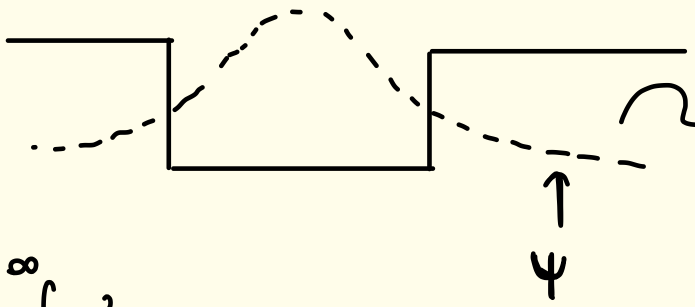


In region II, $E > -V_0$,

$\therefore E > PE \Rightarrow KE > 0$

But in regions I & III,

$E < 0,\ PE = 0 \Rightarrow KE < 0$



Non-zero probability of particle being located here $\Rightarrow$ K.E. -ve

$$\int_{-\infty}^{\infty} |\psi^2|\, dx = 1$$

$$\frac{p^2}{2m} = -|E|$$

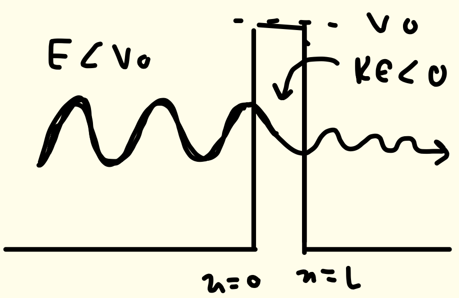
$$\Rightarrow p = \sqrt{-2m|E|}$$

We can't observe the particle in these spaces, but they do exist

$|\psi^2|\, dx =$ Prob of particle being b/w x & $x+dx$

So momentum is imaginary?

"Quantum Quantum Quantum"



$\Rightarrow$ Non-zero probability of quantum tunneling

Tunneling through a barrier

George Gamow → Explained concept of α decay using quantum tunneling

→ Science

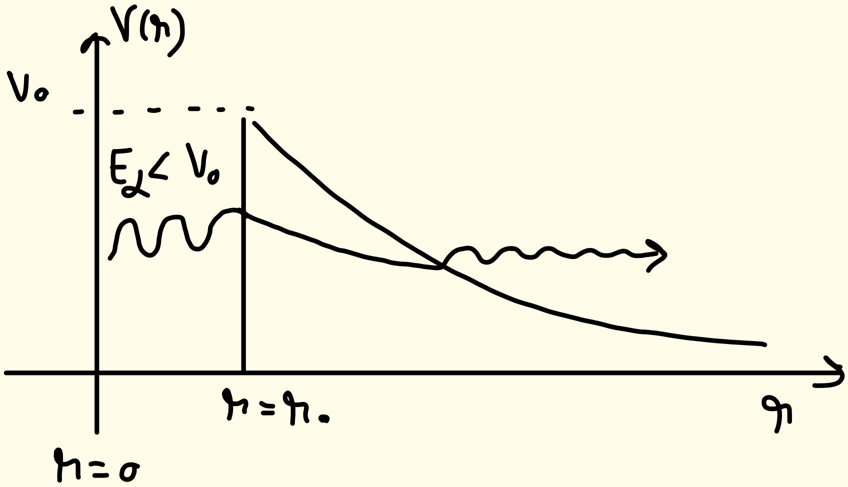
~ 50 years ago

- One, Two, Three Infinity
- Mr. Tompkins in Wonderland

$Pl^{239} \equiv \underbrace{U^{235} + \alpha^{4}}_{}$

loosely bound together $\longrightarrow$ Strong nuclear force

Bound $\equiv$ Potential well



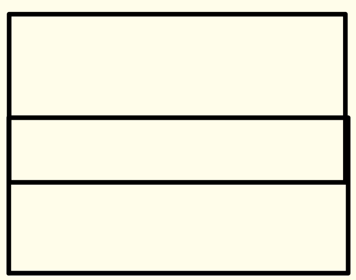
Also explains properties of solids $\rightarrow$ Electrical conduction

