Tutorial Sheet - I

1. Describe all the 2×2 row-reduced echelon matrices.

2. Let
$$A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix}.$$
For which $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ does the system $AX = Y$ have a sloution?

3. Let A be a $n\times n$ matrix.
(i) Suppose there exists a $n\times n$ matrix B such that $BA = I$. Show that A is invertible.
(ii) Suppose there exists a $n\times n$ matrix C such that $AC = I$. Show that A is invertible.

4. Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ -4 & 4 \end{bmatrix}$
Find a matrix C such that $CA = B$.
Is the matrix C unique?

5. If A is an $m\times n$ matrix and B is an $n\times m$ matrix with $n < m$, show that AB is not invertible.

6. Let $B \in M_{n\times n}(\mathbb{C})$.
(i) Show that $B' = \{A \in M_{n\times n}(\mathbb{C}) : AB = BA\}$ is a subspace of $M_{n\times n}(\mathbb{C})$.
(ii) Find B', where $B = \begin{bmatrix} 1 & & & 0 \\ & 2 & & \\ & & 3 & \\ 0 & & & \ddots\; n \end{bmatrix}_{n\times n}$.

7. Let $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in M_{n\times 1}(\mathbb{C})$. Consider the set $LA(X) = \{ A \in M_{n\times n}(\mathbb{C}) : AX = 0 \}$.

(i) Show that $LA(X)$ is a subspace of $M_{n\times n}(\mathbb{C})$.

(ii) Show that $LA(X)$ is a left ideal of $M_{n\times n}(\mathbb{C})$, that is $BC \in LA(X)$ whenever $C \in LA(X)$ and $B \in M_{n\times n}(\mathbb{C})$.

(iii) Show that $LA(X) = M_{n\times n}(\mathbb{C})$ if and only if $X = 0$.

8. Let $A, B \in M_{n\times n}(\mathbb{C})$ be such that $AB = 0$. Give a proof or counter example for each of the following.

(i) $BA = 0$.

(ii) Either $A = 0$ or $B = 0$ (or both).

(iii) If B is invertible then $A = 0$.

(iv) There is a vector $X \neq 0$ such that $BAX = 0$.

Tutorial Sheet - II

1. Are the vectors $a_1 = (1,1,2,4)$, $a_2 = (2,-1,-5,2)$, $a_3 = (1,-1,-4,0)$, $a_4 = (2,1,1,6)$ linearly dependent in $\mathbb{R}^4$?

2. Let V be the vector space of all 2x2 matrices over $\mathbb{C}$.

(i) Prove that V has dimension 4 by exhibiting a basis for V which has four elements.

(ii) Let $W = \{ A \in M_{2\times 2}(\mathbb{C}) : A_{11} + A_{22} = 0 \}$. Show that W is a subspace of V and find the dimension of W.

3. Let X be the vector space of all polynomials $p(t)$ of degree $\leq n$, and let Y be the set of polynomials in X that vanishes at $t_1, \dots, t_j$, $j < n$.

(i) Show that Y is a subspace of X.

(ii) Find dim X and dim Y.

(iii) Find $\dim (X/Y)$.

4. Let V be a vector space and A be a linearly independent subset of V. Prove that A is a basis for V if and only if it is a maximal linearly independent subset of V, that is addition of any vector to A will result in a linearly dependent set.

5. Let V be a vector space, and let $\mathrm{Span}(A) = V$, for a subset A of V. Prove that A is a basis for V if and only if it is a minimal spanning set, that is removal of any vector from A will result in a set which does not span V.

Tutorial Sheet - III

Q.1. Let $C([0,1]) = \{ f : [0,1] \to \mathbb{C} : f \text{ is continuous} \}$.
Define $T : C([0,1]) \to \mathbb{C}$ by
$$T(f) = \int_0^1 f(x) x^3 \, dx, \quad \forall f \in C([0,1]).$$
Show that T is a linear map.

