MA401

Linear Algebra

Instructor – Prof. BK Das

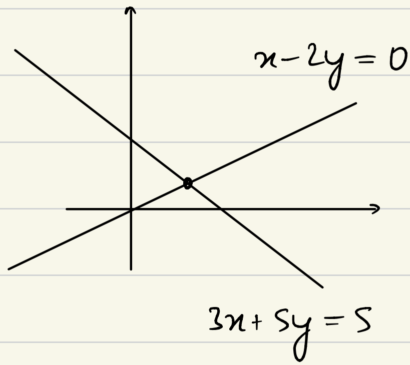
Grading – 2 Surprise Quiz (15 + 15)
Mid Sem (30)
End Sem (40)

References – 1. PD Lax
2. Hoffman & Kunze

System of LE (Linear Eq^{n}s)

eg – 1.

$$x - 2y = 0$$
$$3x + 5y = 5$$



i. Lines in $\mathbb{R}^2$

ii. $x\begin{bmatrix}1\\3\end{bmatrix} + y\begin{bmatrix}-2\\5\end{bmatrix} = \begin{bmatrix}0\\5\end{bmatrix}$

$\Rightarrow \begin{bmatrix}1 & -2\\3 & 5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\5\end{bmatrix}$

$\equiv \quad Ax = b$

2.

$$x + 3y + 0z = 1$$
$$0x + 4y + 2z = 2$$
$$0x + 6y + 5z = 5$$

i. Planes in $\mathbb{R}^3$

ii $\quad x\begin{bmatrix}1\\0\\0\end{bmatrix} + y\begin{bmatrix}3\\4\\6\end{bmatrix} + z\begin{bmatrix}0\\2\\5\end{bmatrix} = \begin{bmatrix}0\\2\\5\end{bmatrix}$

$$\Rightarrow \begin{pmatrix}1 & 3 & 0\\0 & 4 & 2\\0 & 6 & 5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$\equiv \quad Ax = b$$

Gauss Elimination Method (GEM)

eg -
$$\begin{aligned} x + 3y + 0z &= 1 \\ 4y + 2z &= 2 \\ 5z &= 5 \end{aligned}$$

Sol^n for such a sys. (upper or lower triangular) can easily be found using back substitution.

GEM extends this to more general sys.
The idea is to perform row transformations w/o changing sol^n space called elementary row op^{n}s (ERO)

eg -

$$\begin{aligned} 3y + 2z &= 0 \\ 2x + y + 3z &= 1 \\ 3x + 2y - z &= 5 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 0 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 0 & 3 & 2 & 0 \\ 2 & 1 & 3 & 1 \\ 3 & 2 & -1 & 5 \end{array}\right]$$ → Augmented matrix $[A \mid \underline{b}]$

$R_1 \leftrightarrow R_2$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 3 & 2 & 0 \\ 3 & 2 & -1 & 5 \end{array}\right]$$

$R_3 \to R_3 - 3/2 R_1$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 3 & 2 & 0 \\ 0 & 1/2 & -11/2 & 7/2 \end{array}\right]$$

$R_3 \to R_3 - R_2/6$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & -35/6 & 7/2 \end{array}\right]$$

Circled entries are called pivots

All entries below a pivot are zero.

We can represent these EROs by left matrix multip^n of elementary row matrices (EM)

$$R_1 \leftrightarrow R_2 : \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \to kR_1 \ (k \neq 0) : \quad \begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \to R_1 + R_2 : \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In GEM, we successively multiply a matrix A by E_K s to transform it into a triangular matrix T.

Invertible : A sq. matrix is invertible if $\exists\, A^{-1}$ s.t

$$A^{-1}A = I = AA^{-1}$$

- $(AB)^{-1} = B^{-1}A^{-1}$

- EMs are invertible

eg - 1. $R_2 \to R_2 - aR_1$ $\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Inv : $R_2 \to R_2 + aR_1$ $\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. $R_1 \leftrightarrow R_2$ $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Inv : $R_1 \leftrightarrow R_2$

LU Decomposition

Consider a matrix A s.t applying GEM on it doesn't req. permutation matrices (EMs involving row swapping)

Since rest of the EMs are lower triangular.

$$(E_n \dots E_1) A = U \Rightarrow A = (E_1^{-1} \dots E_n^{-1}) U$$

$$\Rightarrow A = LU \quad , \quad L = E_1^{-1} \dots E_n^{-1}$$

(lower triangular matrix)

To generalize this to all matrices, we write

$$PA = LU$$

(permutation matrix)

Row-reduced Echelon form :

- If a col. has pivot, all entries above & below it are 0.

- All pivots need to be 1

- If ith pivot appears in k_ith col., then

$$k_1 < k_2 < \dots < k_n$$

- All the non-zero rows appear at the bottom

Suppose $A_{(n\times n)}$ has n pivots, then RREF of A is identity.

If A has a zero-row, then $Ax = 0$ has a non-trivial sol^n.

This is because pivots < col $\Rightarrow$ free variables

<u>Thm</u> : For $A_{(n\times n)}$, the following are eq.

1. A is invertible
2. $Ax = 0 \Leftrightarrow x = 0$ (only trivial sol^n)
3. A is product of EMs
4. $Ax = b$ has a sol^n $\forall\, b \in \mathbb{R}^n$
5. A is left-invertible

Pf : $1 \Rightarrow 2$ $A^{-1}A = I$

$$Ax = 0 \Rightarrow A^{-1}Ax = A^{-1}0 \Rightarrow Ix = 0 \Rightarrow x = 0$$

$2 \Rightarrow 3$ Since RREF is identity, $\underset{\substack{\uparrow \\ \text{prod. of EM}}}{E}A = I$

$\Rightarrow \quad A = E^{-1}$

$3 \Rightarrow 1$ $A = E^{-1} \Rightarrow A^{-1} = E$ (prod. of inv. matrices)

$1 \Rightarrow 4$ $\forall\, b \in \mathbb{R}^n$, $\quad A(A^{-1}b) = b$

So, $x = A^{-1}b$ is the sol^n to $Ax = b$

$4 \Rightarrow 1$ (contra+)

If A is non-invertible, last row of its RREF is zero

For $b = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$, $1 \neq 0$, $Rx = b$ has no sol^n.

$\because$ Sol^n space of $Rx = b$ & $Ax = b$ is same

$\therefore$ $Ax = b$ has no sol^n.

$1 \Rightarrow 5$ Def^n

$5 \Rightarrow 2 \quad \exists\, B$ s.t $\quad BA = I$

Suppose $\exists\, y \neq 0$ s.t $\quad Ay = 0$

$$\Rightarrow BAy = B0$$
$$\Rightarrow Iy = 0$$
$$\Rightarrow \underline{y = 0}$$

Contd^n

Gauss - Jordan Method

Augment A & $I \longrightarrow [A|I]$

Reduce A to RREF by muliplying both sides successively by EMs.

If A is invertible, its RREF will be I.

$$[A|I] \longrightarrow [\underbrace{EA}_{I} | \underbrace{E}_{A^{-1}}] \Rightarrow [I|A^{-1}]$$

- A is inv. $\Leftrightarrow$ A^{-1} is inv.

$A^{-1}A = I \Rightarrow (A^{-1}A)^T = I^T \Rightarrow A^T(A^{-1})^T = I$

$\Rightarrow (A^T)^{-1} = (A^{-1})^T$

Vector Space

A non-empty set V is a v.s over the field $\mathbb{F}$ if the following hold.

1. Vector add^n: $\forall\ a, b \in V$

- $a+b \in V$
- Comm. & Assoc.
- Unit: $0 + a = a$
- Inv: $a + (-a) = 0$

2. Scalar Multip^n: $\forall\ \alpha \in \mathbb{F},\ v \in V$

- $\alpha v \in V$
- $1_{\mathbb{F}} v = v$
- $\alpha_1 (\alpha_2 v) = (\alpha_1 \alpha_2) v$
- $\alpha (v_1 + v_2) = \alpha v_1 + \alpha v_2$
- $(\alpha + \beta) v = \alpha v + \beta v$

eg — 1. $V = \mathbb{F}$

2. $V = \mathbb{F}^n$

$(a_1, a_2 \cdots a_n) + (b_1, b_2 \cdots b_n)$
$= (a_1 + b_1, \cdots, a_n + b_n)$

$\alpha (a_1, a_2 \cdots, a_n) = (\alpha a_1, \alpha a_2 \cdots, \alpha a_n)$

3. $V = M_{n \times m}(\mathbb{F})$: $n \times m$ matrices with elements in $\mathbb{F}$

4. $V = P_n$: Set of all one-variable polynomials of degree atmost n

Its coeff. belong to a field $\mathbb{F}$

5. $V = \ell^{\infty}(\mathbb{C})$: Complex valued bounded sequences

$= \{ (x_n)_{n \in \mathbb{N}} : |x_n| \leq M \ , \ n \in \mathbb{N} \}$

or

$\sup (x_n)$ is finite

$x = (x_n) \ , \ y = (y_n) \Rightarrow x+y = (x_n + y_n)$

$\alpha (x_n) = (\alpha x_n)$

6. $V = C([0,1])$: Set of cont. fn's on the interval $[0,1]$

<u>Subspace</u> : A non-empty subset W of a v.s V is a subspace if W is also a v.s over the same field.

Prop^n : If V is v.s. , then a non-empty subset W of V is a subsp. iff.

$$\alpha v_1 + v_2 \in W \,, \quad \forall \alpha \in \mathbb{F} \,, \; v_1, v_2 \in W$$

Pf : ($\Rightarrow$) Trivial

($\Leftarrow$) Consider $\alpha = -1$, $v_1 = v_2 = v$

$$\Rightarrow \quad -v + v = 0 \in W$$

For $v_1, v_2 \in W$, $\alpha \in \mathbb{F}$

Add^n : $v_1 + v_2 = (1)v_1 + v_2 \in W$

S. Multip^n : $\alpha v_1 = \alpha v_1 + 0 \in W$

$[\because 0 \in W]$

Inv : $-v = (-1)v + 0 \in W$

Rest of the ppts. are inherited from V.

$\therefore$ W is a v.s

eg - 1. $V = \mathbb{F}$, $W = \{0\}, \mathbb{F}$

2. $V = \mathbb{R}^n$, $W =$ lines through origin

3. $V = \mathbb{R}^n$

Let $A \in M_{n \times n}(\mathbb{R})$

Consider $W = \{ b \in \mathbb{R}^n : \exists x \in \mathbb{R}^n \text{ s.t } Ax = b \}$
(Sol^n space or Col space)

Suppose $b_1, b_2 \in W$, then $\exists x_1, x_2 \in \mathbb{R}^n$
$Ax_1 = b_1$ & $Ax_2 = b_2$

$$A(\alpha x_1 + x_2) = \alpha A x_1 + A x_2 = \alpha b_1 + b_2 \in W$$

4. $V = \mathbb{F}^n$

Let $A \in M_{m\times n}(\mathbb{F})$

$N = \{ x \in \mathbb{F}^n : Ax = 0 \}$
(Null space)

Suppose $v, w \in N$.

$A(\alpha v + w) = \alpha(Av) + Aw = 0 + 0 = 0$
$\Rightarrow \alpha v + w \in N$

- Let W_1 & W_2 be subsp. of V.

1. $W_1 \cup W_2$ is <u>not</u> necessarily a subsp.

2. $W_1 \cap W_2$ is a subsp.

3. $W_1 + W_2 = \{ x + y : x \in W_1, y \in W_2 \}$
is a subsp.

4. $W_1 \times W_2 = \{ (x, y) : x \in W_1, \ y \in W_2 \}$
is a subsp. of $V \times V$

Pf: 2. Consider $v, w \in W_1 \cap W_2$, $\alpha \in \mathbb{F}$

$$\left.\begin{array}{lll} v, w \in W_1 & \Rightarrow & (\alpha v + w) \in W_1 \\ v, w \in W_2 & \Rightarrow & (\alpha v + w) \in W_2 \end{array}\right\} \ (\alpha v + w) \in W_1 \cap W_2$$

3. Consider $\underset{x_1 + y_1}{v}, \underset{x_2 + y_2}{w} \in W_1 + W_2$, $\alpha \in \mathbb{F}$

$$\alpha v + w = \alpha(x_1 + y_1) + x_2 + y_2 = \underbrace{\underbrace{(\alpha x_1 + x_2)}_{\in W_1} + \underbrace{(\alpha y_1 + y_2)}_{\in W_2}}_{\in W_1 + W_2}$$

4. $V \times V$ is v.s : $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$
$\alpha(x_1, y_1) = (\alpha x_1, \alpha x_2)$

Consider $\underset{(x_1, y_1)}{v}, \underset{(x_2, y_2)}{w} \in W_1 \times W_2$, $\alpha \in \mathbb{F}$

$$\alpha v + w = \alpha(x_1, y_1) + (x_2, y_2) = (\underbrace{\alpha x_1 + x_2}_{\in W_1}, \underbrace{\alpha y_1 + y_2}_{\in W_2})$$

$$\in W_1 \times W_2$$

<u>Span</u>: For any subset S of V,

$$\mathrm{span}(S) = \left\{ \sum_{i=1}^{n} \alpha_i x_i \ : \ \alpha_i \in \mathbb{F}, \ x_i \in S \right\}$$

<u>Claim</u>: $\mathrm{span}(S)$ is a subsp. of V

<u>Pf</u>: Consider $v = \sum_{i=1}^{n} \alpha_i x_i$, $w = \sum_{i=1}^{m} \beta_i y_i \in \mathrm{span}(S)$
$\gamma \in \mathbb{F}$

$$\gamma v + w = \sum_{i=1}^{n} \gamma \alpha_i x_i + \sum_{i=1}^{m} \beta_i y_i \in \mathrm{span}(S)$$

- Let $A \in M_{m \times n}(\mathbb{F})$, $A = [c_1 \ c_2 \ \cdots \ c_n]$, $c_i \in \mathbb{F}^m$

Col. sp. of A = $\mathrm{span}\{c_1, c_2, \dots, c_n\}$

Linear Independence : Let $x_1, \dots, x_n \in V$

$\{x_1, \dots, x_n\}$ is lin. dep. if $\exists\ \alpha_1, \dots, \alpha_n$ not all zero s.t $\sum \alpha_i x_i = 0$

$$\Updownarrow$$

$$\begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = 0$$

has a non-trivial solution.

Let $x_1, \dots, x_n \in V$

$\{x_1, \dots, x_n\}$ is lin. indep. if

$$\sum \alpha_i x_i = 0 \Rightarrow \alpha_i = 0 \quad \forall\ i$$

$$\Updownarrow$$

$$\begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = 0$$

has only the trivial solution.

Let $S \subset V$.

S is lin. dep. if $\exists$ finite set of vecs. in S which is dep.

S is lin. indep. if all finite set of vecs. in S are lin. indep.

eg – 1. $V = \mathbb{R}^3$

$S = \{(1,0,0), (0,1,0), (0,0,1)\}$ is l.i.

$$\alpha(1,0,0) + \beta(0,1,0) + \gamma(0,0,1)$$
$$= (\alpha, \beta, \gamma) = (0,0,0) \Rightarrow \alpha = \beta = \gamma = 0$$

2. $V = \mathbb{R}[x]$

$S = \{1, x, x^2, \dots\} = \{x^n : n \in \mathbb{N}\}$ is l.i.

Consider $\{x^k : k = 1, \dots, m\} \subset S$

Suppose $\sum_{k=1}^{m} \alpha_k x^k = 0$.

$\because$ Poly. can have atmost m roots.

$\therefore$ LHS is identically zero

$\Rightarrow \quad \alpha_i = 0 \quad \forall\, i = 1, \dots, m$

3. Let $A = [c_1 \dots c_n]$ be an inv. matrix.

Then, $\{c_1, \dots, c_n\}$ is l.i.

4. $\ell^\infty = \{ x = (x_n) : \sup|x_n| < \infty \}$

$e^{(i)} = (e^{(i)}_j)$, $\qquad e^{(i)}_j = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$

Then $\{ e^{(i)} : i \in \mathbb{N} \}$ is li

Basis : A subset $S \subseteq V$ of a v.s is a basis if S is li & $\mathrm{span}(S) = V$

Dimension : Cardinality of a basis of the v.s.
$\dim(V)$

Thm : If $\mathrm{span}\{\alpha_1, \ldots, \alpha_n\} = V$, then any li set of vecs. in V has cardinality $\leq n$.

Cor : 1. If V has a finite basis, then the cardinality of any two basis are the same.

2. If $\dim(V) = n$, then any set of vecs. with more than n elems. is ld.

3. If $\dim(V) = n$, then any set of vecs. containing less than n-elems. cannot span V.

Pf: Let $V = \mathbb{R}^l = \text{span}\{\alpha_1, \dots, \alpha_m\}$

Consider $\{\beta_1, \dots, \beta_n\}$, $\beta_k \in V$
It suffices to show if $n > m$,
this set is lin. dep.

$\because$ $\beta_i \in V$ & α_ks span V

$$\therefore \quad \beta_i = \underset{l \times m}{[\alpha_1 \ \dots \ \alpha_m]} \underbrace{\begin{bmatrix} x_{i1} \\ \vdots \\ x_{im} \end{bmatrix}}_{x_1}, \quad \forall\, i = 1, \dots, n$$

$$\Rightarrow \underbrace{[\alpha_1 \ \dots \ \alpha_n]}_{A}{}_{l \times m} \underbrace{[x_1 \ \dots \ x_n]}_{B}{}_{m \times n} = \underbrace{[\beta_1 \ \dots \ \beta_n]}_{C}$$

$\{\beta_1 \dots \beta_n\}$ is ld $\Leftrightarrow$ $Cx = 0$ has a non-tvl sol^n

$\because$ $n > m$ $\Rightarrow$ $Bx = 0$ has a non-tvl sol^n
$\therefore$ $Cx = 0$ has a non-tvl sol^n.

