MA403

Real Analysis

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References: 1. Apostol
2. Rudin

Grading: Homework (weekly ungraded)
Quiz (weekly during tutorial) (30%)
Midsem (30%)
Endsem (40%)

Calculus

- Limits
- Continuity
- Differentiability
- Integration

- Intermediate Value Thm
- Mean Value Thm
- Taylor's Thm

The bgd for all this is $\mathbb{R}$.

Why $\mathbb{R}$?

Why <u>not</u> $\mathbb{Z}$ or $\mathbb{Q}$?

What makes calculus work is some spl. ppts of $\mathbb{R}$, especially completeness.

eg - $\sqrt{2}$ is not rational

Pf : Suppose $\exists\ p \in \mathbb{Q}$ s.t $p^2 = 2$

$p = \frac{m}{n}$, $(m, n) = 1$

$\Rightarrow p^2 = \frac{m^2}{n^2} \Rightarrow m^2 = 2n^2$

$\Rightarrow 2 \mid m^2 \Rightarrow m = 2k,\ k \in \mathbb{Z}$

$\Rightarrow 4k^2 = 2n^2 \Rightarrow n^2 = 2k^2$

$\Rightarrow 2 \mid n^2$

$\Rightarrow n = 2l,\ l \in \mathbb{Z}$

Contd^n

$\because (m, n) = 1$

Hence, $\nexists\ p \in \mathbb{Q}$ s.t $p^2 = 2$

Let us investigate more.

Let $A = \{ p \in \mathbb{Q} : p \geq 0 \ \& \ p^2 < 2 \}$
$B = \{ p \in \mathbb{Q} : p \geq 0 \ \& \ p^2 > 2 \}$

Claim: A contains no largest no. &
B contains no smallest no.

Note, $\mathbb{Q}_{\geq 0} = A \cup B$

Pf: It suffices to show,

- $\forall \ p \in A, \ \exists \ q \in A$ s.t $p < q$
- $\forall \ p \in B, \ \exists \ q \in B$ s.t $q < p$

For $p \in \mathbb{Q}_{\geq 0}$, let
$$q = \frac{2p+2}{p+2}$$
$$= p - \left(\frac{p^2 - 2}{p+2} \right)$$

Then, $q^2 - 2 = \dfrac{2(p^2-2)}{(p+2)^2}$

If $p \in A \Rightarrow p^2 < 2 \Rightarrow q > p$ & $q^2 < 2$
$\Rightarrow q \in A$

If $p \in B \Rightarrow p^2 > 2 \Rightarrow q < p$ & $q^2 > 2$
$\Rightarrow q \in B$

This sort of 'bad behaviour' doesn't occur with $\mathbb{R}$.

Real Number System

There are some fundamental ppts from which all other ppts & results of calculus can be derived.

1. Field axioms

On the set $\mathbb{R}$, there are two ops. add^n & multi^n s.t $\forall\, x, y \in \mathbb{R}$,
$x+y \in \mathbb{R}$ & $xy \in \mathbb{R}$

1.1. (Commutativity)

$$x+y = y+x \quad , \qquad xy = yx$$

1.2. (Associativity)

$$x+(y+z) = (x+y)+z \quad , \quad x(yz) = (xy)z$$

1.3. (Distributivity)

$$x(y+z) = xy + xz$$

1.4 $\exists!\ 0 \in \mathbb{R}$ s.t $x + 0 = x \quad \forall x \in \mathbb{R}$

1.5 $\forall x \in \mathbb{R}$, $\exists!\ (-x) \in \mathbb{R}$ s.t $x + (-x) = 0$

1.6 $\exists!\ 1 \in \mathbb{R}$ s.t $x \cdot 1 = x \quad \forall x \in \mathbb{R}$

1.7 $\forall x \in \mathbb{R} \setminus \{0\}$, $\exists!\ x^{-1} \in \mathbb{R}$ s.t $x x^{-1} = 1$

1.8. $0 \neq 1$

2. Order axioms

On $\mathbb{R}$, there is a rel^n $<$ (less than) with ppts.

2.1 If $x, y \in \mathbb{R}$, one and only one of the following hold

$$x < y \qquad x = y \qquad y < x$$

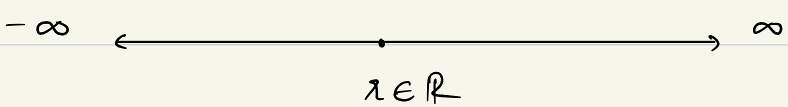
2.2. If $x < y$, $\forall z \in \mathbb{R}$, $x + z < y + z$

2.3. If $x > 0$ & $y > 0$, then $xy > 0$

2.4. If $x < y$ & $y < z$, then $x < z$

Geometric rep

Real nos. are often represented as points on a line (real line or real axis)



Intervals

For $a, b \in \mathbb{R}$, the set of all real nos. b/w a & b is called an interval

If $a < b$,

Open : $(a, b) = \{x : a < x < b\}$

Closed : $[a, b] = \{x : a \leq x \leq b\}$

Half-open : $(a, b] = \{x : a < x \leq b\}$

$[a, b) = \{x : a \leq x < b\}$

Infinite : $(a, \infty) = \{x : x > a\}$

$[a, \infty) = \{x : x \geq a\}$

$(-\infty, a) = \{x : x < a\}$

$(-\infty, a] = \{x : x \leq a\}$

Note, ∞ & $-\infty$ are <u>not</u> pts. in $\mathbb{R}$, but rather symbols.

Thm : Consider $a, b \in \mathbb{R}$ s.t $a \leq b + \epsilon \quad \forall \epsilon > 0$

Then $a \leq b$.

Pf : Suppose $b < a$

Consider $\epsilon = (a-b)/2$,

$b + \epsilon = (a+b)/2 < (a+a)/2 \Rightarrow b + \epsilon < a \rightarrow$ Contd^n

Upper bound : Let $S \subset \mathbb{R}$. If $\exists\, b \in \mathbb{R}$ s.t $\forall x \in S$, $x \leq b$ then b is called upper bound of S.

Also, S is said to be bounded above by b.

– A set can have several ub.

Max. element : If b is ub. & also an elem. of S, then b is called the max. elem. of S

$$b = \max S.$$

A set with no ub is said to be **unbounded above**.

The concepts of lower bound, a set being bounded/unbounded below, min. elem. can be defined similarly.

Least Upper Bound : For a set S bounded above, the real no. b is called a lub. if

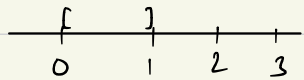
1. b is ub. of S
2. if $a < b$, then a is not ub of S.

- If $\exists$ a lub, it is unique.
  Hence, we refer to it as <u>the</u> lub.

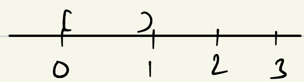
- lub is also referred to as supremum ($\sup(S)$)

- If S has max. elem, then $\sup(S) = \max(S)$

eg - $S_1 = (0, 1]$



$S_2 = (0, 1)$



$\max(S_1) = 1$ $\qquad$ $\sup(S_1) = 1$

$\max(S_2)$ doesn't exist $\qquad$ $\sup(S_2) = 1$

Similarly, greatest lower bound referred ~~to~~ as infimum ($\inf(S)$) can be defined.

3. Completeness axiom

Every non-empty set S of real nos. which is bounded above, has a sup.

Ppts of sup

Thm: (Approximation ppt.)
For a set S with sup. b, $\forall\ a<b$, $\exists\ x \in S$ s.t $a < x \leq b$

Pf: Since $b = \sup(S)$, $\forall\ x \in S$, $x \leq b$

If $x \leq a$ $\forall\ x \in S$, then a is ub. of S.
T'fore, $b \leq a$ $\rightarrow$ Contd^n

Thm: (Additive ppt)

Given non-empty $A, B \subseteq \mathbb{R}$.

Let $C := \{ x+y : x \in A, y \in B \}$

If each of A & B has a sup, then C has a sup. and

$$\sup(C) = \sup(A) + \sup(B)$$

Pf: Let $a = \sup(A)$ & $b = \sup(B)$

$\forall z \in C, \exists x \in A, y \in B$ s.t $z = x+y$

$$z = x+y \leq a+b$$

So, $a+b$ is u.b of C.

Since, C is bounded, by completeness axiom, $\exists c = \sup(C)$

So, $c \leq a+b$

$\forall \epsilon > 0, \quad a - \epsilon < a$

By approx. ppt., $\exists\, x \in A$ s.t $a - \epsilon < x \leq a$

Similarly $\exists\, y \in B$ s.t $b - \epsilon < y \leq b$

$$\Rightarrow \quad (a+b) - 2\epsilon < x+y \leq a+b$$

Since $x+y \in C$, $\quad x+y \leq c$

$$\Rightarrow \quad (a+b) - 2\epsilon < c$$

$$\Rightarrow \quad (a+b) < c + 2\epsilon \qquad \forall \epsilon > 0$$

Hence, $(a+b) \leq c$

$$\Rightarrow \quad c = a+b$$

Thm: (Comparison ppt)
Given non-empty $S, T \subseteq \mathbb{R}$ s.t $s \leq t$ $\forall s \in S, t \in T$. If T has a sup., then S has a sup. and $\sup S \leq \sup T$

Pf: Let $q = \sup(T)$.
$\because$ $s \leq t \leq q$ $\forall s \in S, t \in T$
$\therefore$ q is ub of S & S is bounded.

By completeness, S has a sup.
Let $p = \sup(S)$.

Suppose $q < p$ $\Rightarrow$ q is not an ub. for S ($\because p = \sup(S)$)
$\rightarrow$ Contd^n

$\therefore$ $p \leq q$ $\Rightarrow$ $\sup(S) \leq \sup(T)$

Thm: (Archimedean ppt.)
If $x, y \in \mathbb{R}$ with $x > 0$, $\exists\, n \in \mathbb{Z}_{>0}$ s.t

$$nx > y$$

Pf: Let $A = \{ nx : n \in \mathbb{Z}_{>0} \}$

Suppose $\forall\, n \in \mathbb{Z}_{>0}$, $nx \leq y$

$\Rightarrow$ y is u.b for A.

By completeness, $\exists\, a = \sup(A)$.