Earlier theory $\Rightarrow$ free electrons

Using quantum mechanics $\Rightarrow$ <u>Band theory</u> of solids

Conductors

$\downarrow$



Conduction Band

Valence Band

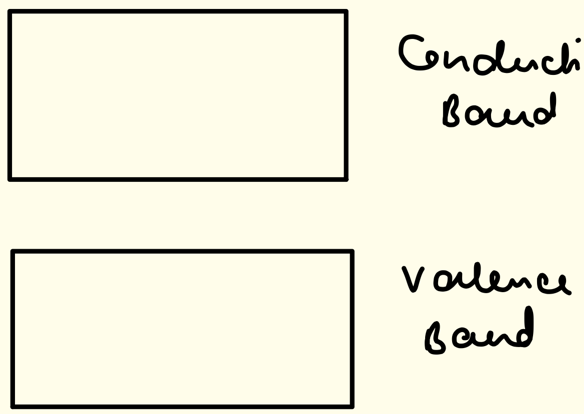
$\downarrow$

Highest energy band above which there are no e's

$\downarrow$

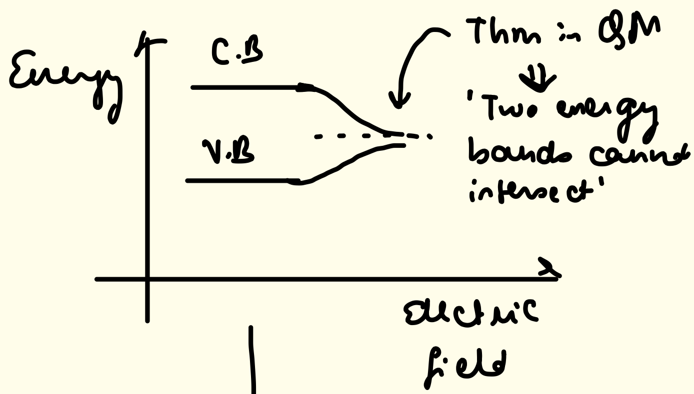
Allowed energy levels form continuous bands

Insulators
↓



Zener diode

Zener assumed the existence of materials where by modulating strength of electric field, the energy levels get affected



↓

Eventually such materials were discovered with diode properties imagined by Zener

---

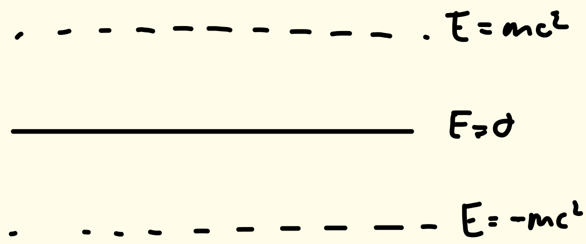
⊛ Dirac → Relativistic version of Schrödinger's Equation
|||
Dirac Equation

↓

Suggestions of negative energy particles

⇓

why don't we see -ve energy electrons around us?