For a general v.s. V,

$$\beta_i = \sum_{j=1}^{m} A_{ji}\, \alpha_j \ , \qquad A_{ji} \in \mathbb{F} \ , \quad \forall\, i = 1, \dots, n$$

$$\sum_{i=1}^{n} a_i \beta_i = \sum_{i=1}^{n} a_i \left(\sum_{j=1}^{m} A_{ji}\, \alpha_j \right)$$

$$= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} A_{ji}\, a_i \right) \alpha_j$$

$$A = [A_{ji}]_{m \times n} \ , \qquad a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$\sum_{i=1}^{n} A_{ji}\, a_i = Aa$$

$\because$ $n > m$ $\Rightarrow$ $Aa = 0$ has a non-tvl sol^n

$\therefore$ $\exists$ $b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \neq 0$ s.t $Ab = 0$

$$\Rightarrow \sum_{i=1}^{n} A_{ji}\, b_i = 0$$

$$\Rightarrow \sum_{j=1}^{m} \left(\sum_{i=1}^{n} A_{ji} b_i \right) \alpha_j = 0$$

$$\Rightarrow \sum_{i=1}^{n} b_i \beta_i = 0$$

eg : 1. $V = \mathbb{R}^n$

$\mathcal{B} = \{ e_k : e_k = (0 \ldots \underset{\uparrow}{1} \ldots 0) , k = 1, \ldots, n \}$

k th place

2. $V = P_n$

$\mathcal{B} = \{ x^k : k = 0, 1, \ldots, n \}$

Lem. : If $\{\alpha_1, \ldots, \alpha_n\}$ is li vecs. in V and $\beta \in V \setminus \text{span}\{\alpha_1, \ldots, \alpha_n\}$, then $\{\alpha_1, \ldots, \alpha_n, \beta\}$ is li.

Pf : If $\{\alpha_1, \ldots, \alpha_n, \beta\}$ is ld,

$$\sum_{i=1}^{n} a_i \alpha_i + b\beta = 0 , \quad \text{where } \alpha_i \neq 0$$
$$b \neq 0$$

Note, $b \neq 0$ otherwise $\alpha_i = 0 \ \forall \ i = 1, \dots, n$
$\rightarrow$ Contd^n

$$\Rightarrow \quad \beta = \sum_{i=1}^{n} (-a_i/b)\, \alpha_i$$

$$\Rightarrow \quad \beta \in \text{span}\{\alpha_1, \dots, \alpha_n\} \rightarrow \text{Contd}^n$$

<u>Thm</u>: Every (finite-dim.) v.s has a basis

Pf: Let $\alpha_1 \in V$, $\alpha_1 \neq 0$
If $\mathbb{F}\alpha_1 = \text{span}(\{\alpha_1\}) = V$, we're done.
Else, take $\alpha_2 \in V \setminus \text{span}(\{\alpha_1\})$

By prev. lemma, $\{\alpha_1, \alpha_2\}$ is li
If $\text{span}(\{\alpha_1, \alpha_2\}) = V$, we're done.
Otherwise continue the process.

If $\dim(V)$ is finite, the process stops after finitely many steps and we get a basis.

- If W is subsp. of V, then any basis of W can be extended to a basis of V.
- If W is subsp. of V, $\dim(W) \leq \dim(V)$
  Equality holds iff $W = V$

<u>Direct Sum</u> : For W, W_1, W_2 subsp. of V, $W = W_1 \oplus W_2$ if every vec. in W can be uniquely expressed as a sum of vec. from W_1 & W_2.

$$W = W_1 + W_2 \quad \& \quad W_1 \cap W_2 = \{0\}$$

Pf: $w = x + y = a + b$, $\quad x, a \in W_1$, $y, b \in W_2$

$$\Rightarrow \underbrace{(x-a)}_{\in W_1} = \underbrace{(b-y)}_{\in W_2}$$

$$\Rightarrow x - a = 0 = b - y \quad (\because W_1 \cap W_2 = \{0\})$$

Let $A \in M_{m\times n}(\mathbb{F})$, R be its RREF,

$$EA = R$$

Suppose R has r-many pivot elems.

$\dim(C(A)) = r$

$\dim(R(A)) = r$ $\quad [\because R(A) = R(R)]$

$\dim(N(A)) = n-r$ $\quad [\because N(A) = N(R)]$

$\dim(N(A^T)) = m-r$

$\uparrow$ left null-space (sol^n of $x^TA = 0$ or $A^Tx = 0$)

Note, $N(A), R(A) \subset \mathbb{F}^n$ & $\dim(R(A)) + \dim(N(A)) = n$

$C(A), N(A^T) \subset \mathbb{F}^m$ & $\dim(C(A)) + \dim(N(A^T)) = m$

- Find free parameters using R.
  Then e_{k_i}, $i = 1, 2, \ldots, n-r$ where col. k_i is pivotless, forms a basis of $N(A)$.

- If k_i, $i = 1, \ldots, r$ are pivotal cols. of R, then k_i cols. of A form basis of $C(A)$

- If k_i, $i = 1, 2, \ldots, m-r$ are zero-rows of R, then rows k_i of E form the basis of $N(A^T)$.

The non-zero rows of R form a basis of $R(A)$.

Lem: Let V be a subsp. of $\mathbb{R}^n$, and let $A \in M_{n \times n}(\mathbb{R})$ be an inv. mat.

Then $\dim(V) = \dim(A(V))$

where $A(V) = \{ Av : v \in V \}$

Pf: Consider a basis $\{v_1, \ldots, v_m\}$ of V

Claim: $\{Av_1, \ldots, Av_m\}$ is basis of $A(V)$.

1. For $Av \in A(V)$,

$$Av = A\left(\sum_{i=1}^{m} a_i v_i\right) = \sum_{i=1}^{m} a_i (Av_i)$$

$\Rightarrow$ $\text{span}\{Av_1, \ldots, Av_n\} = A(V)$

2. Let $\sum a_i Av_i = 0 \Rightarrow A(\sum a_i v_i) = 0$

$\Rightarrow \sum a_i v_i = A^{-1} 0 = 0 \Rightarrow a_i = 0 \quad \forall\, i = 1, \ldots, m$

($\because$ v_i s form a basis)

$\Rightarrow \{Av_1, \dots, Av_n\}$ are lin. indep.

Hence, $\{Av_1, \dots, Av_n\}$ form a basis of $A(V)$.

To prove $\dim(C(A)) = r$, we just need to show $E(C(A)) = C(R)$

$\because EA = R \Rightarrow E(C(A)) \subseteq C(R)$

Now, take $y \in C(R)$. $\Rightarrow y = Rx$

$\Rightarrow y = EAx \in E(C(A))$

$\Rightarrow C(R) \subseteq E(C(A))$

Hence, $E(C(A)) = C(R)$

Thm: for W_1, W_2 subsp. of V,

$$\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

Cor: $\dim(W_1 \oplus W_2) = \dim(W_1) + \dim(W_2)$

Pf: Let $\{\alpha_1, \ldots, \alpha_n\}$ be a basis of $W_1 \cap W_2$

Extend it to a basis of W_1 & W_2 resp. – $\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m\}$ &
$\{\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_r\}$

Claim: $\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_r\}$ is a basis of $W_1 + W_2$

1. Clearly, $\operatorname{span}\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_r\}$
$= W_1 + W_2$

2. Consider $\sum_{i=1}^{n} x_i \alpha_i + \sum_{i=1}^{m} y_i \beta_i + \sum_{i=1}^{r} z_i \gamma_i = 0$

$$\Rightarrow \sum_{i=1}^{r} z_i \gamma_i = -\left(\sum_{i=1}^{n} x_i \alpha_i + \sum_{i=1}^{m} y_i \beta_i\right)$$

$\in W_2 \qquad\qquad \in W_1$

$\Rightarrow \sum_{i=1}^{r} z_i \gamma_i \in W_1 \cap W_2$

$\because \{\alpha_1, \dots, \alpha_n\}$ form a basis of $W_1 \cap W_2$

$\therefore \sum_{i=1}^{r} z_i \gamma_i = \sum_{i=1}^{n} x_i' \alpha_i$

$\Rightarrow \sum_{i=1}^{n} (x_i + x_i') \alpha_i + \sum_{i=1}^{m} y_i \beta_i = 0$

$\because \{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m\}$ form a basis of W_1

$\therefore (x_i + x_i') = 0$ & $y_i = 0 \;\; \forall i$

$\Rightarrow \sum_{i=1}^{n} x_i \alpha_i + \sum_{i=1}^{r} z_i \gamma_i = 0$

$\because \{\alpha_1, \dots, \alpha_n, \gamma_1, \dots, \gamma_r\}$ form a basis of W_2

$\therefore x_i = 0$ & $z_i = 0 \;\; \forall i$

Quotient space : For a subsp. Y of X, the v.s

$X/Y = \{ x + Y : x \in X \}$ is called quotient sp.

$x_1 + Y = x_2 + Y \iff x_1 - x_2 \in Y$

Alt : $x_1 \sim x_2 \iff x_1 - x_2 \in Y$
defines an eq. rel^n on X.

$X/Y = X/\sim$ i.e. the set of eq. classes.

- Vec. add^n : $(x_1 + Y) + (x_2 + Y) := (x_1 + x_2) + Y$

- Scalar multip^n : $\alpha(x + Y) = \alpha x + Y$

<u>Lem</u> : Let Y be subsp. of v.s X (finite).

$$\dim(X/Y) = \dim(X) - \dim(Y)$$

Pf : Let $\{y_1, \dots, y_r\}$ be basis of Y.
Extend it to $\{y_1, \dots, y_r, x_1, \dots, x_m\}$ to get a basis of X.

<u>Claim</u> : $\{x_1 + Y, \dots, x_m + Y\}$ form a basis of X/Y.

<u>1.</u> Suppose $\exists\, \alpha_i$ s.t $\sum_{i=1}^{m} \alpha_i (x_i + Y) = 0 + Y$

$$\Rightarrow \left(\sum_{i=1}^{n} \alpha_i x_i \right) + Y = 0 + Y$$

$$\Rightarrow \sum_{i=1}^{n} \alpha_i x_i \in Y$$

$$\Rightarrow \sum_{i=1}^{m} \alpha_i x_i = \sum_{j=1}^{r} \beta_j y_j \qquad (\because y_j \text{ form a basis of } Y)$$

$$\Rightarrow \begin{matrix} \alpha_i = 0, & i = 1, \dots, m \\ \beta_j = 0, & j = 1, \dots, r \end{matrix} \qquad (\because x_i \,\&\, y_j \text{ form a basis of } X)$$

So, $\{x_1 + Y, \dots, x_m + Y\}$ are lin. indep.

2. Consider $x + Y \in X/Y$.

$$x + Y = \left(\sum_{i=1}^{m} \alpha_i x_i + \sum_{j=1}^{r} \beta_j y_j \right) + Y$$

$$= \left(\sum_{i=1}^{m} \alpha_i x_i \right) + Y$$

$$= \sum_{i=1}^{m} \alpha_i \, (x_i + Y)$$

So, $\text{span}(\{x_1 + Y, \ldots, x_m + Y\}) = X/Y$

<u>Linear map</u> : A map $T : X \to Y$ is linear if

$$T(x_1 + \alpha x_2) = T(x_1) + \alpha T(x_2) \;\; \forall \; x_1, x_2 \in X, \; \alpha \in \mathbb{F}$$

A bij. lin. map is called an isomorphism.

eg : $A \in M_{m \times n}(\mathbb{F})$

$$T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$$
$$x \longmapsto Ax$$

$$T_A(x_1+\alpha x_2) = A(x_1+\alpha x_2) = Ax_1 + \alpha A x_2$$
$$= T_A(x_1) + \alpha T_A(x_2)$$

<u>Null space</u> : $N(T) = \{ x \in X : T(x) = 0 \} \subseteq X$

$\dim(N(T))$ is called <u>nullity</u> of T.
T is inj. iff. $N(T) = \{0\}$

<u>Range space</u> : $R_T = \{ y \in Y : \exists x \in X \text{ s.t } T(x) = y \} \subseteq Y$

$\dim(R_T)$ is called <u>rank</u> of T.
T is surj. iff. $R_T = Y$

<u>Lem</u> : Let $T : X \to Y$. The lin. map induced by T, $\tilde{T} : X/N(T) \hookrightarrow Y$ is injective.
$$x + N(T) \longmapsto T(x)$$

Pf: If $x_1 + N(T) = x_2 + N(T)$, then

$x_1 - x_2 \in N(T) \Rightarrow T(x_1 - x_2) = 0$

$\Rightarrow T(x_1) = T(x_2)$

$\Rightarrow \tilde{T}(x_1) = \tilde{T}(x_2)$

So, the map is well-defined.

$$\tilde{T}((x_1 + N(T)) + \alpha(x_2 + N(T))) = \tilde{T}((x_1 + \alpha x_2) + N(T))$$
$$= T(x_1 + \alpha x_2)$$
$$= T(x_1) + \alpha T(x_2)$$
$$= \tilde{T}(x_1 + N(T)) + \alpha \tilde{T}(x_2 + N(T))$$

So, the map is linear.

If $\tilde{T}(x + N(T)) = 0$, then $T(x) = 0$

$\Rightarrow x \in N(T)$

$\Rightarrow x + N(T) = 0 + N(T)$

$\Rightarrow N(\tilde{T}) = \{0 + N(T)\}$

So, the map is inj.

$$\begin{array}{ccccc} X & \xrightarrow{\pi} & X/N(T) & \xrightarrow{\tilde{T}} & Y \\ x & \longmapsto & x+N(T) & \longmapsto & T(x) \end{array}$$

π : Canonical quotient map.

$T = \tilde{T} \circ \pi$

Let $\{x_1, \dots, x_n\}$ be a basis of X.
Then any lin. map. $T: X \to Y$ is uniquely determined by the images of x_i under T.

$$x = \sum_{i=1}^{n} \alpha_i x_i \quad \Rightarrow \quad T(x) = \sum_{i=1}^{n} \alpha_i T(x_i)$$

Lem: Let $T: X \to Y$ be a lin. map.

1. T is inj. iff. image of any lin. indep. subset of X under T is lin. indep.

2. T is surj. iff. image of any spanning subset of X under T spans Y.

Pf: 1. $(\Rightarrow)$ Let $\{x_1, \dots, x_n\} \subseteq X$ be lin. indep.

Suppose $\sum \alpha_i T(x_i) = 0 \Rightarrow T(\sum \alpha_i x_i) = 0$

$\Rightarrow \sum \alpha_i x_i = 0$ ($\because$ inj. T)

$\Rightarrow \alpha_i = 0 \quad \forall\ 1 \leq i \leq n$

$\therefore \{T(x_1), \dots, T(x_n)\}$ are lin. indep.

$(\Leftarrow)$ Suppose $T(x) = 0$

Let $\{x_1, \dots, x_n\}$ be a basis of X.

$x = \sum \alpha_i x_i \Rightarrow T(\sum \alpha_i x_i) = 0$

$\Rightarrow \sum \alpha_i T(x_i) = 0$

$\Rightarrow \alpha_i = 0 \ \forall\ 1 \leq i \leq n$ $\left(\because T(x_i)\text{'s are lin. indep.} \right)$

$\Rightarrow x = 0$

$\therefore$ T is inj.

<u>Alt</u>: $\because \{x\}$ is lin. indep. for $x \neq 0$

$\Rightarrow \{T(x)\} = \{0\}$ must be lin. indep.

$\longrightarrow$ Contdn

So, $x = 0$

<u>2.</u> $(\Leftarrow)$ $\because X$ is a spanning subset of X.

$T(X)$ is spanning subset of Y

$\text{span}(T(X)) = T(X) = Y$

$\therefore$ T is surj.

$(\Rightarrow)$ Let X_0 be a spanning subset of X.

Consider $y \in Y$.

$\because$ T is surj, $\exists\ x \in X$ s.t $T(x) = y$

$\because X$ is a spanning set $\Rightarrow$ $x = \sum \alpha_i x_i$, $x_i \in X_0$

$$\Rightarrow \quad y = T(x) = T(\sum \alpha_i x_i) = \sum \alpha_i T(x_i)$$

$\therefore$ $T(X_0)$ is a spanning set for Y.

Thm: $T : X \to Y$ is an isomorphism, iff image of basis of X under T forms a basis of Y.

eg: 1. Let X be v.s. with $\dim(X) = n$

$X \cong \mathbb{F}^n$

Consider basis $\{x_1, \ldots, x_n\}$ & $\{e_1, \ldots, e_n\}$ of X & $\mathbb{F}^n$ resp.

$$T : X \longrightarrow \mathbb{F}^n$$
$$x_i \longmapsto e_i$$

Suppose $T(x) = 0 \Rightarrow T(\sum \alpha_i x_i) = 0$

$$\Rightarrow \sum \alpha_i T(x_i) = 0$$

$$\Rightarrow \sum \alpha_i e_i = 0 \Rightarrow \alpha_i = 0, \ 1 \leq i \leq n$$

$$T(\sum \alpha_i x_i) = \sum \alpha_i e_i = (\alpha_1, \ldots, \alpha_n)$$

Thm: (Rank-Nullity Thm)

Let X be a finite dim. v.s, and $T : X \to Y$ be a lin. map. Then

$$\dim(X) = \dim(N_T) + \dim(R_T)$$

Pf:
$$X \xrightarrow{\pi} X/N_T \xrightarrow{\tilde{T}} R_{\tilde{T}} \subseteq Y$$

$$\tilde{T}(x+Y) = T(x).$$

$\tilde{T}$ is inj. but <u>not</u> surj. on Y.
However, $\tilde{T}$ is also surj. on $R_{\tilde{T}}$

$\because \quad R_T = R_{\tilde{T}}$

$\therefore \quad X/N_T \cong R_T \quad \Rightarrow \quad \dim(X/N_T) = \dim(R_T)$

$(\text{bij } \tilde{T}) \quad \Rightarrow \quad \dim(X) = \dim(N_T) + \dim(R_T)$

Rem :

- If X is finite dim. & $T: X \to X$ is inj.,
  then T is surj.

$$\dim(X) = \underbrace{\dim(N_T)}_{0} + \dim(R_T)$$

$$\Rightarrow \quad R_T = X$$

- If $T: X \to Y$, $\dim(Y) < \dim(X)$
  then $\exists\, x \neq 0$ s.t $T(x) = 0$

$$\dim(X) = \dim(N_T) + \dim(R_T)$$

$$\Rightarrow \dim(N_T) = \dim(X) - \dim(R_T) > 0$$

$$[\because \dim(R_T) \leq \dim(Y)]$$

Any lin. map. $T: \mathbb{R}^n \to \mathbb{R}^m$ is of the form $T = T_A$ for some $A \in M_{m\times n}(\mathbb{R})$

$$A = [T(e_1) \quad T(e_2) \quad \dots \quad T(e_n)]_{m\times n}$$

$$T(e_k) = T_A(e_k) \quad \forall \quad k = 1, \dots, n$$

Linear Functional : A lin. fxn on X is a lin. map from X to $\mathbb{F}$.