Since $x > 0$, $a - x < a$
By approx. ppt., $\exists\, s \in A$ s.t $a - x < s < a$

i.e $\exists\, m \in \mathbb{Z}_{>0}$ s.t $a - x < mx \leq a$

$\Rightarrow$ $a < (m+1)x \in A \longrightarrow$ Contd^n

Thm: ($\mathbb{Q}$ is dense in $\mathbb{R}$)
If $x, y \in \mathbb{R}$, $x<y$, $\exists\, p \in \mathbb{Q}$ s.t

$$x < p < y$$

Pf: Since $x<y$, $(y-x) > 0$
By Archimedean ppt on $(y-x), 1$,
$\exists\, n \in \mathbb{Z}_{>0}$ s.t

$$n(y-x) > 1$$

Apply Archimedean ppt. on $1, nx$ &
$1, -nx$ i.e $\exists\, m_1, m_2 \in \mathbb{Z}_{>0}$ s.t

$$m_1 > nx$$
$$m_2 > -nx$$

$$\Rightarrow \quad -m_2 < nx < m_1$$

T'fore, $\exists\, m \in \mathbb{Z}$ with $-m_2 \leq m \leq m_1$
s.t $\quad m-1 \leq nx < m$

So, we have $$nx < m \leq 1 + nx < ny$$

$$x < \frac{m}{n} \qquad \frac{m}{n} < y$$

Hence, $$x < \frac{m}{n} < y$$

Thm: (n^{th} root)

$\forall\, x \in \mathbb{R}_{>0}$, $\forall\, n \in \mathbb{Z}_{>0}$, $\exists !\, y \in \mathbb{R}_{>0}$ s.t

$$y^n = x$$

Pf: Let $E = \{ t > 0 : t^n < x \}$

If $t_0 = \dfrac{x}{1+x}$, then $t_0 < x$ & $0 < t_0 < 1$

Since $t_0^n < t_0 < x$, $t_0 \in E$, so E is non-empty.

If $t_1 \in \mathbb{R}$ s.t $t_1 > 1+x$, then $t_1 > 1$

Since $t_1^n > t_1 > x$, so $t_1 \notin E$

Hence, $1+x$ is u.b of E

By completeness, $\exists\ y = \sup(E)$

Note, $y > t_0$, so $y > 0$

Now, we'll show that y satisfies $y^n = x$

Let us consider the possibilities $y^n > x$ & $y^n < x$.

Note, the identity

$$(b^n - a^n) = (b-a)(b^{n-1} + b^{n-2}a + \dots + ba^{n-2} + a^{n-1})$$

So, if $0 < a < b$, then $(b^n - a^n) < (b-a)\, n b^{n-1}$

Case 1: Suppose $y^n < x$

Choose h s.t $0 < h < 1$ and $h < \frac{x - y^n}{n(y+1)^{n-1}}$

Using the above identity with $a = y$, $b = y+h$,

$$\begin{aligned}(y+h)^n - y^n &< hn(y+h)^{(n-1)} \\ &< hn(y+1)^{(n-1)} && [\because h < 1] \\ &< x - y^n && \left[\because h < \frac{x-y^n}{n(y+1)^{n-1}}\right]\end{aligned}$$

$\Rightarrow \quad (y+h)^n < x$

$\Rightarrow \quad y+h \in E \rightarrow$ Contd^n

$\because y = \sup(E)$

Case 2: Suppose $y^n > x$

Choose $k = \frac{y^n - x}{ny^{(n-1)}}$, then $0 < k < y$

If $t_2 \in \mathbb{R}$ be s.t $t_2 \geq y - k$,

$$y^n - t_2^n \leq y^n - (y-k)^n$$
$$< kny^{n-1} = y^n - x$$

$\Rightarrow \quad t_2^n > x$

$\Rightarrow \quad t_2 \notin E$

$\because \quad t \in E \Rightarrow t < y-k \Rightarrow y-k$ is u.b

$\rightarrow$ Contd^n $\because$ $y = \sup(E)$

Since, both $y^n > x$ & $y^n < x$ given contd^n.
T'fore, $y^n = x$.

This proves existence of such a y.

To prove uniqueness, observe for any $0 < y_1 < y_2$, we have $y_1^n < y_2^n$.

Hence y is also unique.

Decimal representation

Let $r \in \mathbb{R}$ be of the form,

$$r = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \dots + \frac{a_n}{10^n}$$

where $a_0 \in \mathbb{Z}$, $a_0 \geq 0$, and $a_1, \dots, a_n \in \mathbb{Z}$, $0 \leq a_i \leq 9$

Then, $r = a_0 . a_1 a_2 \dots a_n$ is called a finite decimal rep. of r.

- Any r of this form is rational.
- Not all rational nos. have such a finite decimal rep.

eg - $r = 1/3 \neq a/10^k$, $a \in \mathbb{Z}$
$\because 3 \nmid 10^k$

Thm: (finite dec. approximation)

Let $x \in \mathbb{R}$, $x \geq 0$, Then $\forall n \in \mathbb{Z}$, $n \geq 1$, there is a finite decimal $r_n = a_0 . a_1 \ldots a_n$ s.t

$$r_n \leq x < r_n + 1/10^n$$

- If $E = \left\{ a_0 + \frac{a_1}{10} + \cdots + \frac{a_k}{10^k} : k \in \mathbb{Z}_{\geq 0} \right\}$, then $x = \sup(E)$

Infinite dec. rep.

For $a_0, a_1, \ldots$ in the previous discussion, we write $x = a_0 . a_1 a_2 \ldots$ to mean that for each k, a_k is the largest int. satisfying

$$a_0 + \frac{a_1}{10} + \cdots + \frac{a_k}{10^k} \leq x < a_0 + \frac{a_1}{10} + \cdots + \frac{a_k + 1}{10^k}$$

Absolute values

If $x \in \mathbb{R}$, the abs. val. of x, denoted by $|x|$ is

$$|x| := \begin{cases} x, & x \geqslant 0 \\ -x, & x < 0 \end{cases}$$

<u>Thm</u>: If $a \geqslant 0$, $|x| \leq a$ iff $-a \leq x \leq a$

<u>Thm</u>: (Triangle Inequality)
For any $x, y \in \mathbb{R}$, we have $|x+y| \leq |x| + |y|$

Other forms
- $|a-b| \leq |a-c| + |c-b|$
- $|a| - |b| \leq |a+b|$ & $|b| - |a| \leq |a+b|$

$\Rightarrow$ $\big||a| - |b|\big| \leq |a+b|$

By ind^n, in general

$|x_1 + x_2 + \dots + x_n| \leq |x_1| + \dots + |x_n|$ &

$\big||x_1| - |x_2| \dots - |x_n|\big| \leq |x_1 + \dots + x_n|$

Thm: (Cauchy-Schwarz Inequality)

If $a_1, \dots, a_n$ & $b_1, \dots, b_n$ are any real nos.,

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

Moreover, if some $a_i \neq 0$, then equality holds iff $\exists\, x \in \mathbb{R}$ s.t $a_i x + b_i = 0 \quad \forall\, i = 1, 2, \dots, n$

Pf: $\sum_{i=1}^{n} (a_i x + b_i)^2 \geq 0$ is true $\forall x$.

Equality holds iff $a_i x + b_i = 0 \quad \forall\, i = 1, 2, \dots, n$

Expanding, we get $Ax^2 + 2Bx + C = 0$,

$A = \sum_{i=1}^{n} a_i^2$, $B = \sum_{i=1}^{n} a_i b_i$, $C = \sum_{i=1}^{n} b_i^2$

If $A > 0$, for $x = -B/A$, $B^2 \leq AC$ which is the req. ineq.

Else $A = 0$, so $a_i = 0$ & the ineq. is is trivially satisfied.

Function / Mapping / Map

$f : A \to B$

- Range : set of all values $f(x)$, $x \in A$
- For $E \subseteq A$, $f(E)$ is the image of E under f.

- If Range of $f = B$, f is onto.

- For $E \subseteq B$, then the set of all $x \in A$ s.t $f(x) \in E$ is denoted $f^{-1}(E)$ and is called the pre-image (or inverse) of E under f.

- $f^{-1}(y)$ for $y \in B$ is the set of all $x \in A$ s.t $f(x) = y$

- If $\forall\ x_1 \neq x_2 \in A$, $f(x_1) \neq f(x_2)$, f is one-one

- If there exist a one-one map of A onto B, then A & B are said to be in correspondence

& have the same cardinality i.e. $A \sim B$.

The rel^n $\sim$ on the collection of sets is an equivalence rel^n.

- Reflexive : $A \sim A$
- Symmetric : $A \sim B \iff B \sim A$
- Transitive : $A \sim B$ & $B \sim C \Rightarrow A \sim C$

Finite & Infinite Sets

- A set S is finite and said to contain n elems. if $\{1, \dots, n\} \sim S$

n is called the cardinality of S.

- Sets which are <u>not</u> finite are called infinite.

- A set S is <u>countably inf.</u> if $\mathbb{Z}_{>0} \sim S$

$$s : \mathbb{Z}_{>0} \longleftrightarrow S$$

$$S = \{ s(1), s(2), \dots \}$$

or

$$= \{ s_1, s_2, \dots \}$$

Cardinality of count. inf. set is $\aleph_0$
(aleph nought)

- Sets which are <u>not</u> finite or count. inf.
(i.e countable) are called <u>uncountable</u>.

<u>eg</u> : <u>1.</u> $\mathbb{Z}$ is countable

$$\mathbb{Z} = \{ 0, 1, -1, 2, -2, \dots \}$$
$$\mathbb{Z}_{>0} \ \{ 1, 2, 3, 4, 5, \dots \}$$

$f : \mathbb{Z}_{>0} \longleftrightarrow \mathbb{Z}$

$$f(n) = \begin{cases} n/2, & n \text{ is even} \\ -\frac{(n-1)}{2}, & n \text{ is odd} \end{cases}$$

2. Set of even pos. int. is countable

$$f : \mathbb{Z}_{>0} \longleftrightarrow 2\mathbb{Z}_{>0}$$
$$n \longmapsto 2n$$

Thm : Every subset of a countable set is countable

Pf : Let S be a countable set.
Consider $A \subseteq S$.

Case 1 : A is finite.
By def^n A is count.

Case 2 : A is inf.

$\Rightarrow S$ is count. inf.

i.e $S = \{s_1, s_2, \dots\}$

Def. a fx^n k on $\mathbb{Z}_{>0}$ as follows.

- Let $k(1)$ be the smallest m s.t $s_m \in A$
- Suppose $k(1), \dots, k(n-1)$ are defined.
  Let $k(n)$ be the smallest $m > k(n-1)$
  s.t $s_m \in A$

Then, $m_2 > m_1 \Rightarrow k(m_2) > k(m_1)$

$\Rightarrow k$ is one-one

Hence, the composite of s & k satisfies :

- $$\mathbb{Z}_{>0} \xrightarrow{k} \mathbb{Z}_{>0} \xrightarrow{s} S$$

range of $s \circ k$ is A.

- $s \circ k$ is one-one

Since $s_{k(m)} = s_{k(n)} \Rightarrow k(m) = k(n)$

$\Rightarrow m = n$

So, $s \circ k : \mathbb{Z}_{>0} \longleftrightarrow A$

Hence, A is count. inf.