Q2. Let X be a vector space over F. If $x \in X$ is a non-zero vector, show that there exists a linear map $T : X \to F$ such that
$$T(x) \neq 0.$$

Q3. Let $C([-1,1]) = \{ f : [-1,1] \to \mathbb{R}, f \text{ is continuous} \}$.
Define $P : C([-1,1]) \to C([-1,1])$ by
$$(Pf)(x) = \frac{f(x) + f(-x)}{2}, \quad \forall x \in [-1,1] \ \& \ f \in C([-1,1]).$$
Show that P is a linear map and P is a projection, that is $P^2 = P$.

Q4. Let X be the vector space of polynomials with complex co-efficients of degree less than n.
Let $z_0, \dots, z_n$ be distinct complex numbers.
Define a map $T : X \to \mathbb{C}^{n+1}$ by
$$T(p) = (p(z_0), \dots, p(z_n)), \quad \forall p \in X.$$
(i) Show that T is a linear map.
(ii) The null space of T is trivial and T is not a surjective map.

(iii) Show that there exist n numbers $m_1, \dots, m_n$ such that $\int_0^1 p(x)\,dx = m_1 p(t_1) + \dots + m_n p(t_n) \;\forall\; p \in X$

where $t_1, \dots, t_n$ are n distinct points in $[0, 1]$.

(iv) • Show that the map $T': X \to \mathbb{C}^n$ defined by

$$T'(p) = (p(s_1), \dots, p(s_n))$$ is surjective.

• For $(\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$, find a polynomial p in X such that

$$p(s_i) = \alpha_i, \;\forall\; i = 1, \dots, n.$$

Tutorial Sheet - IV

Q.1. Let X be a finite dimensional vector space. A linear map $M \in \mathcal{L}(X,X)$ is similar to $N \in \mathcal{L}(X,X)$ if $\exists$ an invertible element $S \in \mathcal{L}(X,X)$ such that

$$M = SNS^{-1}.$$

(i) Show that similarity is an equivalence relation.

(ii) If either M or N in $\mathcal{L}(X,X)$ is invertible, then MN and NM are similar.

(iii) If A and B are similar show that $\dim(R_A) = \dim(R_B)$ and $\dim(N_A) = \dim(N_B)$.

Q2. Let X be a vector space over F with basis $x = \{x_1, \dots, x_n\}$. For each $1 \le i \le n$, define a linear functional $l_i : X \to F$ by taking linear extension of the following: $l_i(x_j) = \delta_{i,j}$, $\forall\, j = 1, \dots, n$

where
$$\delta_{i,j} = \begin{cases} 1_F, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}.$$

Show that $l = \{l_1, \dots, l_n\}$ is a basis for X^*.
Such a basis l is called the dual basis of x.

Q3: Let X be an n-dimensional vector space over F, and let l be a non-zero linear functional on X.

(i) What is the dimension of N_l?

(ii) If f is a linear functional on X, show that $f = \alpha l$ for some $\alpha \in F$ if and only if

$N_g \subseteq N_f$.

(iii) Let V be a subspace of X with $\dim(V) = n-1$ (such a subspace is called a hyperspace). Find a linear functional $g : X \to F$ such that
$$N_g = V.$$

(iv) Find a linear functional g on $\mathbb{R}^2$ such that N_g is the line $ax + by = 0$; $a, b \in \mathbb{R}$.

Q4. Let T be a linear map on $\mathbb{R}^2$ defined by
$$T(x_1, x_2) = (-x_2, x_1).$$

(i) Find the matrix representation of T in the standard ordered basis of $\mathbb{R}^2$.

(ii) Find the matrix representation of T in the ordered basis $\{(1,2), (1-1)\}$.

Tutorial Sheet - V

Q.1. Let A be an $n \times n$ matrix over a $\mathbb{F}$. For any $\alpha \in \mathbb{F}$, show that $\det(\alpha A) = \alpha^n \det(A)$.