Dual space : The set of all lin. fxn on X is called the dual space and is denoted by X^*. X^* is also a v.s over the same field.

eg : $X = \mathbb{R}^n$, $\quad T: \mathbb{R}^n \to \mathbb{R}$

$$x \mapsto Ax$$

$$A_{1\times n} = [\alpha_1 \ \alpha_2 \dots \alpha_n] = [T(e_1) \ T(e_2) \ \dots \ T(e_n)]$$

$\dim(\mathbb{R}^{n*}) = n$, $\quad \mathbb{R}^{n*} = \mathbb{R}^n$

So elems. of $(\mathbb{R}^n)^*$ are dot prod. with vecs.

Let X be a fin. dim. v.s with $\dim(X) = n$

$X \cong \mathbb{R}^n$ wrt. a basis

$X \xrightarrow{\psi} \mathbb{R}^n \xrightarrow{T} \mathbb{R}$, $T \circ \psi \in X^*$

$\Rightarrow$ $X^* \cong \mathbb{R}^{n^*}$, $X^* = \{ T \circ \psi : T \in \mathbb{R}^{n^*} \}$

$\dim(X^*) = \dim(X) = \dim(\mathbb{R}^n) = n$

In general, if $X \cong Y \Leftrightarrow X^* \cong Y^*$
(not necessarily finite dimensional)

$$\begin{tikzcd} X \arrow[r, shift left, "\psi"] \arrow[d, "l"'] & Y \arrow[l, shift left, "\psi^{-1}"] \\ \mathbb{F} & \end{tikzcd}$$

$$X^* \longrightarrow Y^*$$
$$l \longmapsto l \circ \psi^{-1}$$

(inj) $l \circ \psi^{-1} = 0 \Rightarrow l \circ \psi^{-1} \circ \psi = 0 \circ \psi \Rightarrow l = 0$

(surj.) for $\varphi \in Y^*$, $\varphi = \underbrace{(\varphi \circ \psi)}_{\in X^*} \circ \psi^{-1}$

Orthogonal complement : subsp $Y \subseteq \mathbb{R}^n$

$$Y^\perp = \{ x \in \mathbb{R}^n : x \cdot y = 0 \ \forall \ y \in Y \}$$

Annihilator : subsp $Y \subseteq V$

$$Y^\perp = \{ l \in V^* : l(y) = 0 \ \forall \ y \in Y \}$$

For $V = \mathbb{R}^n$, these notions are eq. since $(\mathbb{R}^n)^* = \mathbb{R}^n$

- $Y \cap Y^\perp = \{0\}$

$\because$ $v \in Y \cap Y^\perp \Rightarrow v \cdot v = 0 \Rightarrow v = 0$

- $Y \oplus Y^\perp = \mathbb{R}^n$

Consider
$$\begin{tikzcd} X \arrow[r, "T"] \arrow[dr, dotted] & Y \arrow[d, "l"] \\ & \mathbb{F} \end{tikzcd}$$

$$\begin{aligned} T' : \ & Y^* \longrightarrow X^* \\ & l \longmapsto l \circ T \end{aligned}$$

T' is called <u>adjoint</u> of T

For $X = \mathbb{R}^n$ & $Y = \mathbb{R}^m$,

$$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m \quad , \qquad A \in M_{m \times n}(\mathbb{R})$$

$$\begin{aligned} A' : \ & \mathbb{R}^m \longrightarrow \mathbb{R}^n \qquad \left(\because \ (\mathbb{R}^n)^* = \mathbb{R}^n \right) \\ & x \longmapsto x \circ A \end{aligned}$$

$$\begin{aligned} (x \circ A)(y) &= x \cdot (Ay) \quad , \qquad y \in \mathbb{R}^n \\ &= \sum_{i=1}^{m} x_i (Ay)_i \\ &= \sum_{i=1}^{m} x_i \sum_{j=1}^{n} a_{ij} y_j \quad , \qquad A = (a_{ij}) \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} x_i \right) y_j \\ &= \sum_{j=1}^{n} (A^T x)_j \, y_j \\ &= (A^T x) \cdot y \end{aligned}$$

$$\Rightarrow \quad A'(x) = x \circ A = A^T x$$

$$\Rightarrow \quad A' = A^T$$

<u>Thm</u> : Let v.s X, Y & lin map. $T: X \to Y$

$$R_T^{\perp} = N_{T'}$$

<u>Pf</u> : Let $l \in Y^*$ s.t $l(y) = 0 \quad \forall\, y \in R_T$ (i.e $l \in R_T^{\perp}$)

$$\begin{aligned}
&\Leftrightarrow \quad l(T(x)) = 0 \quad \forall\, x \in X \\
&\Leftrightarrow \quad (l \circ T)(x) = 0 \\
&\Leftrightarrow \quad T'(l)(x) = 0 \\
&\Leftrightarrow \quad T'(l) = 0 \\
&\Leftrightarrow \quad l \in N_{T'}
\end{aligned}$$

<u>Cor</u> : $\dim(R_T) = \dim(R_{T'})$

<u>Pf</u> : By RNT, $\quad \dim(R_{T'}) + \dim(N_{T'}) = \dim(Y^*)$

$$= \dim(Y)$$

$\because$ $\boxed{R_T^{\perp} = (Y/R_T)^*}$ $\Rightarrow \quad \dim(R_T) + \dim(R_T^{\perp}) = \dim(Y)$

$\because$ $\dim(R_T^{\perp}) = \dim(N_{T'}) \quad \Rightarrow \quad \dim(R_T) = \dim(R_{T'})$

- $L(X, Y)$: set of linear maps from X to Y
  $L(X, Y)$ is a v.s.

- $L(X, X) := L(X)$
  $L(X)$ is an algebra

  $S \cdot T \in L(X)$ s.t $S \cdot T := S \circ T$

- $GL(X)$: set of all invertible linear maps on X
  $GL(X)$ is a group

Consider the action of gp. G on $L(X)$

$$\alpha : G \times L(X) \longrightarrow L(X)$$

For $g \in G$, $\alpha_g : L(X) \longrightarrow L(X)$ is an automorphism (bij. lin. map which is also a homomp.)

$$\begin{aligned} \alpha : \quad G &\longrightarrow \mathrm{Aut}(L(X)) \\ g &\longmapsto \alpha_g \end{aligned}$$

is a gp. homomp.

For $G = GL(X)$, $\quad \alpha : GL(X) \longrightarrow \mathrm{Aut}(L(X))$

$$P \longmapsto \alpha_P$$

for $P \in GL(X)$, $\quad \alpha_P : L(X) \longrightarrow L(X)$

$$T \longrightarrow PTP^{-1}$$

$\alpha_P \in \mathrm{Aut}(L(X))$

(inj) $\quad PTP^{-1} = 0 \Rightarrow T = 0$

(surj.) $\quad$ for $T \in L(X)$, $\quad P(\underbrace{P^{-1}TP}_{\in L(X)})P^{-1} = T$

(homomp.) $\alpha_P(ST) = PSTP^{-1} = (PSP^{-1})(PTP^{-1}) = \alpha_P(S)\alpha_P(T)$

(α: homomp.)

$$\alpha_{PQ}(T) = PQ\,T(PQ)^{-1} = P(QTQ^{-1})P^{-1}$$
$$= P(\alpha_Q(T))P^{-1} = \alpha_P(\alpha_Q(T))$$
$$= \alpha_P \circ \alpha_Q(T)$$

$\Rightarrow \quad \alpha_{PQ} = \alpha_P \circ \alpha_Q$

for $T \in L(X)$,

$$Orb(T) = \{\alpha_g(T) : g \in G\}$$

orbit of T

<u>Similar maps</u>: $T, S \in L(X)$ are similar if $T \in Orb(S)$ or $S \in Orb(T)$ i.e

$T = \alpha_P(S) = PSP^{-1}$ for some $P \in GL(X)$

Similarity defines an eq. relⁿ on $L(X)$.
So, we can write $T \sim S$

For $X = \mathbb{R}^n$, $\quad G = GL_n(\mathbb{R})$, $\quad L(X) = M_{n\times n}(\mathbb{R})$

For $A, B \in M_{n\times n}(\mathbb{R})$, $\quad A \sim B$ if $\exists\, P \in GL_n(\mathbb{R})$
s.t $\quad A = PBP^{-1}$

$$BP^{-1} = P^{-1}A$$

$P^{-1} = [x_1, \dots, x_n]$

$A = [y_1, \dots, y_n]$

$$\begin{aligned} BP^{-1}(e_i) &= P^{-1}A(e_i) \\ \Rightarrow B(x_i) &= P^{-1}(y_i) \\ &= P^{-1}\left(\sum y_{ji} e_i\right) \\ &= \sum y_{ji} P^{-1}(e_i) \\ &= \sum y_{ji} x_i \end{aligned}$$

$$A(e_i) = \sum y_{ji} e_i$$

Hence, A & B represent the same linear map under diff. basis.

So, $\mathcal{E} = \{x_1, \cdots, x_n\}$, $\quad B_{\mathcal{E}} = A$

for $E = \{e_1, \cdots, e_n\}$, $\quad B_E = B$

$(\because P^{-1} = [e_1 \cdots e_n] = I_n)$

So, for $T : \mathbb{R}^n \to \mathbb{R}^n$

$$T_E = [\, T(e_1) \quad \cdots \quad T(e_n) \,]$$

$$\mathcal{E} = \{x_1, \cdots, x_n\} \Rightarrow P^{-1} = [x_1 \quad \cdots \quad x_n]$$

$$T(x_i) = \Sigma a_{ji} x_j$$

$$\Rightarrow T_{\mathcal{E}} = \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = [a_1 \quad \cdots \quad a_n]$$

for X, Y with $\dim(X) = n$, $\dim(Y) = m$

$$\mathbb{R}^n \xrightarrow{E} X \xrightarrow{T} Y \xrightarrow{C} \mathbb{R}^m$$

$$\xi = \{x_1, \dots, x_n\} \quad \eta = \{y_1, \dots, y_n\}$$

$C \circ T \circ E : \mathbb{R}^n \to \mathbb{R}^m$
gives the matrix corresponding to T
wrt basis ξ & η

$$C \circ T \circ E(e_i) = C \circ T(x_i) = C\left(\sum a_{ji} y_j\right)$$
$$= \sum a_{ji} e_j$$

$$C \circ T \circ E = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

<u>Ppⁿ</u> : Let subsp. $Y \subseteq X$ (finite). Then

$$\dim(Y^{\perp}) + \dim(Y) = \dim(X).$$

<u>Pf</u> : It suffices to show, $\quad Y^{\perp} \cong (X/Y)^{*}$

$$X \xrightarrow{\pi} X/Y \xrightarrow{l} \mathbb{F}$$

Def. $\quad L : (X/Y)^{*} \longrightarrow Y^{\perp}$

$$l \longmapsto l \circ \pi$$

This map is well-defined as $l \circ \pi$ vanishes on Y

(inj.)

Suppose $l \circ \pi = 0$. Take $x + Y$ for $x \notin Y$

$$l \circ \pi(x) = 0 \Rightarrow l(x+Y) = 0 \quad \forall\, x \in X$$

$$\Rightarrow l = 0$$

Hence, L is injective

(surj.) Let $\varphi \in Y^{\perp}$. Then $\varphi : X \to \mathbb{F}$ s.t

$\varphi(y) = 0 \quad \forall\, y \in Y$

Def. $\tilde{\varphi} : X/Y \longrightarrow \mathbb{F}$

$$\tilde{\varphi}(x+Y) = \varphi(x)$$

$\because$ φ vanishes on Y $\Rightarrow$ $\tilde{\varphi}$ is well-defined

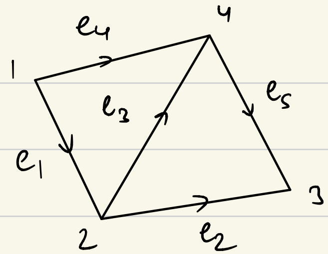
$\tilde{\varphi}$ inherits linearity from φ.

$$\tilde{\varphi} \circ \pi(x) = \tilde{\varphi}(x+Y) = \varphi(x) \quad \forall\, x \in X$$

$$\Rightarrow \quad \varphi = \tilde{\varphi} \circ \pi = L(\tilde{\varphi})$$

Hence, L is surjective.

$$
A = \begin{array}{c c}
\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\
\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} & \begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array}
\end{array}
$$



Note, $\quad C_1 + C_2 + C_3 + C_4 = 0 \Rightarrow$ non-trivial N_A

$$
N_A = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} : \alpha \in \mathbb{R} \right\}
$$

$$
Ax = 0 \Rightarrow A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_4 - x_2 \\ x_4 - x_1 \\ x_3 - x_4 \end{bmatrix} = 0 \Rightarrow x_1 = x_2 = x_3 = x_4
$$

$\Rightarrow \dim(N_A) = 1$

$\Rightarrow \dim(R_A) = 3$

$$A^T = \begin{bmatrix} -1 & 0 & 0 & -1 & 0 \\ 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

$\dim(R_{A^T}) = \dim(R_A) = 3$

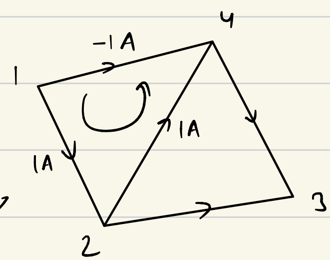
$\Rightarrow \dim(N_{A^T}) = 2$

$$A^T y = 0 \Rightarrow \quad A^T \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} -y_1 - y_4 \\ y_1 - y_2 - y_3 \\ y_2 + y_4 \\ y_3 + y_4 - y_5 \end{bmatrix}$$

If y represents currents, $A^T y = 0$ is Kirchoff's Voltage Law.

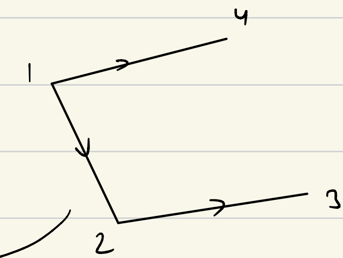
So, we can calc. N_{A^T} by considering loops in the graph

$$N_{A^T} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$



$\because$ Loops are related to N_{A^T}, for R_{A^T}, we consider subgraphs w/o loops.

$$R_{A^T} = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$



So, by Rank-Nullity Thm,

$$\dim(A^T) = \dim(N_{A^T}) + \dim(R_{A^T})$$

$$\Rightarrow \#\text{edges} = \#\text{loops} + (\#\text{nodes} - 1)$$

$$\Rightarrow \#\text{nodes} - \#\text{edges} + \#\text{loops} = 1$$

(Euler's formula)

<u>Multilinear map</u> : A map $T: \overbrace{V \times \dots \times V}^{n \text{ times}} \to X$ is multilinear if it is linear in each variable.

$$T(v_1, \dots, v_{k-1}, \lambda x + y, v_{k+1}, \dots, v_n)$$
$$= \lambda T(v_1, \dots, v_{k-1}, x, v_{k+1}, \dots, v_n)$$
$$+ T(v_1, \dots, v_{k-1}, y, v_{k+1}, \dots, v_n)$$

$\forall \; k = 1, \dots, n$

<u>eg</u> : <u>1.</u> $T: \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$

$T(x, y) = x_1 y_2$, where $x = (x_1, x_2)$ & $y = (y_1, y_2)$

$$T(\lambda x + x', y) = (\lambda x_1 + x_1')y_2 = \lambda x_1 y_2 + x_1' y_2$$
$$= \lambda T(x, y) + T(x', y)$$

$$T(x, \lambda y + y') = x_1(\lambda y_2 + y_2') = \lambda x_1 y_2 + x_1 y_2'$$
$$= \lambda T(x, y) + T(x, y')$$

2. $T: \mathbb{R}^n \times \cdots \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$$x_i = \begin{bmatrix} x_{1i} \\ \vdots \\ x_{ni} \end{bmatrix}$$

$$T(x_1, \dots, x_n) = x_{f(1)1} \, x_{f(2)2} \cdots x_{f(n)n}$$

where $f: \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$

- Sum & scalar multiple of multilinear maps is multilinear

Consider multilinear $D: \mathbb{R}^n \times \cdots \times \mathbb{R}^n \longrightarrow \mathbb{R}$ s.t

1. $D(x_1, \dots, x_n) = 0$ if $x_i = x_j$ for $i \neq j$

2. $D(e_1, \dots, e_n) = 1$ (normalization), where $\{e_1, \dots, e_n\}$ is a std. basis of $\mathbb{R}^n$.