Thm : $\mathbb{R}$ is uncountable

Pf : By prev. thm, it suffices to show $S = \{x \in \mathbb{R} : 0 < x < 1\}$ is uncountable.

Suppose S is countable. i.e $S = \{s_1, s_2, \dots\}$

Write each s_n as an inf. decimal.

$$s_n = 0.a_{n1}a_{n2}a_{n3}\dots$$

Def. the real no. y as,

$$y = 0.b_1 b_2 b_3 \dots$$

where b_i is determined as follows

$$b_n = \begin{cases} 1 , & a_{nn} \neq 1 \\ 2 , & a_{nn} = 1 \end{cases}$$

Since y differs from s_n at n^{th} dec. place $\forall\, n \in \mathbb{Z}_{>0}$, so y is <u>not</u> equal to any elem. of S.

T'fore, $0 < y < 1$ & $y \notin S$ $\rightarrow$ Contd^n

Thm : $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ is countable

Pf : $$f : \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \hookrightarrow \mathbb{Z}_{>0}$$
$$(n, m) \longmapsto 2^n 3^m$$

f is <u>not</u> onto if codomain is $\mathbb{Z}_{>0}$. But we can take range of f as the codomain which will make f a bijection.

$\because$ range of $f \subseteq \mathbb{Z}_{>0} \Rightarrow$ countable

$\therefore$ $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ is countable

Set Algebra

Let F be an arbitrary collection of sets.

- The union of all sets in F is def. to be the set of elems. which belong to at least one of the sets in F.

$$\bigcup_{A \in F} A$$

- The intersection of all sets in F is def. to be the set of elems. which belong to all of the sets in F.

$$\bigcap_{A \in F} A$$

- Complement of A relative to B

$$B \setminus A = \{ x \in B : x \notin A \}$$

Thm : 1. $$B - \bigcup_{A \in F} A = \bigcap_{A \in F} (B - A)$$

2. $$B - \bigcup_{A \in F} A = \bigcap_{A \in F} (B - A)$$

Thm : If F is a count. coll. of count. sets, then the union of all sets in F is also count.

Pf : Case 1 : F is a count. coll. of disjoint sets

Let $F = \{A_1, A_2, \dots\}$ s.t

$A_n = \{a_{1n}, a_{2n}, \dots\} \quad \forall\, n = 1, 2, \dots$

Let $$S = \bigcup_{k=1}^{\infty} A_k$$

If $x \in S$, then $x \in A_n$ for some n.
So, $x = a_{mn}$ for some m.

Since the coll. of sets F is disjoint, the nos. m, n are uniquely determined by x.

Def $\quad f : S \longleftrightarrow \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$

$x \longmapsto (m, n)$

if $x = a_{mn}$

$\because \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ is count.

$\therefore S$ is countable

Case 2 : $\mathcal{F} = \{A_1, A_2, \ldots\}$ is any count. coll. of count. sets

Consider $\mathcal{G} = \{B_1, B_2, \ldots\}$ s.t

$B_1 = A_1$

$B_2 = A_2 - A_1$

$\vdots$

$B_n = A_n - \bigcup_{k=1}^{n-1} A_k$

By construction, $\mathcal{G}$ is coll. of disjoint sets &

$$\bigcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} A_k$$

By Case 1, $\bigcup_{k=1}^{\infty} B_k$ is countable

T'fore, $\bigcup_{k=1}^{\infty} A_k$ is countable

Thm: $\mathbb{Q}$ is countable

Pf: Let $A_n = \{ p \in \mathbb{Q} : p > 0 \ \& \ p = m/n,$ for some $n \}$

Then we can def. a bij. b/w A_n & subset of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ $\Rightarrow$ A_n is countable.

$\because \mathbb{Q}_{>0} = \bigcup_{n=1}^{\infty} A_n$

$\therefore \mathbb{Q}_{>0}$ is countable

Now, $\mathbb{Q} = \mathbb{Q}_{<0} \cup \{0\} \cup \mathbb{Q}_{>0}$

$\mathbb{Q}_{<0} \to \sim \mathbb{Q}_{>0}$, $\{0\} \to$ finite, $\mathbb{Q}_{>0} \to$ countable

Hence, $\mathbb{Q}$ is countable.

Euclidean space $\mathbb{R}^n$

$x = (x_1, \ldots, x_k, \ldots, x_n)$

x_k: k^{th} component/coordinate

x: vector with n components/coordinates

The set of all vecs. with n comps. is called n-dim Euclidean sp. $\mathbb{R}^n$

For $x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n) \in \mathbb{R}^n$

- $x + y := (x_1 + y_1, \ \ldots, \ x_n + y_n)$
- $ax := (ax_1, \ldots, ax_n)$
- $0 = (0, \ldots, 0)$

- Dot/Inner prod : $x \cdot y = \sum_{i=1}^{n} x_i y_i$

- Length/Norm : $\|x\| = (x \cdot x)^{1/2} = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$

- Distance b/w x & y : $\|x - y\|$

Thm : for $x, y \in \mathbb{R}^n$,

1. $\|x\| > 0$ & $\|x\| = 0$ iff $x = 0$
2. $\|ax\| = |a| \, \|x\|$, $\forall \ a \in \mathbb{R}$
3. $\|x - y\| = \|y - x\|$
4. $\|x \cdot y\| \leq \|x\| \, \|y\|$ (Cauchy-Schwarz ineq.)
5. $\|x + y\| \leq \|x\| + \|y\|$ (Triangle ineq.)

Open balls & open sets in $\mathbb{R}^n$

Let $a \in \mathbb{R}^n$ & $r > 0$.

The set of all pts. $x \in \mathbb{R}^n$ s.t $\|x - a\| < r$
is called a ball with center a & radius r.
It is denoted by $B(a, r)$

- Interior pt : Let $S \subseteq \mathbb{R}^n$, and $a \in S$.
  a is called an int. pt. of S if $\exists\, r > 0$
  s.t $B(a, r) \subseteq S$

- The set of all int. pts. of S is called the interior of S & is denoted by $\text{int}(S)$.

- Open set : $S \subseteq \mathbb{R}^n$ is an open set if $S = \text{int}(S)$

  i.e all pts. of S are int. pts.

eg: - In $\mathbb{R}$, any open interval is an open set.

Moreover, union of open intervals is an open set.

A closed interval $[a,b] \subseteq \mathbb{R}$ is <u>not</u> an open set since a & b are <u>not</u> interior pts. of $[a,b]$.

$\{a\}$ is <u>not</u> an open set in $\mathbb{R}$.

- In $\mathbb{R}^2$, any open ball is an open set.

<u>Pf</u>: Let $x \in B(a,r)$ s.t $\|x-a\| = d$.

<u>Claim</u>: for $R < r-d$, $B(x,R) \subseteq B(a,r)$

For $y \in B(x,R)$,

$$\|y-a\| \leq \|y-x\| + \|x-a\|$$
$$\leq R+d < r$$

$\Rightarrow$ $y \in B(a,r)$

An open rectangle $(a,b)\times(c,d) \subseteq \mathbb{R}^2$ is an open set.

- In $\mathbb{R}^n$, $S=\emptyset$ & $S=\mathbb{R}^n$ are open sets.
  (vacuously)

Thm: The union of any collection of open sets is an open set.

Pf: Let F be a collection of open sets.
Let $S = \bigcup_{A\in F} A$

Let $x \in S$. Then $x \in A$ for some $A \in F$.
$\because$ A is open $\Rightarrow \exists\, r>0$ s.t $B(x,r) \subseteq A$
$\Rightarrow B(x,r) \subseteq S$

$\therefore$ x is an int. pt. of S, $\forall\, x \in S$

Hence, S is an open set.

Thm : The intersection of any finite collection of open sets is open.

Pf : Let F be a finite collection of open sets.
Let $S = \bigcap_{k=1}^{m} A_k$

Let $x \in S$. Then $x \in A_k \ \forall k$.
Each A_k is open, i.e $\exists\, r_k > 0$ s.t $B(x, r_k) \subseteq A_k$

Let $r = \min\{r_1, \dots, r_m\}$.
Then $B(x, r) \subseteq B(x, r_k) \subseteq A_k \ \forall k$
$\Rightarrow B(x, r) \subseteq S$

$\therefore$ x is an int. pt. of S, $\forall x \in S$

Hence, S is an open set.

Why does it fail for infinite collection of sets?

eg: Let $A_n = \left(\frac{-1}{n}, \frac{1}{n}\right)$

$$S = \bigcap_{n=1}^{\infty} A_n = \{0\} \rightarrow \text{Closed set}$$

Open sets in $\mathbb{R}$

Let S be an open subset of $\mathbb{R}$. An open interval I (not necessarily finite) is called a component interval of S if $I \subseteq S$ & $\nexists$ open interval $J \neq I$ s.t $I \subseteq J \subseteq S$.

eg: If $S = (1,5) \cup (7,10)$, then $(1,5)$ & $(7,10)$ are comp. intervals of S.

Thm 1: Every pt. of a non-empty open set $S \subseteq \mathbb{R}$ belongs to exactly one comp. interval of S.

Thm 2: Every non-empty open set $S \subseteq \mathbb{R}$ is the union of a countable coll. of disjoint comp. intervals of S.

Pf: (Thm 1)

Let $x \in S$. Since $S \subseteq \mathbb{R}$ is open, x is contained in some open interval I, with $I \subseteq S$ (possibly many)

Let $I_x = (a_x, b_x)$, where

$a_x = \inf\{a : (a, x) \subseteq S\}$

$b_x = \sup\{b : (x, b) \subseteq S\}$

Here, a_x & b_x may possibly be $-\infty$ & ∞ resp.

Claim: $\nexists$ open $J \neq I_x$ s.t $I_x \subseteq J \subseteq S$

i.e I_x is comp. interval of S containing x.

If J_x is another comp. interval of S containing x, then $I_x \cup J_x$ is an open interval contained in S s.t $I_x \subseteq I_x \cup J_x$ & $J_x \subseteq I_x \cup J_x$

$\Rightarrow$ $I_x = I_x \cup J_x$ & $J_x = I_x \cup J_x$ $\left(\because I_x \text{ \& } J_x \text{ are comp. intervals}\right)$

$\Rightarrow$ $I_x = J_x$ $\rightarrow$ Contd^n

Hence, I_x is the only comp. interval of S containing x.

Pf: (Thm 2)

For $x \in S$, let I_x be the comp. interval of S containing x.

If $I_x \cap I_y \neq \emptyset$, then $I_x \cup J_x$ is an open interval contained in S s.t $I_x \subseteq I_x \cup J_x$ & $J_x \subseteq I_x \cup J_x$

$\Rightarrow$ $I_x = I_x \cup J_x$ & $J_x = I_x \cup J_x$ $\left(\because I_x \text{ \& } J_x \text{ are comp. intervals}\right)$

$\Rightarrow$ $I_x = J_x$ $\rightarrow$ Contd^n

T'fore, the (distinct) intervals I_x form a disjoint collection.