Q2. Let A be a 2×2 real matrix be such that $A^2 = 0$. Show that $\det(cI - A) = c^2$ for any real number c.

Q3. An $n \times n$ matrix A over $\mathbb{F}$ is skew-symmetric if $A^T = -A$. If A is a skew-symmetric $n \times n$ matrix with complex entries and n is odd, show that $\det(A) = 0$.

Q4. An $n \times n$ matrix over $\mathbb{F}$ is orthogonal if $AA^T = I$. If A is orthogonal show that $\det(A) = \pm 1$. Give an example of an orthogonal matrix with $\det(A) = -1$.

Q.5. Let S_n be the permutation group of n-symbols. Show that $\operatorname{sgn} : S_n \to \{\pm 1\}$ defined by $\sigma \longmapsto \operatorname{sgn}(\sigma)$ is a group homomorphism, that is $\operatorname{sgn}(\sigma_1 \circ \sigma_2) = \operatorname{sgn}(\sigma_1)\cdot \operatorname{sgn}(\sigma_2)$, $\forall\ \sigma_1, \sigma_2 \in S_n$.

Q.6. Find an eigenvalue of the matrix

$$\begin{bmatrix} 2022 & 2 & 3 \\ 2 & 2025 & 6 \\ 3 & 6 & 2030 \end{bmatrix}.$$

Q7. Let $T: \mathbb{F}^n \to \mathbb{F}^n$ be a linear map.

Show that

$$\det(T(\alpha_1), \ldots, T(\alpha_n)) = c \det(\alpha_1, \ldots, \alpha_n).$$

for all $\alpha_1, \ldots, \alpha_n \in \mathbb{F}^n$, where

$$c = \det(T(e_1), \ldots, T(e_n)), \text{ and}$$

$e_1, \ldots, e_n$ is the standard basis for $\mathbb{F}^n$!

Q8. Find matrices $A, B \in M_{n\times n}(\mathbb{C})$ and scalars $\lambda, \beta \in \mathbb{C}$ such that

(i) λ is an eigenvalue of A, β is an eigenvalue of B but $\lambda + \beta$ is not an eigenvalue of $A+B$.

(ii) $\lambda\beta$ is not an eigenvalue of AB.

Q.9. Find determinant of the matrix

$$\begin{bmatrix} 7 & 2 & 2 & 2 & 2 \\ 2 & 7 & 2 & 2 & 2 \\ 2 & 2 & 7 & 2 & 2 \\ 2 & 2 & 2 & 7 & 2 \\ 2 & 2 & 2 & 2 & 7 \end{bmatrix}.$$

Q1. Let $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix} \in M_{4\times 4}(\mathbb{R})$.

Under what condition on a, b and c is A diagonalizable?

Q2. If A is a 2×2 real matrix such that $A = A^T$ (symmetric), then show that A is diagonalizable.

Q3. Let A and B be $n\times n$ matrices over $\mathbb{F}$. Show that A and B has same eigenvalues.

Q4. Let N be a 2×2 real matrix such that $N^2 = 0$. Show that N is similar to either 0 or $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

Q5. Let V be the space of all real-valued continuous function on $\mathbb{R}$. Define $T : V \to V$ by

$$T(f)(x) = \int_0^x f(t)\,dt, \quad \forall\, x \in \mathbb{R}.$$

Prove that the operator T (known as Volterra operator) has no eigenvalues.

Tutorial Sheet VIII

Q.1. Let $A \in M_{n\times n}(\mathbb{C})$ be a rank one matrix. Show that either A is diagonalizable, or A is nilpotent, not both.

Q.2. Give an example of two 4×4 nilpotent matrices which have same minimal polynomial but which are not similar.

Q.3. Let A be a $n\times n$ matrix such that $A^k = 0$ for some positive integer k. Show that $A^n = 0$.