For $n=2$, $\quad D : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$\begin{aligned}
D(x, y) &= D(x_1 e_1 + x_2 e_2 ,\ y_1 e_1 + y_2 e_2) \\
&= x_1 D(e_1, y_1 e_1 + y_2 e_2) + x_2 D(e_2, y_1 e_1 + y_2 e_2) \\
&= x_1 y_1 D(e_1, e_1) + x_1 y_2 D(e_1, e_2) \\
&\quad + x_2 y_1 D(e_2, e_1) + x_2 y_2 D(e_2, e_2) \\
&= x_1 y_2 - x_2 y_1
\end{aligned}$$

Lem : D is alternating, i.e

$$D(x_1, \dots, x_n) = -D(x_1, \dots, x_{i-1}, x_j, x_{i+1}, \dots, x_{j-1}, x_i, x_{j+1}, \dots, x_n)$$

Pf : Let $\widetilde{(a,b)} = (x_1, \dots, x_{i-1}, a, x_{i+1}, \dots, x_{j-1}, b, x_{j+1}, \dots, x_n)$

$$D(\widetilde{(a+b, a+b)}) = 0$$

$$\Rightarrow D(\widetilde{a,a}) + D(\widetilde{a,b}) + D(\widetilde{b,a}) + D(\widetilde{b,b}) = 0$$

$$\Rightarrow D(\widetilde{a,b}) = -D(\widetilde{b,a})$$

<u>Lem</u> : If $\{x_1, \ldots, x_n\}$ is lin. dep., then
$D(x_1, \ldots, x_n) = 0$

<u>Pf</u> : Suppose $\quad x_1 = \sum_{i=2}^{n} a_i x_i$

$$D(x_1, \ldots, x_n) = D\left(\sum_{i=2}^{n} a_i x_i, \ldots, x_n\right)$$
$$= a_2 D(x_2, \ldots, x_n) + \ldots + a_n D(x_n, \ldots, x_n)$$
$$= 0$$

Let us try to expand $D(x_1, \cdots, x_n)$

$$D(x_1, \cdots, x_n) = D\left(\sum_{i=1}^{n} x_{i1} e_i, \ldots, \sum_{i=1}^{n} x_{in} e_i\right)$$
$$= \sum_{f} x_{f(1)1} \cdots x_{f(n)n} \; D(e_{f(1)}, \cdots, e_{f(n)})$$

where $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$

Due to ppt. 1. of D, $D(e_{f(1)}, \cdots, e_{f(n)})$ vanishes on non-bijective maps.

So, $$D(x_1, \dots, x_n) = \sum_{\sigma \in S_n} x_{\sigma(1)1} \cdots x_{\sigma(n)n} D(e_{\sigma(1)}, \dots, e_{\sigma(n)})$$

$\because$ $\sigma = t_1 \dots t_k$, where t_i is a transposition

$$\Rightarrow D(e_{\sigma(1)}, \dots, e_{\sigma(n)}) = \underbrace{(-1)^k}_{\text{sgn}(\sigma)} D(e_1, \dots, e_n) = \text{sgn}(\sigma)$$

$$\Rightarrow D(x_1, \dots, x_n) = \sum_{\sigma \in S_n} \text{sgn}(\sigma)\, x_{\sigma(1)1} \cdots x_{\sigma(n)n}$$

<u>Determinant</u> : for an $n \times n$ matrix $A = [a_1 \dots a_n]$

$$\det(A) = D(a_1, \dots, a_n)$$

Thm: For any pair of $n \times n$ matrices A & B,

$$\det(AB) = \det(A)\det(B)$$

Pf: Suppose $\det(A) \neq 0$

$$\frac{\det(AB)}{\det(A)} = \frac{1}{\det(A)} D(AB(e_1), \dots, AB(e_n))$$

$$= \frac{1}{\det(A)} D(A(b_1), \dots, A(b_n)), \quad B = [b_1 \dots b_n]$$

Def. $C : \mathbb{R}^n \times \dots \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$$C(x_1, \dots, x_n) = \frac{1}{\det(A)} D(Ax_1, \dots, Ax_n)$$

Suppose C is a multilinear map satisfying properties of det.

Then $C(x_1, \dots, x_n) = D(x_1, \dots, x_n)$

$$\therefore \det(B) = D(b_1, \dots, b_n) = C(b_1, \dots, b_n)$$

$$= D(Ab_1, \dots, Ab_n)/\det(A)$$

$$= \det(AB)/\det(A)$$

For $x_i = x_j \quad (i \neq j)$

$$C(x_1, \dots, x_n) = \frac{1}{\det(A)} D(Ax_1, \dots, Ax_n)$$

$\because \; Ax_i = Ax_j \Rightarrow D(Ax_1, \dots, Ax_n) = 0$

$\therefore \; C(x_1, \dots, x_n) = 0$

Also, $C(e_1, \dots, e_n) = \dfrac{1}{\det(A)} D(Ae_1, \dots, Ae_n)$

$$= \frac{1}{\det(A)} D(a_1, \dots, a_n), \quad A = [a_1 \; \dots \; a_n]$$

$$= 1$$

Consider $\quad A(t) = A + tI \quad , \quad t \in [0,1]$

$\det(A+tI) = 0$ is a polynomial in t of degree n

So, it has at-most n-many rts.

$\therefore \det(A+tI) \neq 0 \quad \forall\, t \in (0,\epsilon)$ for some $\epsilon > 0$

Thus, by earlier identity,

$$\det(A(t)B) = \det(A(t))\det(B) \quad \forall\, t \in (0,\epsilon)$$

As $t \to 0$, $\quad \det(A(t)) \to \det(A)$

$$\det(A(t)B) \to \det(AB)$$

$$(A(t)B)_{ij} \to (AB)_{ij}$$

ijth entry: $\sum_k A(t)_{ik} B_{kj} \longrightarrow \sum_k A_{ik} B_{kj}$

$$d(A,B) = \sum_{i,j} |a_{ij} - b_{ij}|$$

$d(A(t), A) \to 0$ as $t \to 0$

Thm : If A & B are similar, then $\det(A) = \det(B)$

Pf : $A = PBP^{-1} \Rightarrow \det(A) = \det(PBP^{-1})$

$$= \det(P)\det(B)\det(P^{-1})$$

$$= \det(B)\det(PP^{-1})$$

$$= \det(B)$$

Thm : A is invertible $\Leftrightarrow \det(A) \neq 0$

Pf : $(\Rightarrow)$ Suppose $BA = I$

$$\Rightarrow \det(B)\det(A) = 1$$

$$\Rightarrow \det(A) \neq 0$$

$(\Leftarrow)$ (contra+)

If A is not invertible, then $\{a_1, \ldots, a_n\}$ is lin. dep. where $A = [a_1 \ \ldots \ a_n]$

$$\Rightarrow \det(A) = D(a_1, \ldots, a_n) = 0$$

Thm : $\det(A) = \det(A^T)$

Pf : Let $A = (a_{ij})$, $A^T = (b_{ij})$

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{\sigma(1)1} \cdots a_{\sigma(n)n}$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, b_{1\sigma(1)} \cdots b_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma^{-1})\, b_{\sigma^{-1}(\sigma(1))\sigma(1)} \cdots b_{\sigma^{-1}(\sigma(n))\sigma(n)}$$

$$= \det(A^T)$$

Note : $\det(A) = \sum_i (-1)^{i+j} a_{ij} \det(A_{ij})$

Trace

$tr : M_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$

$$tr(A) = \sum_i a_{ii}$$

- $tr(AB) = tr(BA)$

$$tr(AB) = \sum_i (AB)_{ii} = \sum_i \sum_k a_{ik} b_{ki} = \sum_k \left(\sum_i b_{ki} a_{ik} \right)$$
$$= \sum_k (BA)_{kk} = tr(BA)$$

Thm: If A & B are similar matrices, then $tr(A) = tr(B)$

Pf: $tr(A) = tr(P^{-1}BP) = tr(BP^{-1}P) = tr(B)$

- $tr(AA^T) = \sum_{i,j} a_{ij}$

- $tr((A-B)(A-B)^T) = \sum_{i,j} |a_{ij} - b_{ij}|^2$

<u>Eigenvalue</u> : Let $T: X \to X$ be a lin. map on a finite dim. v.s. X. Then $\alpha \in \mathbb{F}$ is an eigenvalue of T if $\exists\, v \neq 0 \in X$ s.t

$$T(v) = \alpha v$$

v is called the eigenvector corresponding to the eigenvalue α.

<u>Note</u> : - $N(T-\alpha I)$ is non-trivial

<u>OR</u>

$T-\alpha I$ is non-invertible

- $A \in M_{n\times n}(\mathbb{F})$: α is e-val $\Leftrightarrow$ $(A-\alpha I)$ is non-invertible.

$\Leftrightarrow$ $\det(A-\alpha I) = 0$

- E-vals of A are the roots of polynomial $\det(A-xI)$ which are elems. of $\mathbb{F}$.

- The poly. $p(x) = \det(A-xI)$ is called the <u>characteristic polynomial of A</u>

Lem: If A & B are similar matrices, then char. poly. of A & B are same

A. Let $B = PAP^{-1}$

$$P_A(x) = \det(A - xI) = \det(PBP^{-1} - xPP^{-1})$$
$$= \det(P(B - xI)P^{-1}) = \det(B - xI)$$
$$= P_B(x)$$

Rem: In particular, A & B have same e-vals.

Eigenspace: For an e-val λ of lin. map $T: X \to X$

$$E_\lambda = \{v \in X : T(v) = \lambda v\}$$
$$= N(T - \lambda I)$$

e-sp. corresponding to λ

Spectrum: $\sigma(A) = \{\lambda \in \mathbb{F} : \lambda \text{ is e-val of } A\}$

- Let poly. $q(x) = \sum_{n=0}^{m} a_n x^n$

$$q(A) := \sum_{n=0}^{m} a_n A^n$$

<u>Thm</u> : (Spectral Mapping Thm)

For $A \in M_{n\times n}(\mathbb{F})$ & poly. q,

$$\sigma(q(A)) = q(\sigma(A))$$
$$= \{q(\lambda) : \lambda \in \sigma(A)\}$$

<u>Pf</u> : Let $\lambda \in \sigma(A)$

Then $\exists\, v \neq 0$ s.t $Av = \lambda v$

Note,
$$q(A)v = \left(\sum_{k=1}^{n} a_k A^k\right) v$$
$$= \sum_{k=1}^{n} a_k A^k v = \sum_{k=1}^{n} a_k \lambda^k v$$
$$= \left(\sum_{k=1}^{n} a_k \lambda^k\right) v = q(\lambda) v$$

$\Rightarrow q(\lambda) \in \sigma(q(A))$

$\Rightarrow q(\sigma(A)) \subseteq \sigma(q(A))$

Let $\alpha \in \sigma(q(A))$

Consider $$q(x) - \alpha = \prod_{k=1}^{n} (x - a_k)$$

$$\Rightarrow \underbrace{q(A) - \alpha I}_{\text{non-inv.}} = \prod_{k=1}^{n} (A - \alpha_k I)$$

$\Rightarrow \exists\ 1 \leq m \leq n$ s.t $A - a_m I$ is non-inv.

$\Rightarrow a_m \in \sigma(A)$

& $q(a_m) = \alpha$

$\Rightarrow \alpha \in q(\sigma(A))$

$\Rightarrow \sigma(q(A)) \subseteq q(\sigma(A))$

Lem: E-vecs corresponding to diff e-vals are lin. indep.

Pf: Let $\alpha_1, \dots, \alpha_n$ be distinct e-vals of A.

Let $v_i \in E_{\alpha_i}(A)$

Claim: $\{v_1, \dots, v_n\}$ are lin. indep.

i.e $\sum a_i v_i = 0$

Take a poly. q_j s.t $q_j(\alpha_i) \begin{cases} \neq 0, & i=j \\ = 0, & i \neq j \end{cases}$

$$0 = q_j(A)(0) = q_j(A)\left(\sum a_i v_i\right) = \sum a_i\, q_j(A)\, v_i$$
$$= a_j \underbrace{q_j(\alpha_j)}_{\neq 0}$$

$\Rightarrow a_i = 0 \quad \forall\ 1 \leq i \leq n$

Diagonalizability

A lin. map $T: V \to V$ is diagonalizable if $\exists$ a basis for V which consists of e-vecs. of T.

If $\{x_1, \dots, x_n\}$ is a basis consisting of e-vecs of T.

Then, $T(x_i) = \lambda_i x_i$

$$\Rightarrow T = [T(x_1) \ \dots \ T(x_n)]$$
$$= \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

Diagonalizable is same as being similar to a diagonal matrix.

<u>Diagonalizable matrix</u> : A matrix is diagonalizable if it has a basis consisting of e-vecs of A.

eg: 1. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$p_A(x) = x^2+1$ → no real rts.

⇒ A is not diagonalizable as a linear map on $\mathbb{R}^2$

2. $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$p_A(x) = x^2$ which has 0 as a rt.

⇒ 0 is an eigenvalue of A.

$E_0(A) = N(A)$ & $\dim(N(A)) = 1$

∴ $E_0(A)$ can not generate $\mathbb{R}^2$.

Hence, A is not diagonalizable

3. $A = \begin{bmatrix} 2 & 8 \\ 2 & 2 \end{bmatrix}$

$p_A(x) = (x-2)^2 - 16$ ⇒ E-val : $-2, 6$

∵ E-vecs corresponding to $-2, 6$ are lin. indep.

Hence, A is diagonalizable.

<u>Note</u>: Char. poly. of a diagonalizable lin. map is of the form $(x-\lambda_1)^{d_1}(x-\lambda_2)^{d_2}\cdots(x-\lambda_m)^{d_m}$ where $d_i = \dim(E_{\lambda_i})$

<u>Thm</u>: Let T be a lin. map on a fin. dim v.s. V. Then the following are eq.

1. T is diagonalizable

2. $P_T(x) = (x-\lambda_1)^{d_1} \cdots (x-\lambda_m)^{d_m}$ s.t $d_1 + \cdots + d_m = n$
   & $\dim(E_{\lambda_i}(T)) = d_i$
   $\forall\ 1 \le i \le m$

3. $\dim(V) = \sum_{i=1}^{m} \dim(E_{\lambda_i}(T))$

<u>Pf</u> : $1 \Rightarrow 2$ Trivial

$2 \Rightarrow 3$ $\quad \dim(V) = d_1 + \dots + d_m = \sum_{i=1}^{m} \dim(E_{\lambda_i}(T))$

$3 \Rightarrow 1$

Let $\quad E = E_{\lambda_1}(T) + \dots + E_{\lambda_m}(T)$

$$x = x_1 + \dots + x_m \quad ; \quad x_i \in E_{\lambda_i}(T)$$

$$= y_1 + \dots + y_m \quad ; \quad y_i \in E_{\lambda_i}(T)$$

$\Rightarrow (x_1 - y_1) + \dots + (x_n - y_n) = 0 \quad , \quad y_i - x_i \in E_{\lambda_i}(T)$

$\because$ e-vecs corresponding to diff. e-vals are lin. indep.

$\therefore \quad x_i = y_i \quad \forall \; 1 \leq i \leq m$

$\Rightarrow \quad E = \bigoplus_{i=1}^{m} E_{\lambda_i}(T)$

$\Rightarrow \quad \dim(E) = \sum_{i=1}^{m} \dim(E_{\lambda_i}(T)) = \dim(V)$

$\because$ E is subsp. of V

$\therefore \quad E = V$

Thus, $\exists$ a basis of V which consists of e-vecs.

Let $A \in M_{m\times n}(\mathbb{F})$ be a diagonalizable matrix.

e-vals : $\lambda_1, \dots, \lambda_n$

e-vecs : $x_1, \dots, x_n$ forming basis of $\mathbb{F}^m$

$P = [x_1 \; \dots \; x_n]$

<u>Claim</u> : $P^{-1}AP = D$

$$P^{-1}AP(e_i) = P^{-1}A(x_i) = P^{-1}(\lambda_i x_i) = \lambda_i P^{-1}(x_i) = \lambda_i e_i \quad \forall i$$

for $A \in M_{m\times n}(\mathbb{F})$,

$$p_A(x) = \det(xI - A) \qquad , \qquad B = xI - A = [b_{ij}]$$

$$= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, b_{\sigma(1)1} \cdots b_{\sigma(n)n} \qquad b_{ij} = x\delta_{ij} - a_{ij}$$

$$= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\,(x\delta_{\sigma(1)1} - a_{\sigma(1)1}) \cdots (x\delta_{\sigma(n)n} - a_{\sigma(n)n})$$

$$= \underset{\uparrow}{1} \cdot x^n - \underbrace{(a_{11} + \dots + a_{nn})}_{\mathrm{tr}(A)} x^{n-1} + \dots + (-1)^n \underbrace{\underset{\uparrow}{\sum_{\sigma \in S_n}} \mathrm{sgn}(\sigma)\, a_{\sigma(1)1} \cdots a_{\sigma(n)n}}_{\det(A)}$$

(# σ fixing all $1, \dots, n$ i.e. $\sigma = e$)

(# σ fixing none of $1, \dots, n$)

So, $\sum_{i=1}^{n} \lambda_i = \mathrm{tr}(A)$, $\prod_{i=1}^{n} \lambda_i = \det(A)$

Annhilating ideal

for $T \in \mathcal{L}(V_{\mathbb{F}})$

$$\text{Ann}(T) = \{ p \in \mathbb{F}(x) : p(T) = 0 \}$$

polynomial

Consider $\{1, T, \ldots, T^{n^2}\} \subseteq \mathcal{L}(V_{\mathbb{F}})$

$\because \dim(\mathcal{L}(V_{\mathbb{F}})) = n^2 \Rightarrow \{1, T, \ldots, T^{n^2}\}$ is lin. dep.

$\Rightarrow \exists\, a_i \in \mathbb{F}$ s.t $\sum_{i=0}^{n^2} a_i T^i = 0$

$\Rightarrow q(T) = 0$ where $q(x) = \sum_{i=0}^{n^2} a_i x^i$

Ideal (I) : Subsp. of A (algebra) s.t
$fg \in I$ whenever $f \in I$ & $g \in A$

$\because$ For $p \in Ann(T)$ & $q \in \mathbb{F}(x)$,

$$(pq)(T) = \underbrace{p(T)}_{0} q(T) = 0 \Rightarrow pq \in Ann(T)$$

$\therefore$ $Ann(T)$ is an ideal of $\mathbb{F}(x)$

Morphism : $\pi : A \to B$

π is linear map &

$\pi(ab) = \pi(a)\pi(b)$

$Ker(\pi) = \{ a \in A : \pi(a) = 0 \}$

Clearly, $Ker(\pi)$ is a subsp.