We now show the collection has countably many intervals.

Let $\mathbb{Q} = \{x_1, x_2, \ldots\}$

Def. a fx^n F by $F(I_x) = n$ if
$x_n \in I_x$ with the smallest index n.

$\because$ $F(I_x) = F(I_y) \Rightarrow I_x \cap I_y \neq \emptyset$ so $I_x = I_y$

$\therefore$ F is a bij b/w coll. of I_x & some subset of $\mathbb{Z}_{>0}$

Hence, coll. of I_x is countable.

Closed set : A set $S \subseteq \mathbb{R}^n$ is said to be closed if $\mathbb{R}^n - S$ is open

eg : 1. $[a, b]$ is closed in $\mathbb{R}$

2. $[a_1, b_1] \times \ldots \times [a_n, b_n]$ is closed in $\mathbb{R}^n$

Thm : The union of a finite coll. of closed sets is closed.

The intersection of an arbitrary coll. of closed sets is closed.

eg : 1. $A_n = [1/n, 1-1/n]$ ← closed

$$\bigcup_{n=1}^{\infty} A_n = (0, 1) \leftarrow \text{open}$$

Thm: If A is open & B is closed, then $A-B$ is open & B is closed.

Pf:

$$A-B = \underset{\substack{\uparrow \\ \text{open}}}{A} \cap \underset{\substack{\uparrow \\ \text{open}}}{(\mathbb{R}^n - B)} \Rightarrow A-B \text{ is open}$$

$$B-A = \underset{\substack{\uparrow \\ \text{closed}}}{B} \cap \underset{\substack{\uparrow \\ \text{closed}}}{(\mathbb{R}^n - A)} \Rightarrow B-A \text{ is closed}$$

Note: A set could be both open & closed

eg - $\emptyset$, $\mathbb{R} \subseteq \mathbb{R}$, $\mathbb{R}^n \subseteq \mathbb{R}^n$

Also, a set could be neither open nor closed

eg - $\mathbb{Q} \subseteq \mathbb{R}$, $(a, b]$

Accumulation / Limit pt : If $S \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$, then x is called an acc./lt. pt. of S if every ball $B(x)$ contains at least one pt. of S distinct from x.

eg: 1. $S = \{\frac{1}{n}, n \in \mathbb{Z}_{>0}\}$ has 0 as lt. pt.

2. Every pt. of set $T = [a, b]$ is a lt. pt. of the set $S = (a, b)$

3. For $S = [0, 1] \cup \{2\}$, all pts. in $[0, 1]$ are lt. pt.

Thm : If x is a lt. pt. of S, then every ball $B(x)$ around x contains infinitely many pts. of S.

Pf: Suppose $\exists$ a ball $B(x)$ which contains finitely many pts. of S say $a_1, a_2, \dots, a_m$

Let $r > 0$ be s.t $r < \min\{\|x-a_1\|, \dots, \|x-a_m\|\}$

Then $B(x, r)$ contains no pts. of S,
which is a contd^n to x being a lt. pt. of S.

Thm: A set $S \subseteq \mathbb{R}^n$ is closed iff it contains all its lt. pts.

Pf: ($\Rightarrow$) Suppose S is closed. Let x be a lt. pt. of S.

Claim: $x \in S$

Suppose $x \notin S$. $\Rightarrow x \in \mathbb{R}^n - S$.
$\because S$ is closed $\Rightarrow \mathbb{R}^n - S$ is open.

So, $\exists$ ball $B(x)$ s.t $B(x) \subseteq \mathbb{R}^n - S$
$\Rightarrow B(x)$ does not contain any pts. of S,
which is a contd^n to x being a lt. pt. of S.

($\Leftarrow$) Suppose S contains all of its lt. pts.

<u>Claim</u>: S is closed or $\mathbb{R}^n - S$ is open.

Suppose $y \in \mathbb{R}^n - S$ s.t any ball $B(y)$, is <u>not</u> completely contained $\mathbb{R}^n - S$.

$\Rightarrow$ $B(y)$ contains some pt. of S

$\Rightarrow$ y is a lt. pt. of S

$\Rightarrow$ $y \in S$ $\rightarrow$ Contd^n

<u>Closure</u>: If S' denote the set of all lt. pts. of S, the closure of S (denote $\bar{S}$) is defined to be the set $\bar{S} = S \cup S'$

<u>Note</u>: $\bar{S}$ is a closed set

<u>eg</u>: <u>1.</u> $S = (0,1) \Rightarrow \bar{S} = [0,1]$

<u>2.</u> $S = \{1/n : n \in \mathbb{Z}_{>0}\}$, $S' = \{0\}$

$\bar{S} = \{0, 1, 1/2, \ldots\}$

Bounded set : A set $S \subseteq \mathbb{R}^n$ is said to be bounded if $\exists\, r > 0$, s.t $S \subseteq B(0, r)$

Thm : (Bolzano-Weierstrass)
Let $S \subseteq \mathbb{R}^n$ be bounded & contain infinitely many pts. Then S has a lt. pt.

Pf : Case 1 : $n = 1$
Bounded $S \subseteq \mathbb{R}$ $\Rightarrow$ $S \subseteq [-a, a]$
($\because \exists\, a$ s.t $S \subseteq B(0, a)$)

At least one of $[-a, 0]$ & $[0, a]$ contain inf. many pts. of S.

Let one such interval be I_1
Bisect I_1. At least one of the two halves must contain inf. many pts. of S.

Pick one such half, say $I_2 = [a_2, b_2]$

By continuing the process, we obtain a count. coll. of subintervals $I_1, I_2, \ldots$ s.t interval $I_m = [a_m, b_m]$ has length $b_m - a_m = a/2^{m-1}$

1. $a_1 \leq a_2 \leq \ldots \leq a$
So, $\{a_1, a_2, \ldots\}$ has a sup. say a_∞

2. $b_1 \geq b_2 \geq \ldots \geq -a$
So, $\{b_1, b_2, \ldots\}$ has an inf. say b_∞

Claim: $a_\infty \leq b_\infty$
Since, each b_i is ub for $\{a_n\}$,
$a_\infty \leq b_i \quad \forall i \in \mathbb{Z}_{>0}$ $\quad$ ($\because$ a_∞ is sup)
$\Rightarrow$ a_∞ is lb. for $\{b_n\}$.
$\Rightarrow$ $a_\infty \leq b_\infty$ $\quad$ ($\because$ b_∞ is inf.)

Claim: $a_\infty = b_\infty$ (Because as $m \to \infty$, $b_m - a_m \to 0$)

Suppose $a_\infty < b_\infty$. Let $r = b_\infty - a_\infty$
Pick large m s.t $\dfrac{a}{2^{m-1}} < r$

$\because$ $b_m \geq b_\infty$ & $a_m \leq a_\infty$ $\Rightarrow$ $b_m - a_m \geq b_\infty - a_\infty$

$\Rightarrow \quad a/2^{m-1} \geq r \longrightarrow$ Contd^n

Let $\quad x = a_\infty = b_\infty$

Claim: x is lt. pt. of S.

Let $B(x, r)$ be some open ball around x

Pick large n s.t $b_n - a_n = \dfrac{a}{2^{n-1}} < r$

$\because \quad b_n \geq x \quad \& \quad a_n \leq x \quad \Rightarrow \quad I_n = [a_n, b_n] \subseteq B(x, r)$

$\because \quad I_n$ contains inf. many pts. of S

$\therefore \quad B(x, r)$ contains a pt. of S diff. from x.

Hence, x is a lt. pt. of S

Case 2 : $n > 1$

$\because$ $S \subseteq \mathbb{R}^n$ is bounded

$\therefore$ $\exists\ a > 0$ s.t $S \subseteq B(0, a)$

$$\Rightarrow S \subseteq [-a, a] \times \dots \times [-a, a] = [-a, a]^n$$

Each $[-a, a]$ can be bisected as $[-a, 0] \cup [0, a]$

So, S can be written as union of 2^n subsets.

$\because$ S is an inf. set, at least one of these subsets (say I_1) contains inf. many pts. in S.

$$I_1 = [a_1', b_1'] \times \dots \times [a_n', b_n']$$

Sim. bisecting each side of I_1, I_1 can be written as a union of 2^n subsets and one of these subsets (say I_2) contains inf. many pts. of S.

$$I_2 = [a_1^2, b_1^2] \times \dots \times [a_n^2, b_n^2]$$

By continuing the process, we obtain a count. coll. of n-dim. subsets $I_1, I_2, \ldots$ s.t $b_k^m - a_k^m = \frac{a}{2^{m-1}}$, $k = 1, 2, \ldots$

Sim. to proof for $\mathbb{R}^1$, we have

$$x_k^\infty = \sup\{a_k^m\} = \inf\{b_k^m\}, \quad \forall\, m, \quad 1 \leq k \leq n$$

Then, the pt. $x^\infty = \{x_1^\infty, \ldots, x_n^\infty\}$ is a lt. pt. of x^∞.

Application of BW thm

<u>Thm</u>: Let $\{A_1, A_2, \dots\}$ be a count. coll. of sets in $\mathbb{R}^n$ s.t each A_k is non-empty, closed, bounded and $A_{k+1} \subseteq A_k$ $\forall k$. Then the intersection $\bigcap_{k=1}^{\infty} A_k$ is closed and non-empty.

<u>Pf</u>: Let $S = \bigcap_{k=1}^{\infty} A_k$. Then S is closed

<u>Case 1</u>: $\exists k_0$ s.t A_{k_0} is a finite set.

(Proof is trivial)

<u>Case 2</u>: $\forall k \in \mathbb{Z}_{>0}$, A_k is inf. set.

$\Rightarrow$ We can select a set of distinct pts.

$T = \{x_1, x_2, \dots\}$ s.t $\forall k \geq 1$, $x_k \in A_k$

$\because$ $T \subseteq A_1$ which is bounded $\Rightarrow$ T is bounded

By BW thm, T has lim. pt. say x_∞

For each k, $x_i \in A_k \ \forall \ i \geq k$

$\Rightarrow$ x_∞ is a lim. pt. of

$T_k = \{x_k, x_{k+1}, \dots\} \subseteq A_k$

So, x_∞ is lim. pt. of A_k.

But A_k is closed, hence $x_\infty \in A_k$

T'fore $x_\infty \in A_k \ \forall \ k \geq 1$

$\Rightarrow$ $x_\infty \in \bigcap_{k=1}^{\infty} A_k = S$ $\Rightarrow$ $S \neq \emptyset$

Cover : A coll. F of sets is said to be a cover (covering) of a given set S if $S \subseteq \bigcup_{A \in F} A$

F is also said to cover S.