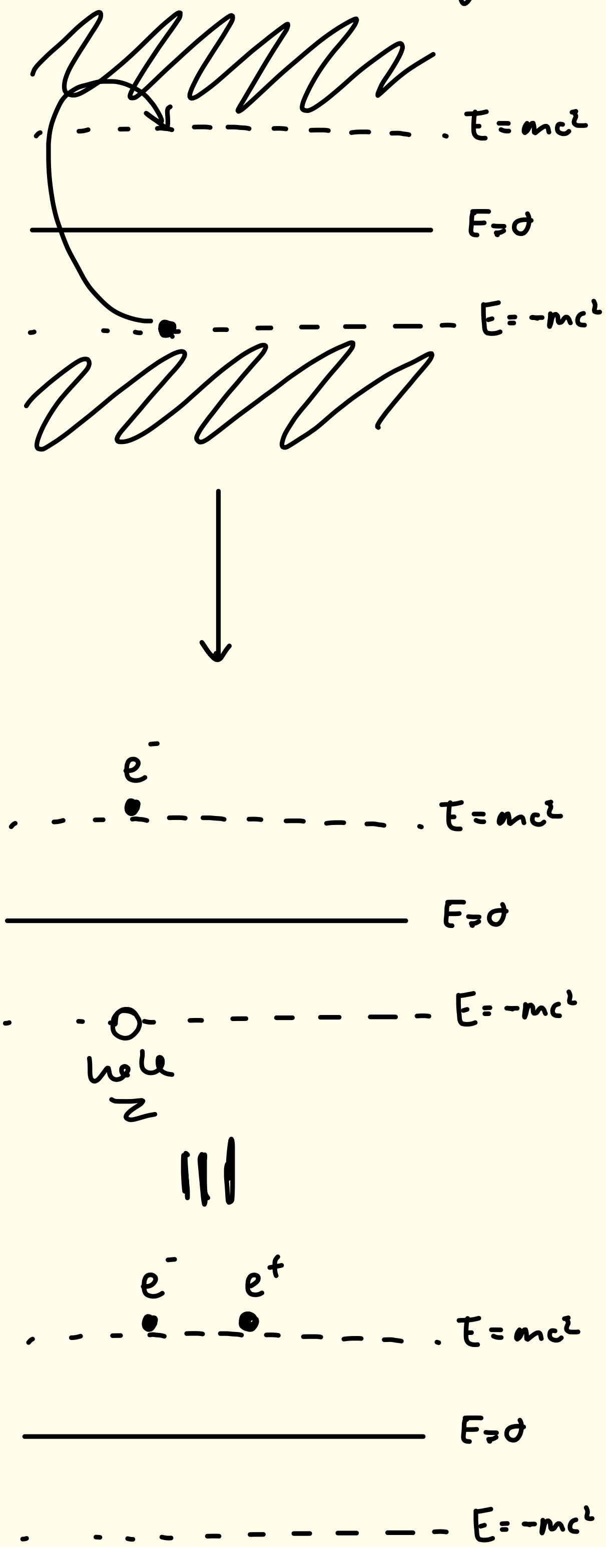


Dirac eqn ⇒ $|E| > mc^2$
⇒ $E > mc^2$ or $E < -mc^2$

Dirac: All -ve energy states are occupied
Everyone: ... wtf?
Dirac: All e's around us are excitations of -ve energy e's that we consider the ground state

Dirac eventually went on to mathematically prove this sh*t

Even more crazy shit



Suppose an e^- in the -ve energy levels was given ~~energ~~ energy to move into a +ve level

↓

Formation of an e^--hole pair

Now how do we comprehend this hole?

Hole = lack of -ve charge
$\equiv$
+ve charge

However, this hole also
↓
lack of -ve energy
$\equiv$
+ve energy

Dirac eqn suggested the existence of a particle with the same physical properties as an electron but with +ve charge

↳ Now called a positron → we eventually observed this particle
↓
Antimatter exists

Measurement in QM

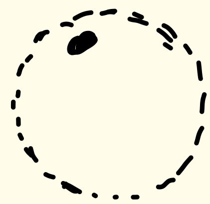
Say we want to tackle the problem of e's bound in an atom

$\Psi(\bar{r})$ describes the 'state' of the electron
↓
Any physical quantity we want to measure

$|\Psi^2(\bar{r})|\, d^3\bar{r} \rightarrow$ Prob of finding an e^- in $d^3\bar{r}$ neighbourhood of $\bar{r}$

Ground state energy eigenvalue of an electron

Position measurement $\longrightarrow$



<u>Collapse of wavefunction</u> into a small region

LEC-26 → ⊛ Bonus - Prof Sai Edition

Early era of Physics - How does the world work?

Modern era of Physics - Can we apply techniques of Physics to stuff very closely related to our daily lives?

Now, Quantum computers

↓

I give you computer ⇒ How to give it ultimate performance

→ Hardware constraints

→ Software constraints

↓

Physicists

:)

If the laws of Physics are true, and the Universe is causal, then

Must be true

↓

Bell's Inequality

"Measurement outcomes are inherently random"

Quantum mechanics ⇒ Can't do 'phase space physics'

↓

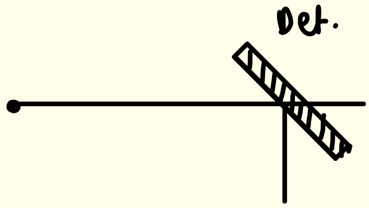
$F = ma$

Physics involving momentum + stuff

If you send a single photon through

↓

Detector will give you one click



output ⇒ inherently random

No amt of previous data can predict future data

'Inherently random'