Moreover, for $a \in Ker(\pi)$, $b \in A$

$$\pi(ab) = \underbrace{\pi(a)}_{0}\pi(b) = 0 = \pi(b)\underbrace{\pi(a)}_{0} = \pi(ba)$$

Hence, $Ker(\pi)$ is an ideal.

Infact, every ideal arises from kernel of a morphism

<u>Generator of ideal</u> : $I = \langle a_1, a_2 \rangle$ means

$\forall f \in I, \exists g_1, g_2 \in A$ s.t $f = a_1 g_1 + a_2 g_2$

$I = \langle a \rangle$ means $\forall f \in I, \exists g \in A$ s.t $f = ag$

<u>Claim</u> : Every ideal of $\mathbb{F}(x)$ is simply generated.

<u>Pf</u> : Let $I \neq \{0\}$ be an ideal.

Take $p \in I$ of min. degree s.t p is monic

(coeff. of highest degree monomial in p is 1)

Such a p is unique.

(If it werent, $p - q = $ (poly. of degree $< \deg p$)

$= 0 \Rightarrow p = q$)

Take $q \in I$. Then $q = ph + r$, $r \in I$ s.t $\deg r < \deg p$

$\Rightarrow r = 0$

$\Rightarrow q = ph$

$\Rightarrow I = \langle p \rangle$

Singly generated ideals are called <u>principle ideal domain</u>.

By the claim, $Ann(T) = \langle m_T \rangle$

We say m_T is the <u>minimal polynomial</u> for T.

<u>Note</u>: If A & B are similar, $m_A = m_B$

$\because$ $A = P^{-1}BP \Rightarrow f(A) = P^{-1} f(B) P$, $\forall f \in \mathbb{F}(x)$

$\Rightarrow Ann(A) = Ann(B)$

$\Rightarrow m_A = m_B$

<u>Thm</u>: $m_T(\alpha) = 0$ iff α is e-val of T.

<u>Pf</u>: $(\Leftarrow)$ $Tv = \alpha v$, for some $v \neq 0$

$\Rightarrow \underbrace{m_T(T)}_{0} v = m_T(\alpha) v$

$\Rightarrow m_T(\alpha) = 0$

$(\Rightarrow)$ $m_T = (x - \alpha) p$

$\underbrace{m_T(T)}_{0} = (T - \alpha I) p(T)$

$\because$ $\deg p < \deg m_T \Rightarrow p(T) \neq 0$

$\therefore$ $\exists\, v \neq 0$ s.t $p(T)v \neq 0$

$$\Rightarrow \underbrace{0v}_{0} = (Tv - \alpha v)\, \underbrace{p(T)v}_{\neq 0}$$

$$\Rightarrow Tv = \alpha v$$

Hence α is an eigenvalue of T.

For any matrix T, $m_T \mid p_T \Leftrightarrow p_T(T) = 0$
(char poly.)

Suppose T has eigenvalues $\lambda_1, \dots, \lambda_m$.

$$p_T(x) = \prod_{i=1}^{m} (1-\lambda_i)^{d_i}$$

$$m_T(x) = \prod_{i=1}^{m} (1-\lambda_i)^{m_i} \quad ; \quad m_i \leq d_i \quad \forall\ 1 \leq i \leq n$$

If T is diagonalizable, $T \simeq \begin{bmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{bmatrix}$

with $\alpha_1, \ldots, \alpha_n$ being distinct eigenvalues.

then, $m_T(x) = (x-\alpha_1) \ldots (x-\alpha_n)$

eg: 1. $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$e_1 \to e_2$
$e_2 \to e_3$
$e_3 \to 0$

So, $A^3 = 0$

$P_A(x) = x^3$

$m_A(x) = x^3$

2. $A = \begin{bmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{bmatrix}$

$e_1 \to e_2$
$e_2 \to e_3$
$e_3 \to ae_1 + be_2 + ce_3$

$\therefore \{e_1, Ae_1, A^2e_1\} = \{e_1, e_2, e_3\}$

Suppose $m_A(x) = x^2 + dx + f$

$\Rightarrow m_A(A)e_1 = A^2e_1 + dAe_1 + fe_1$

$= e_3 + de_2 + fe_1 = 0 \to$ Contd"

$\Rightarrow \deg m_A > 2$

So, $m_A(x) = p_A(x)$

<u>Thm</u>: (Cayley-Hamilton)

For any lin map T on a fin. dim. v.s. V,

$$p_T(T) = 0 \iff m_T \mid p_T$$

If, $p(x) = \sum_{n=1}^{m} a_n x^n$, $q(x) = \sum_{k=0}^{r} b_k x^k$

$$pq(x) = \sum_{l=0}^{m+r} c_l x^l \quad , \quad c_l = \sum_{n+k=l} a_n b_k$$

If , $p(x) = \sum_{n=0}^{m} A_n x^n$, $q(x) = \sum_{k=1}^{r} B_k x^k$

$$= \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & & \vdots \\ p_{n1} & \cdots & p_{nn} \end{bmatrix} \quad , \quad p_{ij} - \left(\begin{matrix} \text{polynomial} \\ \text{in } x \end{matrix} \right)$$

$$pq(x) = \sum_{l=1}^{m+r} C_l x^l \quad , \quad C_l = \sum_{n+k=l} A_n B_k$$

Consider $T \in M_{n \times n}(\mathbb{F})$

$$p(T) = \sum_{n=1}^{m} A_n T^n \quad , \quad q(T) = \sum_{k=1}^{r} B_k T^k$$

$$pq(T) = \sum_{n,k} A_n T^n B_k T^k$$

Suppose, T commutes with B_k $\forall\ k = 1, \dots, r$

$$\Rightarrow \quad pq(T) = \sum_{l=1}^{m+r} C_l T^l, \qquad C_l = \sum_{n+k=l} A_n B_k$$

<u>Pf</u>: (Cayley - Hamilton)

Let $T \in M_{n\times n}(\mathbb{F})$. Consider $q(x) = xI - T$ & suppose $p(x)$ is the transpose of the matrix of cofactors of $q(x)$.

i.e. $$p(x)_{ji} = (-1)^{i+j} \det(\underbrace{\tilde{q}(x)_{ij}})$$

$\tilde{q}(x)_{ij}$ is the matrix obtained from $q(x)$ by i^{th} row & j^{th} col.

Then, $$p(x)\,q(x) = \det(xI - T)\,I = \underbrace{p_T(x)}_{\substack{\text{char. poly.} \\ \text{of } T}}\,I$$

$$\Rightarrow \quad p(T)\,q(T) = p_T(T)\,I$$

as T commutes with coeffs. of q (i.e $xI - T$)

$\Rightarrow \quad 0 = p_T(T)\, I \qquad (\because\ q(T) = 0)$

$\Rightarrow \quad p_T(T) = 0$

Projection

A lin. map. on a fin. dim. v.s V is a proj. if $P^2 = P$

eg : 1. $P = I, 0$

2. $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

- $v \in R_P \iff Pv = v$

$\because\ v = Pw \Rightarrow Pv = P^2 w = Pw = v$

- $v \in N_P \iff Pv = 0$

– $R_P \oplus N_P = V$

Let $x \in R_P \cap N_P \Rightarrow Px = 0 = x$

$\Rightarrow x = 0$

For $x \in V$, $x = \underbrace{Px}_{\in R_P} + \underbrace{(x - Px)}_{\in N_P}$

Hence, P is diagonalizable.

Suppose $V = W_1 \oplus W_2$.

A proj. P is said to <u>project V to W_1</u> if

$P(W_1) = W_1$ & $P(W_2) = \{0\}$

Consider $V = W_1 \oplus \dots \oplus W_n$

Let $P_j : V \to V$ defined by

$$P_j(v = v_1 + \dots + v_n) = v_j \quad , \qquad v_k \in W_k \ \forall\, k = 1, \dots, n$$

<u>Claim</u> : P_j is a proj. with

$$R_{P_j} = W_j \ , \quad N_{P_j} = \bigoplus_{\substack{k=1 \\ k \neq j}}^{n} W_k$$

$$P_j^2(v) = P_j(P_j(v_1 + \dots + v_n)) = P_j(v_j)$$
$$= P_j(0 + \dots + v_j + \dots + 0) = v_j$$
$$= P_j(v) \qquad \forall\, v \in V$$
$\Rightarrow P^2 = P$

$R_{P_j} \subseteq W_j$ & $\because$ $P_j(x) = x \quad \forall\, x \in W_j$
$\Rightarrow R_{P_j} = W_j$

$P_j(v_1, \dots, v_{j-1}, 0, v_{j+1}, \dots, v_n) = 0$ i.e $P(v) = 0$
$\Leftrightarrow v_j = 0$
$\Rightarrow N_{P_j} = \bigoplus_{k=1,\, k \neq j}^{n} W_k$

- $P_i P_j = 0 \quad \forall\ i \neq j$
- $P_1 + P_2 + \dots + P_n = I$

$\because \quad P_i P_j(v) = P_i(v_j) = 0$

$$(P_1 + \dots + P_n)(v) = P_1(v) + \dots + P_n(v)$$
$$= v_1 + \dots + v_n = v$$

Thm : $V = W_1 \oplus \dots \oplus W_n$

$\iff$

$\exists$ proj. $P_1, \dots, P_n$ s.t

1. $P_i P_j = 0 \quad \forall\ i \neq j$
2. $P_1 + \dots + P_n = I$
3. $R_{P_i} = W_i \quad \forall\ i$

Pf : $(\Rightarrow)$ Already proven

($\Leftarrow$) Applying P_i in 2., we get $P_i^2 = P_i$

T'fore P_is are projections.

for $v \in V$, $v = \underbrace{P_1 v}_{\in W_1} + \ldots + \underbrace{P_n v}_{\in W_n}$

$\Rightarrow \quad V = W_1 + \ldots + W_n$

Suppose, $v = v_1 + \ldots + v_n$, $v_i \in W_i$

$$
\begin{aligned}
\Rightarrow \quad P_i v &= P_i v_1 + \ldots + P_i v_n \\
&= P_i v_i \qquad \left[\because P_i P_j = 0 \;\&\; R_{P_j} = W_j \right. \\
&= v_i \quad \forall i \qquad \left. \Rightarrow P_i w = 0 \quad \forall\, w \in W_j, \; j \neq i \right]
\end{aligned}
$$

So, representation of any vector $v \in V$ is <u>unique</u>

Hence, $V = W_1 \oplus \ldots \oplus W_n$

Invariant subspace: Let $T: V \to V$ & let W be a subsp of V. Then W is an invariant subsp for T if $T(x) \in W$ whenever $x \in W$ ie $T(W) \subseteq W$

Consider a lin. map. T on $W_1 \oplus \dots \oplus W_n$ s.t $T(W_i) \subseteq W_i \;\forall\, i$

$n=2$: $V = W_1 \oplus W_2$

Basis of V: $\{\underbrace{x_1, \dots, x_n}_{\text{basis of } W_1}, \underbrace{y_1, \dots, y_m}_{\text{basis of } W_2}\}$

$$T(x_i) = a_{1i} x_1 + \dots + a_{ni} x_n + 0 y_1 + \dots + 0 y_m$$
$$T(y_j) = 0 x_1 + \dots + 0 x_n + b_{1j} y_1 + \dots + b_{mj} y_m$$

$$T = \begin{bmatrix} a_{11} & \cdots & a_{1n} & & & \\ \vdots & & \vdots & & 0 & \\ a_{n1} & \cdots & a_{nn} & & & \\ & & & b_{11} & \cdots & b_{1m} \\ & 0 & & \vdots & & \vdots \\ & & & b_{m1} & \cdots & b_{mm} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

Block-diagonal representation of T

Sim., if for T on $W_1 \oplus \dots \oplus W_n$,
$T(W_i) \subseteq W_i$, $\forall i$,

$$T = \begin{bmatrix} T_1 & & 0 \\ & \ddots & \\ 0 & & T_n \end{bmatrix}$$

where $T_i : W_i \to W_i$ is the restriction of T to W_i i.e $T_i = T|_{W_i}$

<u>Thm</u>: Let T be a lin. map on $W_1 \oplus \dots \oplus W_n$.
Then $T(W_i) \subseteq W_i$ $\forall i \Leftrightarrow P_j T = T P_j$, where P_j's are the projs. as in the earlier thm

<u>Pf</u>: $(\Leftarrow)$ $Tw_j = TP_j w_j = P_j T w_j$
$\Rightarrow Tw_j \in R_{P_j} = W_j$ $\forall w_j \in W_j$, $\forall j$
$\Rightarrow T(W_j) \subseteq W_j$ $\forall j$

($\Rightarrow$) Let $v = v_1 + \dots + v_n$, $v_i \in W_i \ \forall i$

$$\Rightarrow Tv = Tv_1 + \dots + Tv_n$$

$$\Rightarrow P_j T v = P_j T v_1 + \dots + P_j T v_n$$

$$= P_j T v_j \qquad [\because T v_i \in W_i]$$

$$= T v_j$$

$$= T P_j v$$

$$\Rightarrow P_j T = T P_j$$

<u>Note</u>: <u>1.</u> $P_1 = \begin{bmatrix} I & & 0 \\ & 0 & \\ & & \ddots \\ 0 & & 0 \end{bmatrix}$

<u>2.</u> For diagonalizable matrix with e-vals $\lambda_1, \dots, \lambda_n$

$$V = E_{\lambda_1} \oplus \dots \oplus E_{\lambda_n}$$

$$T = \begin{bmatrix} \lambda_1 I & & 0 \\ & \ddots & \\ 0 & & \lambda_n I \end{bmatrix}$$

Thm: (Primary Decomposition Theorem)

Let T be a linear map on finite dim. v.s V. Suppose that $p = p_1^{r_1} \cdots p_k^{r_k}$ is the minimal poly. of T, where p_i's are distinct irreducible poly. over $\mathbb{F}$, and r_i's are positive integers. Then,

1. $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$, where $W_i = N(p_i(T)^{r_i})$
2. W_i is invariant under T
3. If T_i is the map induced by T on W_i, then minimal poly. of T_i is $p_i^{r_i}$ $\forall\ i = 1, \ldots, k$

eg: 1. Let $A \in M_{r \times r}(\mathbb{F})$

$m_T = (x-a)^n (x-b)^m$, $a \neq b$

$\mathbb{F}^r = W_1 \oplus W_2$

$W_1 = N((A - aI)^n)$

$W_2 = N((A - bI)^m)$

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad \begin{aligned} A_1 &= A|_{W_1} \\ A_2 &= A|_{W_2} \end{aligned}$$

<u>Generalized e-vecs</u> : for a lin map T & an eigenvalue α of T, then vecs. in the space $N((T-\alpha)^l)$ are called <u>generalized e-vecs</u>. $l \in \mathbb{Z}_{>0}$

$$N(T-\alpha) \subseteq N((T-\alpha)^2) \subseteq N((T-\alpha)^3) \subseteq \ldots$$

But, since V is fin. dim., after some pt.

$$N((T-\alpha)^n) = N((T-\alpha)^{n+1})$$

The smallest such n is called <u>index of the e-val</u>.

Now,

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} - \underbrace{\begin{bmatrix} aI & 0 \\ 0 & bI \end{bmatrix}}_{\text{diagonal matrix}} = \begin{bmatrix} (A_1 - aI) & 0 \\ 0 & (A_2 - bI) \end{bmatrix} = B$$

So, for $\max(m,n) = n$, $\underbrace{B^n}_{\text{nilipotent matrix}} = 0$

Hence, every matrix whose minimal poly. is a product of linear factors can be expressed as

(or A is a matrix over an algebraically closed field)

$A \sim D + N$

N - nilipotent matrix

D - diagonal matrix

$DN = ND$

Let us try showing

$$N((A-a)^{n-1}) \subsetneq N((A-a)^n) = N((A-a)^{n+1})$$

$$N(A-aI) = N\begin{pmatrix} A_1 - aI & 0 \\ 0 & A_2 - aI \end{pmatrix} \subseteq W_1 \oplus \{0\}$$

$\because$ $(x-a)^n$ is minimal poly. of A_1

$\therefore$ $N((A-a)^n) = W_1$ & $N((A-a)^{n-1}) \subsetneq N((A-a)^n) = W_1$

otherwise minimal poly. would have had lesser degree.