If F is a coll. of open sets that covers S, then F is called an open cover

eg : 1. $F = \{ (1/n, 2/n) : n \in \mathbb{Z}_{>0} \}$ is an open cover of $S = (0,1)$

F is a countable open cover.

2. $S = \mathbb{R}$ is covered by the coll. F of all open intervals (a, b).

F is uncountable open cover. But it has a countable subcoll. that still covers $\mathbb{R}$

eg : $(n, n+2)$, $n \in \mathbb{Z}_{>0}$

Thm: Suppose $A \subseteq \mathbb{R}^n$ and let $\mathcal{F}$ be an open cover of A. Then there is a countable subcoll. of $\mathcal{F}$ which covers A.

Lem: Let $G = \{A_1, A_2, \dots\}$ denote the countable coll. of all balls in $\mathbb{R}^n$ with rational radii & centres at rational coordinates.

Suppose $x \in \mathbb{R}^n$ & let S be an open set in $\mathbb{R}^n$ which contains x. Then $x \in A_k \subseteq S$ for some $A_k \in G$

Pf: (Lem)

$\because$ S is an open set containing x

$\Rightarrow \exists\, r > 0$ s.t $B(x, r) \subseteq S$

If $x = (x_1, \dots, x_n)$, then $y_k \in \mathbb{Q}$ s.t $|y_k - x_k| < r/4n$

Let $y = (y_1, \dots, y_n)$

Then $\|y - x\| \leq |x_1 - y_1| + \dots + |y_n - x_n|$

$< r/4$

Let $q \in \mathbb{Q}$ s.t $r/4 < q < r/2$

Then $\|y - x\| < q \Rightarrow x \in B(y, q)$

Also, if $z \in B(y, q)$, then

$$\|z - x\| \leq \|z - y\| + \|y - x\| < q + q < r$$

$\Rightarrow z \in B(x, r)$

$\Rightarrow B(y, q) \subseteq B(x, r)$

So, $B(y, q)$ is the req. elem. A_k of $\mathcal{G}$.

Pf: (Thm)

Let $G = \{A_1, A_2, \dots\}$ as in the prev. lemma.

Let $x \in A$, let $S \in F$ s.t $x \in S$.

By the lemma $\exists\ A_k \in G$ s.t $x \in A_k \subseteq S$

There will be many such A_k for each x.

Let us choose the one with the smallest index

say $m = m(x)$.

Then $x \in A_{m(x)} \subseteq S$.

The set of open balls $A_{m(x)}$ as x varies in A,

is a countable coll. of open sets which covers A.

For each $A_{m(x)}$, pick one of the sets S of F

which contains $A_{m(x)}$. Then the coll. of such

S is a count. subcoll. of F which covers A.

Thm : (Heine-Borel)

Let F be an open cover of a closed and bounded set $A \subseteq \mathbb{R}^n$. Then a finite subcoll. of F also covers A.

Pf : By prev. thm, $\exists$ count. subcoll. of F say $\{I_1, I_2, \dots\}$ which covers A.

For $m \geq 1$, consider the finite union $S_m = \bigcup_{j=1}^{m} I_j$
This is a union of open sets & hence is open.

Also, $A \subseteq \bigcup_{j=1}^{\infty} I_j$.
We will show $\exists\, m$ s.t $A \subseteq S_m$

For this, def. a count. coll. of sets $\{T_1, T_2, \dots\}$ as follows.

$T_1 = A$
$T_m = A - S_m = A \cap (\mathbb{R}^n - S_m)$

We want to show $\exists\, m$ s.t $T_m = \emptyset$

Note,

1. each T_m is closed
   (intersection of closed sets)
2. $T_{m+1} \subseteq T_m$
3. each T_m is bounded ($T_m \subseteq A$)

If each T_m is non-empty, $\bigcap_{k=1}^{\infty} T_k$ is non-empty.

Then, if $x \in \bigcap_{k=1}^{\infty} T_k = \bigcap_{k=1}^{\infty} (A - S_k)$,
it means that x is an elem. of A which is <u>not</u> contained in any of the sets S_k.

This is a contdn to $A \subseteq \bigcup_{k=1}^{\infty} S_k$.

$\therefore \exists\, m$ s.t $T_m = \emptyset$

<u>Compact set</u> : A set $S \subseteq \mathbb{R}^n$ is compact if every open cover of S has a finite subcover

<u>Thm</u> : Let $S \subseteq \mathbb{R}^n$. Then the following statements are eq.

1. S is compact
2. S is closed & bounded
3. Every infinite subset of S has a lim. pt. in S.

<u>Pf</u> : $2 \Rightarrow 1$ By Heine-Borel thm

$1 \Rightarrow 2$ Suppose S is a compact set.
The coll. of balls $B(0, m)$, $m \in \mathbb{Z}_{>0}$ is an open cover of S.

$\because$ S is compact $\Rightarrow$ it has a finite subcover say $\{B(0, m_1), \ldots, B(0, m_k)\}$

Let $R = \max\{m_1, \ldots, m_k\}$. Then $S \subseteq B(0, R)$
So, S is bounded

Suppose S is not closed. Then $\exists$ a lim. pt. y of S s.t $y \notin S$.

For each $x \in S$, def. $r_x = \dfrac{\|x-y\|}{2} > 0$

The coll. $F = \{B(x, r_x) : x \in S\}$ is an open cover of S.

$\because$ S is compact $\Rightarrow$ $\exists$ a finite subcover say $\{B(x_1, r_1), \dots, B(x_m, r_m)\}$

Let $r = \min\{r_1, \dots, r_m\}$

Then $B(y, r) \cap B(x_k, r_k) = \emptyset$

$\Rightarrow B(y, r) \cap S = \emptyset$ $\rightarrow$ Contd^n

$\because$ y is a lim. pt. of S

So, S is closed

eg - 1. $[a, b] \subseteq \mathbb{R}$ is compact in $\mathbb{R}$

2. Let $S = \{a_1, \dots, a_n\}$ be a finite set in $\mathbb{R}$.
Then S is compact.

3. $\mathbb{Z} \subseteq \mathbb{R}$ is bounded but not closed.
So $\mathbb{Z}$ is not compact.

4. $\left\{ \frac{1}{n} : n = 1, 2, \dots \right\}$ is bounded but not closed.

So, not compact

(Continuing the proof)

$2 \Rightarrow 3$. If T is an infinite subset S, then T is bounded ($\because S$ is bounded)

By Bolzano-Weierstrass theorem, T has a lim. pt. x.
Then x is also a lim. pt. of S.
$\because S$ is closed $\Rightarrow x \in S$

$3 \Rightarrow 2$ First we show S is bounded.

If S is unbounded, then $\forall\, m > 0$, $\exists\, x_m \in S$ s.t $\|x_m\| > m$. (x_m lies outside $B(0, m)$)

If $T = \{x_1, x_2, \dots\}$, then T is an infinite subset of S.

$\therefore$ By 3, T has a lim. pt. in S, say y.

But, if we choose m large enough s.t $m > 1 + \|y\|$ then

$$\begin{aligned} \|x_m - y\| &\geq \|x_m\| - \|y\| \\ &> m - \|y\| > 1 \end{aligned}$$

i.e $\forall\, m$ larger than $1 + \|y\|$, the pt. x_m lies outside the ball $B(y, 1)$

This contradicts the fact that y is a lim. pt. of T.

Next, we show that S is closed. To do this, let x be a lim. pt. of S. We will show that $x \in S$.

Consider open balls $B(x, 1/k)$, $k = 1, 2, \dots$

$\because$ each open ball around x contains infinitely many pts. of S, we obtain a countable set of distinct pts, say $T = \{x_1, x_2, \dots\} \subseteq S$ s.t $x_k \in B(x, 1/k)$

Then x is also lim. pt. of T.

We will show that x is the only lim. pt. of T (because then by 3, $x \in S$)

3 will imply that T has a lim. pt. in S.
If we showed that x is the only lim. pt. of T, then x must be in S.

Suppose $y \neq x$ is a lim. pt. of T.
For $x_k \in T$,

$$\|y-x\| \leq \|y-x_k\| + \|x_k - x\|$$
$$< \|y-x_k\| + 1/k$$

Take k_0 large enough s.t $\frac{1}{k} < \frac{1}{2}\|y-x\| \quad \forall\ k \geq k_0$

Then, for $k \geq k_0$, we have

$$\|y-x\| < \|y-x_k\| + \frac{1}{2}\|y-x\|$$

$$\Rightarrow \quad \frac{1}{2}\|y-x\| < \|y-x_k\|$$

So, if we set $\lambda = \frac{1}{2}\|y-x\|$, then this becomes

$$\lambda < \|y-x_k\|$$

i.e $x_k \notin B(y,\lambda) \rightarrow$ Contd^n ($\because$ y is lim. pt. of T)

$\therefore$ x is the only lim. pt. of T
Hence, by prev. discussion $x \in S$ i.e. S is closed.

Metric space

A metric sp. is a non-empty set M together with a fxn $d: M \times M \to \mathbb{R}$ satisfying the following ppts. $\forall x, y, z \in M$

1. $d(x,x) = 0 \quad \forall x \in M$
2. $d(x,y) > 0 \quad \forall x \neq y \in M$
3. $d(x,y) = d(y,x)$
4. $d(x,y) \leq d(x,z) + d(z,y)$

The fxn d is called the dist. fxn of the metric sp. (M, d)

1. $M = \mathbb{R}^n$, $\quad d(x,y) = \|x-y\|$

2. Discrete Metric space

For any non-empty set M, define $d: M \times M \to \mathbb{R}$ as follows

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

discrete metric

3. If (M, d) is a metric space, and non-empty $S \subseteq M$, then (S, d) is also a metric space with the restriction of d to S.

(S, d) is called a metric subsp. of (M, d)

eg - $(\mathbb{Q}, d)$ with $d = |x - y|$ is a metric subsp. of $(\mathbb{R}, d)$

4. $(\mathbb{R}^2, d)$ with $d(x, y) = \sqrt{(x_1 - y_1)^2 + 4(x_2 - y_2)^2}$

Note, $d(x, y) = d_{Euc.}(\tilde{x}, \tilde{y}) = \| \tilde{x} - \tilde{y} \|$
where $\tilde{x} = (x_1, 2x_2)$, $\tilde{y} = (y_1, 2y_2)$

So, $d(x, y) = \| \tilde{x} - \tilde{y} \| \leq \| \tilde{x} - \tilde{z} \| + \| \tilde{z} - \tilde{y} \|$
$= d(x, z) + d(z, y)$

$\Rightarrow d(x, y) \leq d(x, z) + d(z, y)$

5. $M = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$

$d(x,y)$ = length of smaller arc along the unit circle joining x & y

6. $M = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$

$d(x,y)$ = length of the shorter arc along the great circle joining x & y.