Q4. Let $T : M_{n\times n}(\mathbb{F}) \to M_{n\times n}(\mathbb{F})$ be defined by

$$B \longmapsto AB$$

for some fixed $A \in M_{n\times n}(\mathbb{F})$. Is it true that T and A have same eigen values?

Show that the minimal polynomial for T is the minimal polynomial for A.

Q5. Find primary decomposition of the matrix

$$\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

over $\mathbb{R}$.

Q6. Find the projection E which project $\mathbb{R}^2$ onto the space $\mathrm{Span}\{(1,-1)\}$ and $N(E) = \mathrm{Span}\{(1,2)\}$.

Q.7. If E is a projection on a vector space V, then show that $(I-E)$ is also a projection.

How $N(E)$ and $R(I-E)$ are related.

Q8. If E is a projection on V, show that $V = R(E) \oplus N(E)$.

Tutorial Sheet VIII

Q1. Let N_1 and N_2 be 3×3 nilpotent matrices over $\mathbb{F}$. Show that N_1 and N_2 are similar if and only if they have same minimal polynomial.

Q2. If A is a complex matrix with characteristic polynomial $(x-2)^3(x+7)^2$ and minimal polynomial $(x-2)^2(x+7)$, what is the Jordan form of A?

Q3 How many possible Jordan form are there for a complex matrix with characteristic polynomial $(x+2)^3(x-1)^2$?

Q4. Classify up to similarity of all 3×3 complex matrices A such that $A^3 = I$.

Q.5. Let N_1 and N_2 be 6×6 nilpotent matrices over $\mathbb{F}$. Suppose that N_1 and N_2 have the same minimal polynomial and the same nullity $(d(N(N_1)) = d(N(N_2)))$. Prove that N_1 and N_2 are similar.

Q6. Let $J(m)$ be the Jordan matrix of size m. Show that $J(m)$ and $J(m)^T$ is similar. Use Jordan canonical form to prove that every complex matrix A is similar to A^T.

Tutorial Sheet IX

Q.1. Apply Gram-Schmidt process to the vectors $(1,0,1)$, $(1,0,-1)$ and $(0,3,4)$ to obtain an ONB for $\mathbb{R}^3$.

Q2. Consider the vector space $M_{n\times n}(\mathbb{C})$. Show that the map $\langle \cdot , \cdot \rangle : M_{n\times n}(\mathbb{C}) \times M_{n\times n}(\mathbb{C}) \to \mathbb{C}$ defined by

$$\langle A, B \rangle = tr(AB^*)$$

is an inner product on $M_{n\times n}(\mathbb{C})$. Let Y be the space of all diagonal matrices.

(a) find an ONB for Y, and find $Y^\perp$.

(b) For real $n\times n$ matrices A and B, show that

$$tr(AB^T)^2 \leq tr(AA^T)\, tr(BB^T).$$

(c) Consider the linear functional defined by

$$\varphi : M_{n\times n}(\mathbb{C}) \longrightarrow \mathbb{C}$$

$$A = (a_{ij}) \longmapsto \sum_{i,j=1}^{n} a_{ij}.$$

Check the validity of the Riesz representation theorem, that is, find the unique matrix B such that

$$\varphi(A) = \langle A, B \rangle, \quad \forall\, A \in M_{n\times n}(\mathbb{C}).$$

(d) Let J be the $n\times n$ matrix with all the entries equal to 1. Find the best approximation of J in Y.

Q3. Let U be a unitary map on $\mathbb{R}^2$, with the standard inner product. Show that the matrix representation of U with respect to the standard basis is

either $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ (rotation by θ) or $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ (reflection followed by rotation)

for some $0 \le \theta \le 2\pi$.

Q4. For the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

find a orthogonal matrix U s.t. $U^t A U = D$, where D is a diagonal matrix.

Q5. Show that T is normal if and only if $T = T_1 + iT_2$ for some commuting self-adjoint maps T_1 and T_2.