Also, $N((A-a)^{n+1}) \subseteq W_1 = N((A-a)^n)$

$\Rightarrow N((A-a)^n) = N((A-a)^{n+1})$

2. A complex matrix is diagonalizable iff the minimal poly. is product of linear factors.

$\underline{\text{Pf}}$: (Primary Decomposition Theorem)

Def. $\quad f_i = p/p_i^{r_i} \quad \forall\ i = 1, \dots, k$

Then $\ f_1, \dots, f_k$ are relatively prime i.e

$\gcd(f_1, \dots, f_k) = 1$

$$\langle f_1, f_2, \dots, f_k \rangle = \langle d \rangle$$
$$\Rightarrow \quad f_i = d h_i \quad \forall\ i = 1, \dots, k$$
$$\Rightarrow \quad d = 1$$

$$\Rightarrow \quad 1 = \sum_{i=1}^{k} f_i g_i \quad \text{for some polys. } g_i$$

Def. $\quad P_i = f_i(T)\, g_i(T) \quad \forall\ i = 1, \dots, k$

$$\Rightarrow \quad 1 = P_1 + P_2 + \dots + P_k$$

$$P_i P_j = f_i(T)\, g_i(T)\, f_j(T)\, g_j(T), \qquad i \neq j$$
$$= f_i(T)\, f_j(T)\, g_i(T)\, g_j(T)$$

$\because\ p \mid f_i f_j \Rightarrow f_i(T) f_j(T) = 0 \Rightarrow P_i P_j = 0$

(minimal poly)

Now, we need to show $\text{Range}(P_i) = W_i = N(p_i(T)^{r_i})$

For $x \in \text{Range}(P_j)$,

$$
\begin{aligned}
p_j(T)^{r_j} x &= p_j(T)^{r_j} P_j x \\
&= \underbrace{p_j(T)^{r_j} f_j(T)}_{0} g_j(T)\, x = 0
\end{aligned}
$$

$\Rightarrow \text{Range}(P_j) \subseteq W_j$

Take $x \in W_i \Rightarrow p_i^{r_i}(T) x = 0$

$$P_j(x) = \underbrace{f_j(T) g_j(T)}_{\text{div. by } p_i^{r_i}(T)} x = 0$$

$\Rightarrow P_j(W_i) = 0, \quad i \neq j$

Then, for $x \in W_j$,

$$x = Ix = P_1(x) + \dots + P_n(x) = P_j(x)$$

$\Rightarrow W_j \subseteq \text{Range}(P_j)$

$\Rightarrow W_j = \text{Range}(P_j)$

for $x \in W_i$, $\quad 0 = T\, P_i(T)^{r_i} x = P_i(T)^{r_i}(Tx)$

$\Rightarrow \; Tx \in W_i$

$\Rightarrow \; W_i$ is invariant under T.

$$T = \begin{bmatrix} T_1 & & 0 \\ & \ddots & \\ 0 & & T_k \end{bmatrix}, \qquad T_i = T\big|_{W_i}$$

So, $P_i^{r_i}(T_i) = P_i^{r_i}(T)\big|_{W_i}$

$$\Rightarrow P(T) = \begin{bmatrix} P(T_1) & & 0 \\ & \ddots & \\ 0 & & P(T_k) \end{bmatrix}$$

Let g be a poly. s.t $g(T_i) = 0$

$$g(T)\, f_i(T) = \begin{bmatrix} g(T_1) f_i(T_1) & & 0 \\ & \ddots & \\ 0 & & g(T_k) f_i(T_k) \end{bmatrix}$$

$$f_i(T_j) = f_i(T)\big|_{W_j}, \quad i \neq j$$
$$= 0$$

So, only $g(T_i)\, f_i(T_i)$ remains. But $g(T_i) = 0$

$\Rightarrow \quad g(T)\, f_i(T) = 0$

As p is minimal poly. for T, $\quad p \mid g f_i$

$\Rightarrow p_i^{r_i} f_i \mid g f_i$

$\Rightarrow p_i^{r_i} \mid g$

Let $T : V \to V$ with $m_T(x) = p_1^{r_1} \cdots p_k^{r_k}$

& $\quad p_T(x) = p_1^{d_1} \cdots p_k^{d_k}$

(char. poly.)

Then, $\quad \dim(V) = \sum_{i=1}^{k} \deg(p_i^{d_i})$

<u>Claim</u>: $\quad \dim(W_i) = \deg(p_i^{d_i}) = d_i$ (if p_i is linear)

$T \simeq D + N$, $\qquad D = \begin{bmatrix} \lambda_1 I_{W_1} & & \\ & \ddots & \\ & & \lambda_k I_{W_k} \end{bmatrix}$

$p_{N_i}(x) = x^{n_i}$ $\qquad N = \begin{bmatrix} N_1 & & \\ & \ddots & \\ & & N_k \end{bmatrix}$

$\Rightarrow p_N(x) = \prod_{i=1}^{k} x^{n_i}$

$\sum n_i = \dim(V)$

$N_i \simeq T_i - D_i = T_i - \lambda_i I$, $\qquad p_{N_i} = x^{n_i}$

$\Rightarrow \quad T_i \simeq N_i + \lambda_i I$

$$\begin{aligned} p_{T_i} &= \det(T_i - xI) \\ &= \det(N_i - (x-\lambda_i) I) \\ &= (x-\lambda_i)^{n_i} \end{aligned}$$

$$p_T = \prod_{n=1}^{k} (x-\lambda_i)^{n_i} \quad \Rightarrow \quad d_i = n_i = \dim(W_i)$$

eg: 1. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$p_A(x) = (x+1)^2(x-2)$

$m_A(x) = (x+1)(x-2)$

$W_1 = N(A+I) = \text{span}\{(1,-1,0), (0,1,-1)\}$

$W_2 = N(A-2I) = \text{span}\{(1,1,1)\}$

$\mathcal{B} = \{(1,-1,0), (0,1,-1), (1,1,1)\}$

$A(1,-1,0) = (-1,1,0) = -(1,-1,0)$

$A(0,1,-1) = (0,-1,1) = -(0,1,-1)$

$A(1,1,1) = (2,2,2) = 2(1,1,1)$

$$A_{\mathcal{B}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A_2 = [2]$$

2. $$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$p_A(x) = (x-1)^3$

$m_A(x) = (x-1)^3$

To find m_T, consider $N(A-I)$ nullity $=1$. So, $(x-1)$ can't be m_T. We want nullity of m_T to be 3

$V = W = N((A-I)^3)$

$\mathcal{B} = \{e_1, e_2, e_3\}$

$$A_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}}_{N}$$

3. $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

$p_A(x) = (x-1)^2(x-2)$

$m_A(x) = (x-1)^2(x-2)$ $\quad [\because \dim N(A-I) = 1]$

$W_1 = N((A-I)^2)$

$= \operatorname{span}\{(1,-2,0), (0,1,-1)\}$

$A - I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$W_2 = N(A-2I)$

$= \operatorname{span}\{(0,0,1)\}$

$(A-I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix}$

$\mathcal{B} = \{(1,-2,0), (0,1,-1), (0,0,1)\}$

$A - 2I = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

$A(1,-2,0) = (1,-1,-1)$

$= (1,-2,0) + (0,1,-1)$

$A(0,1,-1) = (0,1,-1)$

$A(0,0,1) = (0,0,2) = 2\cdot(0,0,1)$

$A_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$, $\quad A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$A_2 = [2]$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{N}$$

4. $A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$

$p_A(x) = x(x^2+3)$

$m_A(x) = x(x^2+3)$

$N(A) = \text{span}\{(1,-1,1)\}$

$N(A^2+3I) = \text{span}\{(1,1,0),(0,1,1)\}$

$A^2+3I = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

$\mathcal{B} = \{(1,-1,1),(1,1,0),(0,1,1)\}$

$A(1,-1,1) = (0\ 0\ 0) = 0\cdot(1,-1,1) + 0\cdot(1,1,0) + 0\cdot(0,1,1)$

$A(1,1,0) = (-1,1,2) = -(1,1,0) + 2(0,1,1)$

$A(0,1,1) = (-2,-1,1) = -2(1,1,0) + (0,1,1)$

$$A_B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad A_1 = [0], \quad A_2 = \begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix}$$

Can't write $A_B = D + N$ since x^2+3 isn't linear.

For $A \in M_{n\times n}(\mathbb{C})$.

$$A \simeq D + N, \qquad D = \begin{bmatrix} \lambda_1 I_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k I_k \end{bmatrix}$$

$$N = \begin{bmatrix} N_1 & & 0 \\ & \ddots & \\ 0 & & N_k \end{bmatrix}$$

<u>Nilpotent map</u>: A map $T: V \to V$ is nilpotent if $\exists\, \lambda \in \mathbb{Z}_{>0}$ s.t $T^{\lambda} = 0$.

The order of nilpotency is the smallest +ve integer n s.t $T^n = 0$ but $T^{n-1} \neq 0$.

If $T: V \to V$ nilpotent s.t order of nilpotency $= \dim(V)$
$= m$

$\Rightarrow \exists\, x \in V$ s.t $T^m(x) = 0$ but $T^{m-1}(x) \neq 0$

Claim: $\{T^{m-1}x, T^{m-2}x, \ldots, Tx, x\}$ is lin. indep.

Suppose $\sum_{i=0}^{m-1} a_i T^i x = 0$

Let a_j be the first non-zero coeff.

$\Rightarrow T^{(m-j-1)} \sum_{i=0}^{m-1} a_i T^i x = 0 \Rightarrow a_j T^{(m-1)} x = 0 \to$ Contd^n

$\Rightarrow a_i = 0 \quad \forall\, i = 0, \ldots, m-1.$

Hence, $\{T^{m-1}x, \ldots, Tx, x\}$ are lin. indep.
& thus form a basis (say B)
$\Rightarrow V = \text{span}\{T^{m-1}x, \ldots, Tx, x\}$

$T(T^{(m-1)}x) = 0 = 0 \cdot T^{(m-1)}x + \ldots + 0 \cdot x$
$T(T^{(m-2)}x) = T^{(m-1)}x = 1 \cdot T^{(m-1)}x + \ldots + 0 \cdot x$
$\vdots \qquad \vdots \qquad \vdots$
$T(x) = Tx = 0 \cdot T^{(m-1)}x + \ldots + 1 \cdot Tx + 0 \cdot x$

So, $$T_B = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ & & & & 1 \\ 0 & 0 & \cdots & & 0 \end{bmatrix}$$

This is called <u>Jordan block</u>.

<u>Cyclic map</u> : A lin. map. T on V is cyclic if $\exists\ x (\neq 0) \in V$ s.t span$\{x, Tx, \ldots, T^{n-1}x\} = V$ for some $n \in \mathbb{Z}_{>0}$. In such a case, x is called cyclic vector for T.

<u>eg</u>: <u>1.</u> $T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\because\ Tx = 0\ \ \forall x \Rightarrow [x, Tx]$ cannot form a basis

So, T is <u>not</u> cyclic.

<u>2.</u> $T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $T^2 = 0$

$\Rightarrow \{x, Tx, T^2x\}$ cannot form a basis

So, T is <u>not</u> cyclic.

How to characterize non-cyclic nilpotent matrices?

Let N be a nilpotent map on V.

Suppose $NI(N) = m_1 < \dim(V)$

$\uparrow$

order of nilpotency

or

nilpotency index

Then $\{N^{m_1-1}x, N^{m_1-2}x, \ldots, Nx, x\}$ is lin. indep.

Let $V_1 = \mathrm{span}\{N^{m_1-1}x, \ldots, x\} \subsetneq V$

Note that V_1 is an invariant subsp. under N.

The space V_1 has an N-invariant complement, i.e.

N-invariant subsp. W_1 s.t $V = V_1 \oplus W_1$ *

Thus, we get an N-invariant subsp. W_1 s.t $V = V_1 \oplus W_1$

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}, \quad \begin{aligned} N_1 &= N|_{V_1} : V_1 \to V_1 \\ N_2 &= N|_{W_1} : W_1 \to W_2 \end{aligned}$$

N_1 is a cyclic nilpotent map.

* proven later

Moreover, $NI(N_2) \leq m_1$

If N_2 is cyclic, we're done.

Else $NI(N_2) = m_2 < \dim(W_1)$, then $\exists\, y$ s.t

$\{N_2^{m_2-1}y, \dots, N_2 y, y\}$ is lin. indep.

Set $V_2 = \operatorname{span}\{N_2^{m_2-1}y, \dots, N_2 y, y\}$

Then, $\exists$ an N-invar. subsp. W_2 s.t $W_1 = V_1 \oplus W_2$

Then, $V = V_1 \oplus V_2 \oplus W_2$ where V_1, V_2, W_2 are N-invar subsp. s.t $N|_{V_1}$ & $N|_{V_2}$ are cyclic.

If $N|_{W_2}$ is <u>not</u> cyclic, we repeat the process.

$\because$ V is finite dim., after finitely many steps, we get $V = V_1 \oplus \dots \oplus V_r$ s.t each V_i is invar. under N & $N|_{V_i}$ is cyclic $\forall\, i = 1, \dots, r$.

<u>Thm</u>: (Nilpotent Cyclic Decomposition)

Every nilpotent map can be decomposed into nilpotent cyclic maps (as proved above)

<u>Rem</u>: $N = \begin{bmatrix} N_1 & & 0 \\ & N_2 & \\ 0 & & \ddots \ N_r \end{bmatrix}$, $m_1 = NI(N_1) \geq NI(N_2) \geq \dots \geq NI(N_r)$

Consider $T = \begin{bmatrix} T_1 & & 0 \\ & \ddots & \\ 0 & & T_n \end{bmatrix}$, $V = W_1 \oplus \cdots \oplus W_n$

$$T \simeq D + N, \qquad D = \begin{bmatrix} \lambda_1 I_{W_1} & & 0 \\ & \ddots & \\ 0 & & \lambda_n I_{W_n} \end{bmatrix}$$

$$\Rightarrow \quad T - D \simeq \begin{bmatrix} N_1 & & 0 \\ & \ddots & \\ 0 & & N_n \end{bmatrix}$$

$$T_i - \lambda_i I_{W_i} \simeq \begin{bmatrix} J(m_1) & & 0 \\ & \ddots & \\ 0 & & J(m_r) \end{bmatrix}$$

by Nilpotent
Cyclic Decomposition

$$T_i \simeq \begin{bmatrix} J(m_1) + \lambda_i I_{W_i} & & 0 \\ & \ddots & \\ 0 & & J(m_r) + \lambda_i I_{W_i} \end{bmatrix} = \begin{bmatrix} J_{\lambda_i}(m_1) & & 0 \\ & \ddots & \\ 0 & & J_{\lambda_r}(m_r) \end{bmatrix}$$

$$J(m_1) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}, \qquad J(m_1) + \lambda_i I_{W_i} = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & \lambda_i \end{bmatrix}$$

Thm : (Jordan Canonical Form)

Let T be a linear map over a complex vector sp. V, and let $\lambda_1, \dots, \lambda_n$ be the distinct e-vecs. of T. Then $\exists$ a basis of V w.r.t. which T is a block diagonal matrix & each of the block is of the form $J_{\lambda_i}(m_i)$.

Moreover, Jordan block corresponding to each of the e-vals appears atleast once.

eg : 1. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$p_A(x) = (x-1)^3$

$m_A(x) = (x-1)^2$

$A - I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$N = A - I$, $\quad (A-I)^2 = 0$

$NI(N) = 2 < 3 \rightarrow$ not cyclic.

$V_1 = span\{Nx, x\} = span\{(1,0,0), (0,1,0)\}$

$V_2 = span\{(0,1,-1)\}$

$$N \simeq \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \quad A \simeq D+N \simeq \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} J_1(2) & 0 \\ 0 & J_1(1) \end{bmatrix}$$

2. $A = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix}$

$p_A(x) = (x+1)^3$

$m_A(x) = (x+1)^3$

$N = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

$N^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$N = A + I$.

$NI(N) = 3 = \dim(V) \rightarrow$ cyclic.

$x = (0,0,1)$ is a cyclic vector for N.

$$N \simeq \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A \simeq D+N \simeq \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= J_{-1}(3)$$

for $m > 1$

$$J(m) = \begin{bmatrix} 0 & 1 & \cdots & & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ & & \ddots & \ddots & 1 \\ 0 & \cdots & & & 0 \end{bmatrix}, \quad \dim(N(J(m))) = 1$$

$$(J(m))^2 = \begin{bmatrix} 0 & 0 & 1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 \\ & & & \ddots & 0 \\ 0 & \cdots & & & 0 \end{bmatrix}, \quad \dim(N((J(m))^2)) = 2$$

$$\vdots$$

$$\dim(N((J(m))^{m-1})) = m-1$$

$$\dim(N((J(m))^{m})) = m$$

$$\dim(N((J(m))^{m+1})) = m$$

Let A s.t $\quad p_A(x) = (x-2)^7(x-3)$

$$m_A(x) = (x-2)^3(x-3)$$

$V = W_1 \oplus W_2$, $\dim(W_1) = 7$

$$\dim(W_2) = 1$$

$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$, $A - D = \begin{bmatrix} A_1 - 2I & 0 \\ 0 & A_2 - 3I \end{bmatrix}$

$$= \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}$$

$NI(N_1) = 3$, $NI(N_2) = 1$

$N_1 = \begin{bmatrix} J(3) & & \\ & J(3) & \\ & & J(1) \end{bmatrix}$ $\left(\because \text{There will be at least one Jordan block of size } 3 \right)$

or

$$\begin{bmatrix} J(3) & & \\ & J(2) & \\ & & J(2) \end{bmatrix}$$

Working Rule to obtain Jordan Canonical Form

- A : matrix, λ : e-val of A
- a_λ : alg. mp'ty of λ, $p_A(x) = (x-\lambda)^{a_\lambda} \cdots$
- k : max. size of the Jordan block corresponding to λ, $m_A(x) = (x-\lambda)^k \cdots$
- l_i : dim. of $N(A-\lambda I)^i$, $i = 1, \dots, k$
- N_i : number of Jordan blocks of size i appearing in Jordan canonical form, $i = 1, \dots, k$

eq^n s:

1. $a_\lambda = N_1 + 2N_2 + \dots + kN_k$

2. $a_\lambda = l_k$ $\quad\left(\because \; W = N\left((A-\lambda I)^k\right) \right)$

3.
$$A - \lambda I = \begin{bmatrix} J(1) & & & & & & \\ & \ddots & & & & & \\ & & J(1) & & & & \\ & & & J(2) & & & \\ & & & & \ddots & & \\ & & & & & J(2) & & \\ & & & & & & \ddots & \\ & & & & & & & J(k) & \\ & & & & & & & & \ddots \\ & & & & & & & & & J(k) \end{bmatrix}$$

$$N(A-\lambda I) = \bigcup_{i=1}^{k} N(J(i)) \qquad \left(\because N\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = N(A) \cup N(B)\right)$$

$$\Rightarrow \quad r_1 = N_1 + N_2 + \dots + N_k$$

$\because$ for nilpotent N, $\quad N(N) \subsetneq N(N^2) \subsetneq \dots$

& because $J(m)^2$ increases nullity by 1 for $m > 1$

$$\Rightarrow \quad r_2 = r_1 + N_2 + \dots + N_k$$
$$= N_1 + 2N_2 + \dots + 2N_k$$

Sim., $\quad r_3 = r_2 + N_3 + N_4 + \dots + N_k$
$$= N_1 + 2N_2 + 3N_3 + \dots + 3N_k$$
$$\vdots$$
$$r_{k-1} = N_1 + 2N_2 + \dots + (k-1)N_{k-1} + (k-1)N_k$$
$$= a_\lambda - N_k$$

So, $\quad r_k = a_\lambda$
$$r_{k-1} = a_\lambda - N_k$$
$$r_{k-2} = a_\lambda - N_{k-1} - 2N_k$$
$$\vdots$$
$$r_1 = a_\lambda - N_2 - 2N_3 \dots - (k-1)N_k$$

eg : 1. $p_A(x) = (x-2)^7(x-3)$

$m_A(x) = (x-2)^3(x-3)$

$\dim(N(A-2I)) = 3$

$7 = 3+3+1 = 3+2+2 = \underbrace{3+2+1+1}_{\times} = \underbrace{3+1+1+1+1}_{\times}$

$\dim(N((A-2I)^2)) = 5 \Rightarrow 7 = 3+3+1$ ✓

(Nullity ↑ by 2 $\because \begin{bmatrix} J(3) & & \\ & J(3) & \\ & & J(1) \end{bmatrix}$)

So, Jordan block corresponding to e-val 2 is $\begin{bmatrix} J(3) & & \\ & J(3) & \\ & & J(1) \end{bmatrix}$

Sim, Jordan block corresponding to e-val 3 is $[J(1)]$

Hence, $A - D = \begin{bmatrix} A_1 - 2I & \\ & A_2 - 3I \end{bmatrix}$

$= \begin{bmatrix} J(3) & & & \\ & J(3) & & \\ & & J(1) & \\ & & & J(1) \end{bmatrix}$

$$\Rightarrow \quad A = \begin{bmatrix} J(3)+2I & & & \\ & J(3)+2I & & \\ & & J(1)+2I & \\ & & & J(1)+3I \end{bmatrix}$$

$$= \begin{bmatrix} J_2(3) & & & \\ & J_2(3) & & \\ & & J_2(1) & \\ & & & J_3(1) \end{bmatrix}$$

2. $p_A(x) = (x-2)^3(x-5)^2$

$m_A(x) = (x-2)^2(x-5)$

$$A = \begin{bmatrix} J_2(2) & & & \\ & J_2(1) & & \\ & & J_5(1) & \\ & & & J_5(1) \end{bmatrix}$$

3. N_1, N_2 - 6×6

same minimal poly.