7. $M = \mathbb{R}^n$

$d(x,y) = |x_1 - y_1| + \dots + |x_n - y_n|$

8. $M = \mathbb{R}^n$

$d(x,y) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}$

In $\mathbb{R}^n$, we had notions like open & closed sets, lim. pt., closure, etc.

These concepts can be extended to an arbitrary metric sp.

<u>Open Ball</u> : If $a \in M$, then the ball with radius $r > 0$ & center a is defined as

$$B(a, r) = \{ x \in M : d(x, a) < r \}$$

Sometimes, we write it as $B_M(a, r)$.

If S is a metric subsp. of M & $a \in S$, then

$$B_S(a, r) = S \cap B_M(a, r)$$

<u>Interior pt.</u> : For $S \subseteq M$, $x \in S$, x is called an int. pt. of S if $\exists\ r > 0$ s.t $B_M(x, r) \subseteq S$

The interior $\text{int}\, S$ is the set of interior pts. of S

<u>Open set</u> : A set S is open in M if
$S = \text{int}\, S$

<u>Closed set</u> : A set S is closed in M if
$M - S$ is open.

eg : 1. Every ball $B_M(a, r)$ in a metric sp. is open in M.

So, in $M = [0,1]$, $d = |x-y|$,
$[0, 1/8)$ & $(1/4, 3/4)$ are open sets.
$= B_M(0, 1/8)$ $\quad = B_M(1/2, 1/4)$

2. In a discrete metric sp. every subset S is open.

$\because$ If $x \in S$, $B(x, 1/2) = \{x\}$ so $\{x\}$ is open in M.

And for any set S, $S = \bigcup_{x \in S} \{x\}$

i.e union of open sets is open (though we have not proved it yet !)

3. In $M = [0,1] \subseteq \mathbb{R}$ every interval of the form $[0, x)$ & $(x, 1]$, where $0 < x < 1$ is an open set in M.

Thm : Let (S,d) be a metric subsp. of (M,d)

Let $X \subseteq S$. X is open in S iff $X = S \cap A$ for some set A open in M.

Pf : $(\Leftarrow)$ Suppose A is open in M & $X = S \cap A$

If $x \in X$, then $x \in A$, so $\exists\, r > 0$ s.t $B_M(x,r) \subseteq A$

Then $B_S(x,r) = S \cap B_M(x,r) \subseteq S \cap A = X$

Hence, X is open.

$(\Rightarrow)$ Suppose X is open in S.
So, $\forall\, x \in X$, $\exists\, r_x > 0$ s.t $B_S(x, r_x) \subseteq X$

Note, $B_S(x, r_x) = B_M(x, r_x) \cap S$

Def. $A = \bigcup_{x \in X} B_M(x, r_x)$

Then A is open in M &

$$A \cap S = \left(\bigcup_{x \in X} B_M(x, r_x) \right) \cap S = \bigcup_{x \in X} B_M(x, r_x) \cap S$$

$$= \bigcup_{x \in X} B_S(x, r_x) = X$$

Thm: Let (S, d) be a metric subsp. (M, d).
Let $Y \subseteq S$ be closed in S iff $Y = S \cap B$ for a set B closed in M.

Pf ($\Leftarrow$) If $Y = S \cap B$ for closed $B \subseteq M$, then $B = M - A$ for some open $A \subseteq M$.

So, $Y = S \cap B = S \cap (M - A) = S - \underbrace{(A \cap S)}_{\text{open in } S \text{ by prev. thm}}$

Hence Y is closed in S.

$(\Rightarrow)$ Suppose Y is closed in S.

Then $Y = S - X$ for open $X \subseteq S$

By prev. thm, $X = S \cap A$ for some open $A \subseteq M$

$$\Rightarrow \quad Y = S - X = S - S\cap A = S \cap M - S \cap A = S \cap \underbrace{(M - A)}_{\text{closed in } M}$$

<u>Accumulation/Lim pt.</u> : For $S \subseteq M$, $x \in M$ is a lim. pt. of S if every ball $B(x, r)$ contains at least one pt. of S diff. from x.

<u>Closure</u> : for S, let S' be the set of all lim pts. of S. The closure $\bar{S}$ of S is defined as

$$\bar{S} = S \cup S'$$

The following thms are valid in every metric space. They are proved exactly the same as we did for $\mathbb{R}^n$, simply by replacing $\|x-y\|$ with $d(x,y)$

1. The union of any coll. of open sets in (M,d) is open & the intersection of any finite coll. of open sets is open.

2. The union of any finite coll. of closed sets in (M,d) is closed & the intersection of any coll. of closed sets is closed.

3. If A is open & B is closed, the $A-B$ is open & $B-A$ is closed

4. For any subset S of M, the following are eq.
   - S is closed in M
   - S contains all its lim. pts.
   - $S = \bar{S}$

eg: Let $M = \mathbb{Q}$ with $d(x,y) = |x-y|$

Let $S = \mathbb{Q} \cap (a,b)$ where $a, b \in \mathbb{R} \setminus \mathbb{Q}$

Then S is closed in $\mathbb{Q}$

Pf: $S = \mathbb{Q} \cap (a,b) = \mathbb{Q} \cap [a,b]$ $\left\{ \because a, b \notin \mathbb{Q} \right\}$

$\because$ $[a,b]$ is closed in $\mathbb{R}$

$\therefore$ S is closed in $\mathbb{Q}$ ($\because$ $\mathbb{Q}$ is metric subsp. of $\mathbb{R}$)

The proof of the following thms used not only the metric ppts. of $\mathbb{R}$ but also special ppts. of $\mathbb{R}$ (coming from completeness axiom), that are not necessarily valid in arbitrary metric spaces.

1. Bolzano - Weierstrass thm

2. If $\{A_n\}_{n=1}^{\infty}$ are closed, non-empty, bounded sets in $\mathbb{R}^n$ s.t $A_k \subseteq A_{k+1}$ $\forall$ k then $\bigcap_{k=1}^{\infty} A_k$ is closed & non-empty

3. For any $A \subseteq \mathbb{R}^n$, any any cover F of A,
F has countable cover.

4. Heine-Borel thm

Open Cover : for a metric sp. (M, d) & $S \subseteq M$.
A coll. F of open subsets of M is said to be an open cover of S if $S \subseteq \bigcup_{A \in F} A$

Compact set : A subset $S \subseteq M$ is compact if every open cover of S has a finite subcover.

Bounded set : S is bounded if $\exists\, a, r \in M$ s.t
$S \subseteq B(a, r)$

<u>Thm</u> : Let S be a compact subset of M.

Then -

1. S is closed and bounded
2. Every infinite subset of S has a lim. pt. in S.

<u>Pf</u> : Proof of 1. is same as $1 \Rightarrow 2$ in earlier proof.

2. Let T be an inf. subset of S s.t no pt of S is a lim. pt. for T.

Then, for any $x \in S$, $\exists$ a ball $B(x)$ which contains no point of T (if $x \notin T$) or exactly one pt. of T (if $x \in T$)

As x ranges over S, the coll. of these balls is an open cover of S.

$\because$ S is compact $\Rightarrow$ $\exists$ a finite subcoll. of these balls that covers S, and hence it also covers T ($\because T \subseteq S$)

which is a contd^n as T is an inf. set. & each ball contains at most 1 pt. of T.

eg: In $M = \mathbb{Q}$, with $d(x,y) = |x-y|$ & $S = \mathbb{Q} \cap (a,b)$ where $a, b \in \mathbb{R} \setminus \mathbb{Q}$.

S is closed & bounded but <u>not</u> compact.

<u>Thm</u>: Let X be a closed subset of a compact metric sp. M. Then X is compact.

<u>Pf</u>: Let $\mathcal{F}$ be an open cover of X,

i.e $X \subseteq \bigcup_{A \in \mathcal{F}} A$

$\because$ X is closed $\Rightarrow$ $Y = M - X$ is open

$\therefore$ The coll. $\tilde{\mathcal{F}} = \mathcal{F} \cup \{Y\}$ is an open cover of M.

$\because$ M is compact, $\exists$ a finite subcoll. of $\tilde{\mathcal{F}}$ which covers M.

wlog, let this finite subcoll. be $\{A_1, \ldots, A_k\} \cup \{Y\}$

In particular, $\{A_1, \ldots, A_k\}$ is a finite open cover of X, which is a finite subcover of $\mathcal{F}$.

<u>Boundary pt</u> : Let subset $S \subseteq M$ metric sp. A pt $x \in M$ is called a b'ry pt of S if every ball $B_M(x, r)$ contains at least one pt. of S & at least one pt. of $M - S$.

The set of all b'ry pts of S is called the <u>boundary</u> of S & is denoted by ∂S.

<u>eg</u> : $M = \mathbb{R}$, $d(x,y) = |x-y|$

$S = \mathbb{Q}$, $\partial S = \mathbb{R}$

<u>Note</u> : $\partial S = \overline{S} \cap \overline{(M-S)}$

Dense set : In a metric sp. M, if A & S are subsets satisfying $A \subseteq S \subseteq \bar{A}$, then we say A is dense in S.

eg: $M = \mathbb{R}$, $S = \mathbb{R}$, $A = \mathbb{Q}$
$\bar{A} = \mathbb{R} \Rightarrow A \subseteq S \subseteq \bar{A}$
So, $\mathbb{Q}$ is dense in $\mathbb{R}$.

Separable metric sp. : A metric sp. M is said to be separable if $\exists$ a countable subset A which is dense in M.

eg: $\mathbb{R}$ is separable because the set $\mathbb{Q}$ is a countable dense subset.

Limit : A seq. $\{x_n\}$ of pts. in a metric sp. (X, d) is said to converge if $\exists$ a pt. $p \in X$ s.t $\forall \epsilon > 0$, $\exists N \in \mathbb{Z}_{>0}$, s.t. $d(x_n, p) < \epsilon$ $\forall n \geq N$

We say that $\{x_n\}$ converges to p & write

$$x_n \to p \quad \text{as} \quad n \to \infty$$

If $\nexists$ such a pt. $p \in S$, then the seq. $\{x_n\}$ is said to diverge.

eg: In $\mathbb{R}$, a seq. $\{x_n\}$ is called increasing if $x_n \leq x_{n+1}$ $\forall n$.

If an increasing seq. is bounded above, if converges to supremum.

Rem: $d(x_n, p) < \epsilon$ whenever $n \geq N$ is eq. to $x_n \in B(p, \epsilon)$ whenever $n \geq N$

Thm: A seq. $\{x_n\}$ in a metric sp. (S, d) can converge to at most one pt. in S.