⇓

It is impossible to predict future outcomes no matter the observer or previous data collected

It is simply impossible to comprehend Physics of small particles through lens of x & p

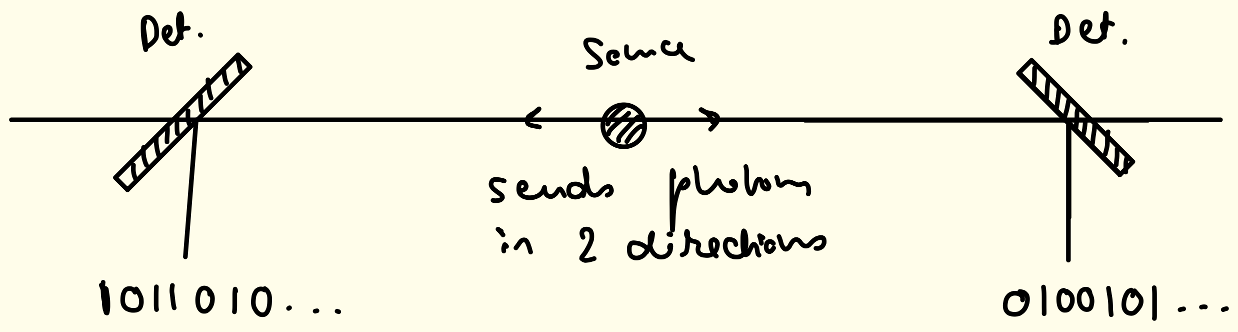
↓ Why?

Law of large numbers

⇓

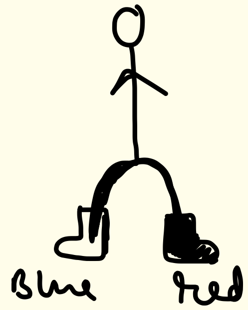
Statistical mech

Proof → $\boxed{\text{EPR paradox}}$



Observation → Results in each direction are anti-correlated

(*) Tangent - Bertlmann's socks



Used to wear different coloured socks in each feet.

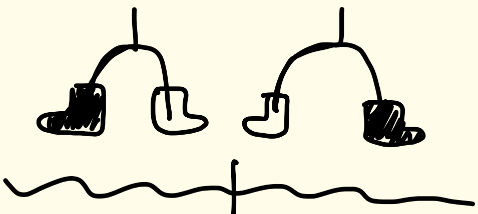
Say someone is a very good friend of Bertlmann's

Has a pair of blue socks

Bertlmann → pair of red socks

They both exchange a sock

⇓



Their socks are 'anti-correlated'

Now, say someone hasn't met them that day, but knows of their peculiar behaviour

↓

Say the person meets Bertlmann's friend. As soon as they see the friend's feet, they instantaneously knew what Bertlmann is wearing

Say Bertlmann is in another galaxy far far away → apparent violation of causality

⇑

No matter how far Bertlmann and his friend are from each other

Differences

→ Preparation uncertainty vs Identical preperation

→ In quantum mechanics, the colour of the socks isn't defined beforehand, it 'pops into existence' as soon as we make a measurement

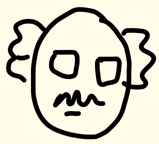
Paradox lies in the fact that identical preperation leads to random outcomes AND anti-correlation still holds

According to E, P & R, maybe there is some 'hidden variable' which affects things

How does stuff hide?

→ Size

→ Lack of interaction strength



– 'Bro eff all this quantum BS classical for the win!'

Einstein said that QM is nothing but CM with some hidden variables

↳ Philosophical + Causality reasons

Bell's intuition → Classical mechanics generate probability theories

↓

Worked at CERN

If classical mechanics is correct

$$\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle < 2$$

Every result → ± 1

Pre-determined reality

If Q.M is correct

$$\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle < 2\sqrt{2}$$

No pre-determined reality

You have to take a (:g?) wide range of outcomes, which expands the range of possibilities.