$\dim(N(N_1)) = \dim(N(N_2))$

$p_{N_1}(x) = p_{N_2}(x) = x^6$

$m_{N_1}(x) = m_{N_2}(x) = x^2$

$6 = 2+2+2 = 2+2+1+1 = 2+1+1+1+1$

$\because$ Null sp. is diff. for the 3 possibilites

(nullity 3, 4, 5 resp.)

$\therefore$ Same JCF $\Rightarrow N_1 \simeq N_2$

Lem: Let $T: V \to V$ & $W \subseteq V$ be a subsp.

Then $T^{-1}(T(W)) = W + N(T)$

Pf: Clearly, $W \subseteq T^{-1}(T(W))$ & $N(T) \subseteq T^{-1}(T(W))$

$\Rightarrow \quad W + N(T) \subseteq T^{-1}(T(W))$

Let $v \in T^{-1}(T(W)) \quad \Rightarrow \quad T(v) \in T(W)$

$\Rightarrow \quad T(v) = T(x), \quad x \in W$

$\Rightarrow \quad T(v - x) = 0$

$\Rightarrow \quad v - x \in N(T)$

& $\quad v = \underbrace{x}_{\in W} + \underbrace{(v - x)}_{\in N(T)}$

Lem: Let $T: V \to V$, & W be a T-invariant subsp.

Then $T^{-1}(W)$ is also T-invariant.

Pf: $T^{-1}(W) = \{ v \in V : Tv \in W \}$

Let $v \in T^{-1}(W) \quad \Rightarrow \quad Tv \in W$

$\Rightarrow \quad T(Tv) \in T(W) = W$

$\Rightarrow \quad Tv \in T^{-1}(W)$

Let N be a nilpotent map on V with nilpotency index m.
$\Rightarrow \exists x$ s.t $N^{m-1}x \neq 0$.

Then $\text{span}\{x, Nx, \ldots, N^{m-1}x\} = V_1$ & we need to
to find an N-invariant subsp. V.

<u>Pf</u>: (Ind^n on nilpotency index m)

<u>Base case</u>: Let N be a nilpotent map with nilpotency 1.
Then $V_1 = \text{span}\{x_1\}$, $x \neq 0$ & we can find a subsp. V_2
$V = V_1 \oplus V_2$

Since N is the zero map, V_2 is invariant under N.

<u>Ind^n Hypothesis</u>: We assume that the proposition is true for any nilpotent map with nilpotency index $m-1$.

Let $N: V \to V$ be a nilpotent map with nilpotency m.
Let $V_1 = \text{span}\{x, Nx, \ldots, N^{m-1}x\}$, with $N^{m-1}x \neq 0$.

$N|_{\text{Ran}(N)} : \text{Ran} N \to \text{Ran} N$

Note that, the nilpotency index of $N|_{\text{Ran} N}$ is $m-1$.

Let $\mathcal{Y}_1 = \text{span}[Nx, N^2x, \ldots, N^{m-1}x] \subseteq \text{Ran} N$

By ind^n hyp., we get an N-invar. subsp. $\mathcal{Y}_2$ s.t

$$\text{Ran} N = \mathcal{Y}_1 \oplus \mathcal{Y}_2$$

<u>Claim</u>: $V = V_1 + N^{-1}(\mathcal{Y}_2)$

Note that $N(V_1) = \mathcal{Y}_1$

Rewriting, we get $N(V) = N(V_1) + \mathcal{Y}_2$

Taking inverse, we get

$$\begin{aligned} V &= N^{-1}(N(V)) \\ &= N^{-1}(N(V_1)) + N^{-1}(\mathcal{Y}_2) \\ &= V_1 + \underbrace{\text{Null}(N) + N^{-1}(\mathcal{Y}_2)} \end{aligned}$$

$\text{Null}(N) \subseteq N^{-1}(\mathcal{Y}_2)$ $[\because 0 \in \mathcal{Y}_2]$

$$= V_1 + N^{-1}(\mathcal{Y}_2)$$

<u>Claim</u>: $V_1 \cap Y_2 = \{0\}$

$$N(V_1 \cap Y_2) \subseteq \underbrace{N(V_1)}_{Y_1} \cap \underbrace{N(Y_2)}_{Y_2} = \{0\}$$

$\Rightarrow \quad V_1 \cap Y_2 \subseteq Null(N)$

$\because \quad V_1 \cap Null(N) = span\{N^{m-1}x\} \subseteq Y_1$

$\Rightarrow \quad V_1 \cap Y_2 \subseteq V_1 \cap Null(N) \subseteq Y_1 \cap Y_2 = \{0\}$

This proves the claim.

Now, $N^{-1}(Y_2) = Y_2 \oplus (V_1 \cap N^{-1}(Y_2)) \oplus Y_3 \quad -(*)$

$\because \; Y_2 \cap (V_1 \cap N^{-1}(Y_2)) = V_1 \cap Y_1 = \{0\}$ by the above claim.

So, $\quad V = V_1 + N^{-1}(Y_2)$

$\quad = V_1 + (Y_2 \oplus (V_1 \cap N^{-1}(Y_2)) \oplus Y_3)$

$\quad = V_1 + (Y_2 \oplus Y_3) \qquad [\because V_1 \cap N^{-1}(Y_2) \subseteq V_1]$

By $(*)$, $\quad V_1 \cap (Y_2 \oplus Y_3) = \{0\}$

$$\Rightarrow \quad V = V_1 \oplus V_2 \quad , \quad V_2 = Y_2 \oplus Y_3$$

$Y_2 \subseteq N^{-1}(Y_2)$, $Y_3 \subseteq N^{-1}(Y_2)$ $\Rightarrow$ $V_2 = Y_2 \oplus Y_3$ is N-invariant.

$T : V \to V$

$c_T(x) = p_1^{m_1} \cdots p_n^{m_n}$, p_is are prime poly. in $\mathbb{F}[x]$

$m_T(x) = p_1^{d_1} \cdots p_n^{d_n}$

$V = W_1 \oplus \cdots \oplus W_n$

$T = \begin{bmatrix} T_1 & & 0 \\ & \ddots & \\ 0 & & T_n \end{bmatrix}$, $m_{T_i}(x) = p_i(x)^{d_i}$

$\dim(W_i) = \deg(p_i^{m_i})$

$T_i - \lambda_i I = N_i$

$T - D = \begin{bmatrix} T_1 - \lambda_1 I & & \\ & \ddots & \\ & & T_n - \lambda_n I \end{bmatrix} = \begin{bmatrix} N_1 & & \\ & \ddots & \\ & & N_n \end{bmatrix}$

$N_i = \begin{bmatrix} J(m_{i1}) & & \\ & \ddots & \\ & & J(m_{ij}) \end{bmatrix}$ $\to$ Cyclic decomposition of nilpotent map.

$T_i : W_i \to W_i$, $\quad W_i = W_i^{(1)} \oplus \dots \oplus W_i^{(\lambda)}$

s.t $T_i\big|_{W_i^{(k)}} : W_i^{(k)} \to W_i^{(k)}$ is cyclic

$$\operatorname{span}\{x, Tx, \dots, T^{\ell}x\} = V$$

$A : V \to V$

$B = \{x, Ax, \dots, A^{\ell}x\}$ is a basis for V

$$A_B = \begin{bmatrix} 0 & 0 & \cdots & a_0 \\ 1 & 0 & & \vdots \\ 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_\ell \end{bmatrix}$$

$$T = \begin{bmatrix} A_i^1 & & \\ & \ddots & \\ & & A_i^k \end{bmatrix}$$

where $A^{\ell+1}x = \sum_{n=0}^{\ell} a_i A^n x$

This is called the <u>Rational Canonical Form</u>.

Inner Product Space

Let V be a vector sp. over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$).
Then an inner product (scalar prod.) is a map.

$$\langle \cdot , \cdot \rangle : V \times V \to \mathbb{F} \qquad \text{s.t}$$

1. **Linearity**: $\langle ax+y, z\rangle = a\langle x,z\rangle + \langle y,z\rangle$,
   $a \in \mathbb{F}$, $x,y,z \in V$

2. **Symmetry**: $\langle x,y\rangle = \overline{\langle y,x\rangle}$

3. **Faithfulness**: $\langle x,x\rangle \geqslant 0$ &
   $\langle x,x\rangle = 0 \iff x = 0$

A vector sp. with an inner prod. is called an inner product space.

Rem: $\langle x, by+z\rangle = \bar{b}\langle x,y\rangle + \langle x,z\rangle$
(Conjugate linearity)

eg: 1. $\mathbb{R}^n$, $\mathbb{C}^n$

$$\langle x, y\rangle = \sum_{k=1}^{n} x_k \bar{y}_k \quad , \quad x = (x_1, \dots, x_n)$$
$$y = (y_1, \dots, y_n)$$

2. $C([0,1])$: cont. fn's on $[0,1]$

$$\langle f, g\rangle = \int_0^1 f(x)\,\bar{g}(x)\,dx$$

3. $\ell^2 = \{ x = (x_n) : \sum_n |x_n|^2 < \infty \}$

$$\langle (x_n), (y_n)\rangle = \sum_n x_n \bar{y}_n$$

Norm : Length of a vector
$$\|x\| = \sqrt{\langle x, x\rangle}$$

Two vectors are said to be orthogonal/perpendicular if $\langle x, y\rangle = 0$

Thm: (Cauchy-Schwarz inequality)

Let V be an inner prod. sp. for $x, y \in V$,

$$|\langle x, y\rangle| \leq \|x\| \, \|y\|$$

Pf: $$v = y - \frac{\overline{\langle x, y\rangle}}{\|x\|^2} x$$

$$\|v\|^2 \geq 0 \Rightarrow \langle v, v\rangle \geq 0$$

$$\Rightarrow \left\langle y - \frac{\overline{\langle x, y\rangle}}{\|x\|^2} x, \; y - \frac{\overline{\langle x, y\rangle}}{\|x\|^2} x \right\rangle \geq 0$$

$$\Rightarrow \left\langle y, \; y - \frac{\overline{\langle x, y\rangle}}{\|x\|^2} x \right\rangle - \frac{\overline{\langle x, y\rangle}}{\|x\|^2} \left\langle x, \; y - \frac{\overline{\langle x, y\rangle}}{\|x\|^2} x \right\rangle \geq 0$$

$$\Rightarrow \|y\|^2 - \frac{\langle x, y\rangle \langle y, x\rangle}{\|x\|^2} - \frac{\overline{\langle x, y\rangle}}{\|x\|^2} \langle x, y\rangle$$

$$+ \frac{\overline{\langle x, y\rangle}}{\|x\|^2} \frac{\langle x, y\rangle}{\|x\|^2} \|x\|^2 \geq 0$$

$$\Rightarrow \|y\|^2 - \frac{|\langle x, y\rangle|^2}{\|x\|^2} - \frac{|\langle x, y\rangle|^2}{\|x\|^2} + \frac{|\langle x, y\rangle|^2}{\|x\|^2} \geq 0$$

$$\Rightarrow \quad |\langle x, y\rangle|^2 \leq \|x\|^2 \|y\|^2$$

$$\Rightarrow \quad |\langle x, y\rangle| \leq \|x\|^2 \|y\|^2$$

Rem: Equality occurs in Cauchy-Schwarz ineq.

iff. $v = 0 \Rightarrow y = \frac{\overline{\langle x, y \rangle}}{\|x\|^2} x$

i.e $y = \lambda x$

Thm: (Triangle inequality)

For $x, y \in V$, $\|x+y\| \leq \|x\| + \|y\|$

Pf: $\|x+y\|^2 = \langle x+y, x+y \rangle$

$$= \|x\|^2 + \langle y, x \rangle + \langle x, y \rangle + \|y\|^2$$

$$= \|x\|^2 + 2\mathrm{Re}(\langle x, y \rangle) + \|y\|^2$$

$$\leq \|x\|^2 + 2|\langle x, y \rangle| + \|y\|^2$$

$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$

$$= (\|x\| + \|y\|)^2$$

$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$

Orthonormal basis: A set $\{x_1, \dots, x_n\}$ in an inner product sp. V is an **orthonormal basis** if

1. $\{x_1, \dots, x_n\}$ is a basis for V

2. $$\langle x_i, x_j \rangle = \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$$

Fourier expansion:

$$x = \sum_{i=1}^{n} \alpha_i x_i$$

$$\langle x, x_j \rangle = \left\langle \sum_{i=1}^{n} \alpha_i x_i , x_j \right\rangle$$
$$= \sum_{i=1}^{n} \alpha_i \langle x_i, x_j \rangle = \alpha_j$$

$$\Rightarrow \quad x = \sum_{i=1}^{n} \langle x, x_i \rangle x_i$$

Bessel's identity: $$\|x\|^2 = \langle x, x \rangle = \left\langle \sum_{i=1}^{n} \langle x, x_i \rangle x_i , x \right\rangle$$
$$= \sum_{i=1}^{n} \langle x, x_i \rangle \langle x_i, x \rangle = \sum_{i=1}^{n} |\langle x, x_i \rangle|^2$$

Thm: (Gram-Schmidt)

Let V be a finite-dim. inner-prod. sp., and let $\{x_1, x_2, \ldots, x_n\}$ be a basis for V. Then $\exists$ an orthonormal basis $\{y_1, \ldots, y_n\}$ s.t $y_k \in \text{span}\{x_1, \ldots, x_k\}$ $k = 1, 2, \ldots, n$

Pf: Take $y_1 = c_1 x_1$ s.t $c_1 = 1/\|x_1\|$

Def. $y_2 = c_2(x_2 - \langle x_2, y_1 \rangle y_1)$ s.t $\|y_2\| = 1$

$$\langle y_2, y_1 \rangle = c_2 \left(\langle x_2, y_1 \rangle - \langle x_2, y_1 \rangle \underbrace{\langle y_1, y_1 \rangle}_{1} \right)$$
$$= 0$$

Suppose we have constructed $y_1, \ldots, y_{k-1}$

To construct y_k,

$$y_k = c_k \left(x_k - \sum_{i=1}^{k-1} \langle x_k, y_i \rangle y_i \right) \quad \text{s.t } \|y_k\| = 1$$

$$\langle y_k, y_j \rangle = c_k \left(\langle x_k, y_j \rangle - \underbrace{\sum_{i=1}^{k-1} \langle x_k, y_i \rangle \langle y_i, y_j \rangle}_{\langle x_k, y_j \rangle} \right)$$
$$= 0$$

Thm: (Riesz - Representation Thm)

If ℓ is a linear functional on V, then $\exists$ a unique vector $y \in V$ s.t $\ell(x) = \langle x, y \rangle$

Pf: Let $\{x_1, \dots, x_n\}$ be an o.n.b. for V.

Let $b_k = \ell(x_k) \quad \forall\ k = 1, 2, \dots, n$

Def. $\quad y = \sum_{k=1}^{n} \bar{b}_k x_k \quad \& \quad \varphi : V \mapsto \mathbb{F}$

$$x \mapsto \langle x, y \rangle$$

Note that,

$$\begin{aligned} \varphi(x_k) &= \langle x_k, y \rangle \\ &= \langle x_k, \sum_{i=1}^{n} \bar{b}_i x_i \rangle \\ &= b_k \\ &= \ell(x_k) \qquad \forall\ k = 1, 2, \dots, n \end{aligned}$$

Since $\varphi(x_k) = \ell(x_k) \quad \forall\ k = 1, 2, \dots, n$

$\Rightarrow \varphi = \ell$

For uniqueness, suppose if $l(x) = \langle x, y_1 \rangle = \langle x, y_2 \rangle$

$$\Rightarrow \langle x, y_1 - y_2 \rangle = l(x) - l(x) = 0 \quad \forall x \in V$$

In particular, for $x = y_1 - y_2$, $\langle y_1 - y_2, y_1 - y_2 \rangle = 0$

$$\Rightarrow y_1 = y_2$$

$l : V \to \mathbb{C}$

$l(x) = \langle x, y_l \rangle \quad \forall x \in V$

So, we can define an isomp.