If a seq. $\{x_n\}$ converges, the unique pt. to which it converges is called the limit of the seq. & is denoted by $\lim_{n \to \infty} x_n$

eg: The seq. $\{1 - \frac{1}{n}\}$ in $\mathbb{R}$ converges to $p = 1$, but it does not converge in the metric subsp. $[0, 1)$

Thm: In a metric sp. (S,d) a seq. $\{x_n\}$ converges iff every subseq. $\{x_{k(n)}\}$ converges

Thm: In a metric sp. (S,d) assume $x_n \to p$ & let $T = \{x_1, x_2, \ldots\}$ be the range of $\{x_n\}$

Then T is bounded

Pf: Let $N \in \mathbb{Z}_{>0}$ s.t $d(x_n, p) < 1 \quad \forall\, n \geq N$

So, $\forall\, n \geq N$, $x_n \in B(p,1)$

Def. $R = \max\{1, d(x_1,p), d(x_2,p), \ldots, d(x_{N-1},p)\}$

Then $T \subseteq B(p,R)$

$\therefore$ T is bounded

Thm: Given a metric sp. (S,d) & a subset $T \subseteq S$.

If a pt. $p \in S$ is a lim. pt. of T, then $\exists$ a seq. $\{x_n\}$ of pts. in T s.t $x_n \to p$

__Thm__ : Suppose $\{x_n\}$ is conv. in a metric sp. (S,d).
Then $\forall\, \epsilon > 0$, $\exists\, N$ s.t $d(x_n, x_m) < \epsilon$ $\forall\, n,m \geq N$

__Pf__ : Let $p = \lim_{n\to\infty} x_n$

Given $\epsilon > 0$, $\exists\, N \in \mathbb{Z}_{>0}$ s.t $d(x_n, p) < \epsilon/2$ $\forall\, n \geq N$

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, p) + d(x_m, p) \\ &< \epsilon/2 + \epsilon/2 \qquad [\because\ n,m \geq N] \\ &\leq \epsilon \end{aligned}$$

__Cauchy seq.__ : A seq. $\{x_n\}$ in metric sp. (S,d) is Cauchy if $\forall\, \epsilon > 0$, $\exists\, N \in \mathbb{Z}_{>0}$ s.t $d(x_n, x_m) < \epsilon$ $\forall\, n,m \geq N$

The thm above states that every conv. seq. is a Cauchy seq. The converse is __not__ true in a general metric sp.

eg: $\left\{1-\frac{1}{n}\right\}$ is Cauchy but not conv. in $[0,1)$

Consider $x_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots$

Not easy to find lim. But easy to show it is Cauchy.

Thm: In $\mathbb{R}^k$, every Cauchy seq. is conv.

Pf: Let $\{x_n\}$ be Cauchy in $\mathbb{R}^k$.

Case 1: The set $T = \{x_1, x_2, \dots\}$ is finite.
Then all but a finite no. of terms are x_n equal.
Hence, $\{x_n\}$ converges.

Case 2: The set $T = \{x_1, x_2, \dots\}$ is infinite.

$\because$ $\{x_n\}$ is Cauchy $\Rightarrow$ $\exists\, N \in \mathbb{Z}_{>0}$ s.t $\|x_n - x_m\| < 1$
$\forall\, n, m \geq N$

Let $R = \max\{1, \|x_1 - x_N\|, \ldots, \|x_{N-1} - x_N\|\}$

Then $T \subseteq B(x_N, R)$, so T is bounded.

By BW thm, T has a lim. pt p in $\mathbb{R}^k$

Claim: $x_n \to p$ as $n \to \infty$

Given $\epsilon > 0$, $\exists\, N_0 \in \mathbb{Z}_{>0}$ s.t $\|x_n - x_m\| < \epsilon/2 \quad \forall\, m, n \geq N_0$

Also, $\exists\, m_0 \geq N_0$ s.t $\|x_{m_0} - p\| < \epsilon/2$

$\therefore\ \forall\, n \geq N_0$,

$$\begin{aligned} \|x_n - p\| &\leq \|x_n - x_{m_0}\| + \|x_{m_0} - p\| \\ &< \epsilon/2 + \epsilon/2 \\ &\leq \epsilon \end{aligned}$$

Hence $\lim\limits_{x \to \infty} x_n = p$

Complete metric sp. : A metric sp. (S, d) is called complete if every Cauchy seq. converges in S.

$T \subseteq S$ is called complete if (T, d) is complete.

eg : 1. $\mathbb{R}^k$ is complete
2. $(0,1)$ is not complete
3. $\mathbb{R}^n$ with $d(x,y) = \max_{1 \le i \le n} |x_i - y_i|$ is complete.

Thm : In any metric sp. (S, d), every compact set is complete.

Pf : Let $\{x_n\}$ be a Cauchy seq. in T.
If $A = \{x_1, x_2, \ldots\}$ is finite, then $\{x_n\}$ conv. to one of the elems. of A.
$\therefore$ $\{x_n\}$ converges in T.

If A is infinite, then since T is compact subset of S, A has a lim. pt. p in T.
Then $x_n \to p$

Suppose we have 2 metric sp. (S, d_S) & (T, d_T).
Let $A \subseteq S$ & $f: A \to T$

<u>Limit of f</u>: If p is a lim. pt. of A & if $b \in T$, then $\lim_{x \to p} f(x) = b$ is def. as

$\forall \epsilon > 0$, $\exists \delta > 0$ s.t $x \in A$, $x \neq p$, $d_S(x, p) < \delta$
$\Rightarrow d_T(f(x), b) < \epsilon$

<u>Thm</u>: Suppose p is a lim. pt. of A & $b \in T$.
Then the following are eq.

1. $\lim_{x \to p} f(x) = b$

2. $\lim_{n \to \infty} f(x_n) = b$ for every seq. $\{x_n\}$ of pts. in $A - \{p\}$ which conv. to p.

Pf : ($\Rightarrow$)

Let $\{x_n\}$ be a seq. in $A-\{p\}$ which conv. to p.

Given $\epsilon>0$, let $\delta>0$ be s.t. $\underset{x\neq p}{d_S(x,p)<\delta} \Rightarrow d_T(f(x),b)<\epsilon$

Let $N\in\mathbb{Z}_{>0}$ be s.t $d_S(x_n,p)<\delta \quad \forall\, n\geq N$

Then $n\geq N \Rightarrow d_S(x_n,p)<\delta \Rightarrow d_T(f(x_n),b)<\epsilon$ i.e

$\lim_{n\to\infty} f(x_n)=b$

($\Leftarrow$) Suppose $\exists\,\epsilon>0$ s.t $\forall\,\delta>0$, $\exists\, x\in A$ s.t

$0<d_S(x,p)<\delta$ but $d_T(f(x),b)\geq\epsilon$

Taking $\delta=1/n$, $n=1,2,\dots$ we get a seq.

$x_1, x_2, \dots$ s.t $0<d(x_n,p)<1/n$ but $d_T(f(x_n),b)\geq\epsilon$

Then $x_n\to p$ but $f(x)$ does <u>not</u> converge to b.

<u>Continuous fxⁿ</u>: Let (S, d_S) & (T, d_T) be two metric sp. & $f : S \to T$. f is said to be continuous at p in S if $\forall \epsilon > 0, \exists \delta > 0$ s.t

$$d_S(x, p) < \delta \Rightarrow d_T(f(x), f(p)) < \epsilon$$

If f is cont. at every pt. of a subset A in S, we say f is cont. on A.

<u>Alt</u>: f is cont. at p iff $\forall \epsilon > 0, \exists \delta > 0$ s.t

$$f(B_S(p, \delta)) \subseteq B_T(f(p), \epsilon)$$

If p is a lim. pt. of S, then if f is cont. at p, then $\lim\limits_{x \to p} f(x) = f(p)$

Thm : Let $f: S \to T$ be a fn from one metric sp (S, d_S) to another (T, d_T) & assume $p \in S$.
Then, f is cont. at p iff. $\forall$ seq. $\{x_n\}$ in S conv. to p, the seq. $\{f(x_n)\}$ in T conv. to $f(p)$

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right)$$

Thm : Let (S, d_S), (T, d_T), (U, d_U) be metric sp. & let $f: S \to T$ & $g: T \to U$. Let $h = g \circ f : S \to U$
If f is cont. at p & g is cont. at $f(p)$, then h is cont. at p.

Pf : Let $b = f(p)$. Given $\epsilon > 0$, $\exists \delta > 0$ s.t
$d_T(y, b) < \delta \Rightarrow d_U(g(y), g(b)) < \epsilon$.

For this δ, $\exists \delta'$ s.t $d_S(x, p) < \delta' \Rightarrow d_T(f(x), f(p)) < \delta$

Combining these statements & taking $y = f(x)$, we get
$d_S(x, p) < \delta' \Rightarrow d_U(g(f(x)), g(f(p))) < \epsilon$
i.e $d_S(x, p) < \delta' \Rightarrow d_U(h(x), h(p)) < \epsilon$
Hence h is cont. at p.

Thm : Let $f: S \to T$. Then f is cont. on S iff $\forall$ open set $Y \subseteq T$, the inverse image $f^{-1}(Y)$ is open in S.

Pf : ($\Rightarrow$)

Let f be cont. on S & $Y \subseteq T$ be open.
Let $p \in f^{-1}(Y)$. Let $y = f(p)$

$\because$ Y is open $\Rightarrow$ $\exists\ \epsilon > 0$ s.t $B_T(y, \epsilon) \subseteq Y$

$\because$ f is cont. at p $\Rightarrow$ $\exists\ \delta > 0$ s.t
$$f(B_S(p, \delta)) \subseteq B_T(y, \epsilon)$$

Hence $B_S(p, \delta) \subseteq f^{-1}(B_T(y, \epsilon)) \subseteq f^{-1}(Y)$
So p is an interior pt. of $f^{-1}(Y)$.

$\because$ this holds for any $p \in f^{-1}(Y)$
$\therefore$ $f^{-1}(Y)$ is open in S.

($\Leftarrow$)

Suppose $f^{-1}(Y)$ is open in S for every open set Y in T.

Let $p \in S$ & $y = f(p)$.
$\forall \epsilon > 0$, $B_T(y, \epsilon)$ is open in T.

$\therefore$ $f^{-1}(B_T(y, \epsilon))$ is open in S.

We have $p \in f^{-1}(B_T(y, \epsilon))$
$\therefore$ $\exists \delta > 0$ s.t $B_S(p, \delta) \subseteq f^{-1}(B_T(y, \epsilon))$ i.e
$f(B_S(p, \delta)) \subseteq B_T(y, \epsilon)$

Hence, f is cont. at p.