$V^* \xrightarrow{\sim} V$

$l \mapsto y_l$

For $Y \subseteq V$, $Y^\perp = \{ l \in Y^* : l(y) = 0 \ \forall y \in Y \}$

$= \{ y_l \in V : \langle y, y_l \rangle = 0 \ \forall y \in Y \}$

<u>Thm</u>: Let V be a fin. dim. inner prod. sp. & $Y \subseteq V$ be a subsp. Then $V = Y \oplus Y^\perp$

Pf: Let $\{y_1, y_2, \dots, y_n\}$ be an o.n.b. for Y.
Extend it to a basis $\{y_1, \dots, y_n, \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_m\}$ be a basis for V.

By Gram-Schmidt, we get an o.n.b. for V as $\{y_1, y_2, \dots, y_n, x_1, \dots, x_m\}$

Then for any $v \in V$,

$$v = \underbrace{\sum_{i=1}^{n} \langle v, y_i \rangle y_i}_{\in Y} + \underbrace{\sum_{j=1}^{m} \langle v, x_j \rangle x_j}_{\in Y^{\perp}}$$

$\Rightarrow \quad V = Y + Y^{\perp}$

Suppose $v \in Y \cap Y^{\perp} \Rightarrow \langle v, v \rangle = 0 \Rightarrow v = 0$

$\Rightarrow \quad Y \cap Y^{\perp} = \{0\}$

$\Rightarrow \quad V = Y \oplus Y^{\perp}$

Define $\quad P_y : V \to V$

$$P_y(v) = \sum_{i=1}^{n} \langle v, y_i \rangle y_i \;, \quad \text{where}$$

$\{y_1, y_2, \dots, y_n\}$ is an o.n.b. for Y.

$$\begin{aligned} P_y^2(v) &= P_y\left(\sum_{i=1}^{n} \langle v, y_i \rangle y_i\right) \\ &= \sum_{i=1}^{n} \langle v, y_i \rangle \underbrace{P_y(y_i)}_{\sum_{j=1}^{n} \langle y_i, y_j \rangle y_j \,=\, y_i} \\ &= \sum_{i=1}^{n} \langle v, y_i \rangle y_i = P_y(v) \end{aligned}$$

Thus, P_y is a proj.

Given a subsp. Y & a vector $v \notin Y$, def.

$$d(v, Y) = \min\{\|v-y\| : y \in Y\}$$

Does $\exists\, x \in Y$ s.t $d(v, Y) = \|v-x\|$?

$$\|v-y\|^2 = \|v_1 + v_2^{\perp} - y\|^2 \quad ; \quad v = v_1 + v_2^{\perp},\ v_1 \in Y,\ v_2^{\perp} \in Y^{\perp}$$

$$= \|(v_1 - y) + v_2^{\perp}\|^2$$

$$= \|v_1 - y\|^2 + \|v_2^{\perp}\|^2$$

($\because$ Pythagoras thm
If $\langle p, q\rangle = 0$,
$\|p+q\|^2 = \|p\|^2 + \|q\|^2$)

For min, choose $y = v_1 = P_Y(v)$

So, $d(v, Y) = \min\{\|v-y\| : y \in Y\}$

$$= \|v_2^{\perp}\| = \|v - v_1\|$$

$$= \|v - P_Y(v)\|$$

Adjoint

$$T : X \to Y$$

$$T^* : Y^* \to X^*$$

$$T^*(\varphi)(x) = \varphi(Tx) \quad , \quad x \in X, \ \varphi \in Y^*$$

For inner prod. sp. X, Y, $\quad X^* = X$ & $Y^* = Y$

(self-dual)

We can identify φ with y_φ

$$T^* : Y \to X$$

$$\langle x, T^*(y_\varphi) \rangle = \langle Tx, y_\varphi \rangle$$

Since this holds $\forall \varphi \in Y^*$

$$\Rightarrow \langle x, T^* y \rangle = \langle Tx, y \rangle \quad \forall x \in X, y \in Y$$

Note, $\quad T_1, T_2 : X \to X$

$$\langle T_1 T_2 x, y \rangle = \langle T_1 x, T_2^* y \rangle = \langle x, T_1^* T_2^* y \rangle \quad \forall x \in X. \ y \in Y$$

So, $(T_1 T_2)^* = T_2^* T_1^*$

$$\langle P_y(v), w\rangle = \left\langle \sum_{i=1}^{n} \langle v, y_i\rangle y_i , w \right\rangle$$

$$= \left\langle \sum_{i=1}^{n} \langle v, y_i\rangle y_i , \sum_{i=1}^{n}\langle w, y_i\rangle y_i + \sum_{j=1}^{m} \langle w, x_j\rangle x_j \right\rangle$$

where $\{y_1, \ldots, y_n, x_1, \ldots, x_m\}$ is an o.n.b. for V.

$$\Rightarrow \langle P_y(v), w\rangle = \left\langle \sum_{i=1}^{n} \langle v, y_i\rangle y_i , \sum_{i=1}^{n} \langle w, y_i\rangle y_i \right\rangle$$

$$= \left\langle \sum_{i=1}^{n} \langle v, y_i\rangle y_i + \sum_{j=1}^{m} \langle w, x_j\rangle x_j , \sum_{i=1}^{n} \langle w, y_i\rangle y_i \right\rangle$$

$$= \langle v, P_y(w)\rangle$$

$$\Rightarrow \quad P_y^* = P_y$$

$A : \mathbb{C}^n \to \mathbb{C}^n \quad , \quad A = (a_{ij})$

$a_{ij} = \langle A(e_j), e_i\rangle$

$A^* : \mathbb{C}^n \to \mathbb{C}^n \quad , \quad A^* = (b_{ij})$

$b_{ij} = \langle A^*(e_j), e_i\rangle$

$= \overline{\langle e_i, A^*(e_j)\rangle} = \overline{\langle A(e_i), e_j\rangle} = \overline{a_{ji}}$

An isomp. b/w inner prod. sp. X & Y is a bij. map.

$$T : X \to Y \quad \text{s.t.}$$

$$\langle Tx_1, Tx_2 \rangle_Y = \langle x_1, x_2 \rangle_X \qquad \forall\, x_1, x_2 \in X$$

Such a map is called a <u>unitary map</u>.

$$\Rightarrow \quad \|Tx\|^2 = \|x\|^2 \quad \forall\, x \in X$$

<u>Thm</u>: Let $U : X \to X$ be a lin. map on a f.d. inner prod. sp. The following are eq.

1. U is unitary
2. $\|Ux\| = \|x\|, \quad \forall\, x \in X$
3. $U^*U = I = UU^*$
4. U maps any o.n.b of X to an o.n.b of X

Pf : $1 \Rightarrow 2$ Trivial

(Take $x_1 = x_2$ & apply def^n)

$2 \Rightarrow 1$

Clearly, U is inj. $\quad (\because \|Ux\| = 0 \Rightarrow \|x\| = 0 \Rightarrow x = 0)$

Hence, it is bij. as X is finite dim.

For $x_1, x_2 \in X$,

$$\|U(x_1 - x_2)\|^2 = \|(x_1 - x_2)\|^2$$

$$\Rightarrow \|Ux_1\|^2 + \|Ux_2\|^2 - 2Re\langle Ux_1, Ux_2\rangle$$

$$= \|x_1\|^2 + \|x_2\|^2 - 2Re\langle x_1, x_2\rangle$$

$$\Rightarrow Re\langle Ux_1, Ux_2\rangle = Re\langle x_1, x_2\rangle \quad \forall\, x_1, x_2 \in X$$

Replace x_1 with ix_1

$$\Rightarrow Re\langle Uix_1, Ux_2\rangle = Re\langle ix_1, x_2\rangle$$

$$\Rightarrow Re\; i\langle Ux_1, Ux_2\rangle = Re\; i\langle x_1, x_2\rangle$$

$$\Rightarrow -Img\langle Ux_1, Ux_2\rangle = -Img\langle x_1, x_2\rangle$$

$$\Rightarrow Img\langle Ux_1, Ux_2\rangle = Img\langle x_1, x_2\rangle \quad \forall\, x_1, x_2 \in X$$

$1 \Leftrightarrow 3$ $\quad \langle Ux_1, Ux_2 \rangle = \langle x_1, x_2 \rangle$

$\Leftrightarrow \langle U^*Ux_1, x_2 \rangle = \langle x_1, x_2 \rangle$

$\Leftrightarrow \langle (U^*U - I)x_1, x_2 \rangle = 0 \quad \forall x_1, x_2 \in X$

$\Leftrightarrow (U^*U - I)x_1 = 0 \quad \forall x_1 \in X$

$\Leftrightarrow U^*U = I$

$\because$ U has a left-inverse, U is inj.

$\therefore$ U is bij. as X is f.d.

$1 \Rightarrow 4$

$\because$ U sends orthogonal vec. to orthogonal vecs.

& preserves length

$\therefore$ U sends o.n.b. to an o.n.b

$4 \Rightarrow 1$

$\because$ U sends basis to basis $\Rightarrow$ U is bij.

Let $\{x_1, x_2, \dots, x_n\}$ be an o.n.b. s.t $\{Ux_1, Ux_2, \dots, Ux_n\}$ is an o.n.b. for X.

Then, $\quad v = \sum_{i=1}^{n} \langle v, x_i \rangle x_i$, $\quad w = \sum_{j=1}^{n} \langle w, x_j \rangle x_j$

$$\langle v, w\rangle = \left\langle \sum_{i=1}^{n} \langle v, x_i\rangle x_i , \sum_{j=1}^{n} \langle w, x_j\rangle x_j \right\rangle$$

$$= \sum_{i=1}^{n} \langle v, x_i\rangle \langle x_i, w\rangle$$

$$\langle Uv, Uw\rangle = \left\langle \sum_{i=1}^{n} \langle v, x_i\rangle Ux_i , \sum_{j=1}^{n} \langle w, x_j\rangle Ux_j \right\rangle$$

$$= \sum_{i=1}^{n} \langle v, x_i\rangle \langle x_i, w\rangle$$

$$= \langle v, w\rangle$$

<u>Orthogonal matrix</u> : $A^T A = I$

<u>Thm</u> : for $A \in M_{n\times n}(\mathbb{C})$ is unitary iff the col. vectors of A is an o.n.b for $\mathbb{C}^n$.

<u>Pf</u> : Choose o.n.b. $\{e_1, \dots, e_n\}$ & apply 4. of prev. thm.

Sim. results holds for rows of unitary matrix by observing that if A is unitary, so if A^*.

$\mathcal{U}(X)$: Set of all unitary maps on X.
It is a gp. under multiplication.

$\mathcal{L}(X)/\mathcal{U}(X)$: Unitary eq. classes of linear maps.

$T_1 \simeq T_2$ if $\exists$ a unitary map U s.t $U^* T_1 U = T_2$

Given T when $\exists$ a unitary U s.t $U^* T U$ is diagonal?

<u>Self-adjoint</u> : A linear map T on an inner prod. sp. is self-adjoint if $T = T^*$

<u>Lem</u> : A self-adj. map has only real e-vals.

<u>Pf</u> : Let $a+ib$ be an e-val of T.
Then $\exists\ x \neq 0$ s.t $Tx = (a+ib)x$
$\Rightarrow (T-aI)x = ibx$

$$\underbrace{\langle (T-aI)x, x\rangle}_{\langle x, (T-aI)^* x\rangle = \langle x, (T-aI)x\rangle} = \langle ibx, x\rangle = ib\langle x, x\rangle$$

$(\because T = T^*)$

$\Rightarrow \quad \langle x, ibx \rangle = ib\langle x, x \rangle$

$\Rightarrow \quad -ib\langle x, x \rangle = ib\langle x, x \rangle$

$\Rightarrow \quad \underline{b = 0}$

<u>Lem</u>: e-vecs corresponding to diff. e-vals of a self-adjoint map are orthogonal.

<u>Pf</u>: Let $Tx = ax$ & $Ty = by$ where $a \neq b$ & $x, y \neq 0$.

$$\begin{aligned} a\langle x, y \rangle = \langle ax, y \rangle &= \langle Tx, y \rangle \\ &= \langle x, T^* y \rangle \\ &= \langle x, Ty \rangle \\ &= \langle x, by \rangle \\ &= b\langle x, y \rangle \end{aligned}$$

$\because \; a \neq b \;\Rightarrow\; \langle x, y \rangle = 0$

Thm: (Spectral Thm for Self-Adj. Maps)

For a self-adj. map T on a f.d. inner prod. sp. V, $\exists$ an o.n.b. for V which consists of eigenvectors of T.

Pf: Suppose, $(T-aI)^n x = 0$ for some $x \neq 0$, $a \in \mathbb{R}$

We want to show, $(T-aI)x = 0$

For $n=2$, $(T-aI)^2 x = 0$

$$\Rightarrow \langle (T-aI)^2 x, x \rangle = 0$$

$$\Rightarrow \langle (T-aI)x, (T-aI)x \rangle = 0$$

$$\Rightarrow (T-aI)x = 0$$

Suppose that the result is true for $n=m$,

& $(T-aI)^{m+1} x = 0$ for some $x \neq 0$, $a \in \mathbb{R}$.

$$\Rightarrow (T-aI)^2 \underbrace{(T-aI)^{m-1} x}_{y} = 0$$

$$\Rightarrow (T-aI)\left((T-aI)^{m-1} x\right) = 0 \quad \text{[By hypothesis for } n=2\text{]}$$

$$\Rightarrow (T-aI)^m x = 0$$

$$\Rightarrow (T-aI)x = 0 \quad \text{[By ind}^n\text{ hypothesis]}$$

So, by primary decomposition thm, $V = W_1 \oplus \dots \oplus W_n$ where each W_i is an eigensp. Moreover, W_i & W_j are orthogonal for $i \neq j$.

Hence, $\exists$ o.n.b of V consisting of e-vecs.

Equivalent form of spectral thm

$$V = E_{a_1} \oplus E_{a_2} \dots \oplus E_{a_n}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\{x_{11}, x_{21}, \dots, x_{m_1 1}\} \qquad \{x_{1n}, x_{2n}, \dots, x_{m_n n}\}$$

$P_1 : V \longrightarrow E_{a_1}$

$$P_1(x) = \sum_{i=1}^{m_1} \langle x, x_{i1} \rangle x_{i1}$$

P_1 – orthogonal proj. $\quad (P_1^2 = P \;\; \& \;\; P_1^* = P_1)$

$\exists$ orthogonal proj. $P_1, \dots, P_n$ s.t $\quad P_1 + \dots + P_n = I$

$$T = a_1 P_1 + a_2 P_2 + \dots + a_n P_n$$

For $x \in V$, $\quad x = x_1 + \dots + x_n$, where $x_i \in E_{a_i}$ $\forall$ $i = 1, \dots, n$

We also know that,

$$P_i(x) = x_i \quad \forall \ i = 1, \dots, n$$

$$Tx = T(x_1 + \dots + x_n)$$
$$= a_1 x_1 + \dots + a_n x_n$$

On the other hand,

$$(a_1 P_1 + \dots + a_n P_n)\, x = a_1 x_1 + \dots + a_n x_n$$

<u>Normal map</u>: A lin. map T on an inner prod. sp. is normal if $TT^* = T^*T$

$T = A + iB$ where A & B are self-adjoint

$T^* = A - iB$

$$\Rightarrow \quad A = \frac{T + T^*}{2}, \quad B = \frac{T - T^*}{2i}$$

T is normal $\Leftrightarrow$ $T = A + iB$ where A & B are self-adj.

& $AB = BA$

$T = A + iB$

$V = E_{a_1} \oplus E_{a_2} \oplus \dots \oplus E_{a_n}$, a_i are e-vals of A.

<u>Claim:</u> $B(E_{a_i}) \subseteq E_{a_i}$

For $x \in E_{a_i}$ $\Rightarrow$ $Ax = a_i x$

$\Rightarrow$ $ABx = BAx = B(a_i x) = a_i Bx$

$\Rightarrow$ $Bx \in E_{a_i}$

By claim, $B = \begin{bmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_n \end{bmatrix}$

$B_i = B\big|_{E_{a_i}}$

<u>Ex:</u> $V = W_1 \oplus W_2$

$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$

B is self-adj $\Leftrightarrow$ B_1 & B_2 are self-adj.

$$V = E_{a_1} \oplus E_{a_2}$$

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

$B_i : E_{a_i} \to E_{a_i}$ (self-adj.)

$E_{a_1} = E_{b_{11}} \oplus E_{b_{21}}$
$E_{a_2} = E_{b_{12}} \oplus E_{b_{22}}$
(Applying spectral thm on B_1 & B_2)

$$V = E_{a_1} \oplus E_{a_2}$$
$$= E_{b_{11}} \oplus E_{b_{21}} \oplus E_{b_{12}} \oplus E_{b_{22}}$$

for $x \in E_{b_{jk}}$, $B_k x = b_{jk} x \Rightarrow Bx = b_{jk} x$
$Ax = a_j x$

$$\Rightarrow Tx = (A + iB)x = (a_j + ib_{jk})x$$

Hence, $\exists$ o.n.b of e-vecs of T.

Thm: A lin. map on a f.d complex inner prod. sp. is unitarily diagonalizable i.e equivalent to a diagonal map iff it is normal

Pf: $(\Rightarrow)$ $\quad U^*TU = D \Rightarrow T = UDU^*$

$$T^* = UD^*U^*$$

$$TT^* = UDD^*U = UD^*DU = T^*T$$

$(\Leftarrow)$ Already proven.