Thm: Let $f: S \to T$. Then f is cont. on S iff $\forall$ closed sets Y in T, the inverse image $f^{-1}(Y)$ is closed in S.

Thm : Let $f : S \to T$ be continuous. Let X be a compact subset of S. Then $f(X)$ is a compact subset of T.

Pf : Let F be an open cover of $f(X)$, so

$$f(X) \subseteq \bigcup_{A \in F} A$$

Then $G = \{ f^{-1}(A) : A \in F \}$ is an open cover of X.

$\therefore$ X is compact $\Rightarrow$ G has a finite subcover say $\{ f^{-1}(A_1), \ldots, f^{-1}(A_k) \}$

Then $\{ A_1, \ldots, A_n \}$ is a finite subcover for $f(X)$.

$\therefore$ $f(X)$ is compact.

Rem : In particular, $f(X)$ is closed & bounded.

Thm : Let $f : S \to \mathbb{R}$ be cont. Let $X \subseteq S$ be compact
Then $\exists$ pts. $p, q \in S$ s.t $f(p) = \inf f(X)$
$f(q) = \sup f(X)$

Pf : By prev. thm, $f(X)$ is compact.
In fact, it is closed & bounded.

Let $m = \inf f(X)$. Then $m \in f(X)$ & since $f(X)$ is closed, $\exists\, p \in X$ s.t $m = f(p)$

Sim., $\exists\, q \in X$ s.t $f(q) = \sup f(X)$

eg : 1. A cont. fxn $f : [a, b] \to \mathbb{R}$ attains its maxi & min.

2. $f : (0, 1] \to \mathbb{R}$, $f(x) = x$ does not attain its min. value.

3. $f : (0, 1] \to \mathbb{R}$, $f(x) = 1/x$ is not even bounded above.

<u>Thm</u> : Let $f : (a, b) \to \mathbb{R}$. Assume that f is cont. at a pt. $c \in (a, b)$ & that $f(c) \neq 0$. Then $\exists\, \delta > 0$ s.t $f(x)$ has the same sign as $f(c)$ in $(c-\delta, c+\delta)$

<u>Pf</u> : Suppose $f(c) > 0$.

$\because$ f is cont. at $c \Rightarrow \forall\, \epsilon > 0,\ \exists\, \delta > 0$ s.t

$x \in (c-\delta, c+\delta) \Rightarrow f(c) - \epsilon < f(x) < f(c) + \epsilon$

Take $\epsilon = f(c)/2$. Then $\exists\, \delta > 0$ s.t $f(c)/2 < f(x) < 3f(c)/2$ for $x \in (c-\delta, c+\delta)$

So, $f(x) > 0$ for $x \in (c-\delta, c+\delta)$

Sim., $f(x) < 0$ for $x \in (c-\delta, c+\delta)$ by taking $\epsilon = |f(c)|/2$ in the case $f(c) < 0$

Thm: Let $f : [a,b] \to \mathbb{R}$ be cont. & suppose $f(a)$ & $f(b)$ have opposite signs. Then there is at least one pt. $c \in (a,b)$ s.t $f(c) = 0$

Pf: Suppose $f(a) > 0$ & $f(b) < 0$

Let $A = \{ x : x \in [a,b] \text{ \& } f(x) > 0 \}$
Then A is non-empty since $a \in A$ & bounded above by b.

Let $c = \sup A$. Then $a < c < b$
We will prove $f(c) = 0$

If $f(c) \neq 0$, then $\exists\, \delta > 0$ s.t on $(c-\delta, c+\delta)$, f has the same sign as $f(c)$.

If $f(c) > 0$, then $\exists\, x > c$ s.t $f(x) > 0$ $\to$ Contd^n
$(x \in (c-\delta, c+\delta))$ $\qquad (\because c = \sup A)$

If $f(c) < 0$, then $\exists\, \delta > 0$ s.t $f(x) < 0$ for $x \in (c-\delta, c+\delta)$.
$\because c = \sup A \Rightarrow \exists\, p \in (c-\delta, c)$ s.t $p \in A \Rightarrow f(p) > 0 \to$ Contd^n
$(\because f(x) < 0 \ \forall\, x \in (c-\delta, c+\delta))$

Thm (Intermediate Value Thm)

Suppose $f:[a,b]\to\mathbb{R}$ is cont. & suppose $f(a)\neq f(b)$. Then f takes every value between $f(a)$ & $f(b)$ in the interval (a,b).

Pf: Let k be a no. b/w $f(a)$ & $f(b)$ & apply the prev. thm to $g(x)=f(x)-k$

As a consequence, the image under a cont. fx^n f of a compact interval $S\subseteq\mathbb{R}$, is another compact interval $[\inf f(S), \sup f(S)]$

<u>Disconnected metric sp.</u> : A metric sp. S is disconnected if $S = A \cup B$ where A & B are disjoint, non-empty, open sets in S.

We can S connected if it is <u>not</u> disconnected.

A subset $X \subseteq S$ is connected if when thought of as a metric subsp. of S, it is connected.

<u>eg</u>: <u>1.</u> $S = \mathbb{R} \setminus \{0\}$ is disconnected

$\because$ $S = (-\infty, 0) \cup (0, \infty)$

<u>2.</u> Every open interval in $\mathbb{R}$ is connected

<u>3.</u> $\mathbb{Q}$ is disconnected.

$\because$ $\mathbb{Q} = A \cup B$, where $A = \{p \in \mathbb{Q} : p^2 < 2\}$

$B = \{p \in \mathbb{Q} : p^2 > 2\}$

Thm: Let $f: S \to M$ be a fxn b/w metric sp. Let X be a connected subset of S. If f is cont. on X, then $f(X)$ is connected subset of M.

Pf: Suppose $f(X)$ is disconnected i.e $f(X) = A \cup B$ where A & B are disjoint, non-empty, open sets in M.

Let $\tilde{f}$ denote the restriction of f to X.

Let $U = \tilde{f}^{-1}(A)$ & $V = \tilde{f}^{-1}(B)$. Then U & V are disjoint open sets in X s.t $X = U \cup V \to$ Contdn

($\because X$ is connected)

Rem: This is a generalization of the Intermediate Value Thm for continuous fxns.

Thm: Let cont. $f : \mathbb{R}^n \to \mathbb{R}^n$. Let $X \subseteq \mathbb{R}^n$ be compact & connected. Then $\exists\ p, q \in X$ s.t $f(p) = \inf f(X)$ & $f(q) = \sup f(X)$

Also, if $p, q \in X$ with $f(a) \neq f(b)$ & c is any number b/w $f(a)$ & $f(b)$, then $\exists\ x \in X$ s.t $f(x) = c$

Uniform continuity : Let $f: S \to T$ be a fx^n from one metric sp (S, d_S) to another (T, d_T). Then, f is said to be uniformly cont. on $A \subseteq S$ if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t if $p, q \in A$,
$d_S(p,q) < \delta \Rightarrow d_T(f(p), f(q)) < \epsilon$

eg: 1. $f: (0,1] \to \mathbb{R}$, $f(x) = 1/x$ is <u>not</u> uniformly cont.

Take $\epsilon = 10$ & let $0 < \delta < 1$. We will find p, q s.t $|p-q| < \delta$ but $|f(p) - f(q)| > \epsilon$

If $p = \dfrac{\delta}{10}$, $q = \dfrac{\delta}{20}$, then $|p-q| = \dfrac{\delta}{10} < \delta$

but $|f(p) - f(q)| = \left| \dfrac{1}{p} - \dfrac{1}{q} \right| = \dfrac{10}{\delta} > 10$

2. $f : (0,1] \to \mathbb{R}$, $f(x) = x^2$ is uniformly cont.

To prove, let $\epsilon > 0$.

$$|f(p) - f(q)| = |p^2 - q^2| = |p+q||p-q|$$
$$\leq (|p| + |q|)|p-q|$$
$$\leq 2|p-q|$$

Choose $\delta = \epsilon/2 \Rightarrow |p-q| < \delta \Rightarrow |p-q| < \epsilon/2$

$$\Rightarrow |f(p) - f(q)| < \epsilon$$

Rem: For uniform cont., δ should depend only on ϵ, not the pts. p, q.

Thm: Let $f: S \to T$ be a fn^n from metric sp. (S, d_S) to (T, d_T). Let A be a compact subset of S & assume that f is cont. on A. Then f is uniformly cont. on A.

Pf: Let $\epsilon > 0$. Let $a \in A$.
Since f is cont. on A, $\exists\, r_a > 0$ s.t $d_T(f(x), f(a)) < \epsilon/2$ whenever $x \in B_A(a, r_a)$

The coll. $F = \{ B_A(a, r_a/2),\ a \in A \}$ is an open cover of A.

Since A is compact, F has a finite subcover.

$$F' = \{ B_A(a_1, r_1/2), \ldots, B_A(a_k, r_k/2) \}$$

Let $\delta = \min\{ r_1/2, \ldots, r_k/2 \}$

Claim: $p, q \in A$ s.t $d_S(p, q) < \delta \Rightarrow d_T(f(p), f(q)) < \epsilon$

Suppose $p, q \in A$ s.t $d_S(p,q) < \delta$

Since F' is a cover of A, $\exists$ some i, $1 \leq i \leq k$ for which $p \in B_A(a_i, r_i/2)$ i.e $d_S(p, a_i) < r_i/2$

Then,

$$d_S(q, a_i) \leq d_S(q,p) + d_S(p, a_i)$$
$$< \delta + r_i/2 \leq r_i/2 + r_i/2$$
$$< r_i$$

i.e $q \in B_A(a_i, r_i)$

$$\Rightarrow \quad d_T(f(q), f(a_i)) < \epsilon/2$$

Also, $p \in B_A(a_i, r_i/2) \subseteq B_A(a_i, r_i)$

$$\Rightarrow \quad d_T(f(p), f(a_i)) < \epsilon/2$$

$$d_T(f(p), f(q)) \leq d_T(f(p), f(a_i)) + d_T(f(q), f(a_i))$$
$$< \epsilon/2 + \epsilon/2$$
$$< \epsilon$$

Let $f: S \to S$ (metric sp.). A pt. p in S is called a <u>fixed pt.</u> of f if $f(p) = p$

<u>Contraction</u> : The fxn f is called a contraⁿ if $\exists$ positive $\alpha < 1$ (called contraⁿ constant) s.t

$$d(f(x), f(y)) < \alpha\, d(x, y) \quad \forall\; x, y \in S$$

<u>Note</u> : A contraⁿ of any metric sp. S is uniformly continuous on S.

Thm : (Contraⁿ Mapping Principle)

A contraⁿ f of a complete metric sp. S has a unique fixed pt.

Pf : First, let us find a fixed pt. Later, we show that it is unique.

Existence

Take any pt. $x_0 \in S$ & consider the seq. obtained by repeatedly applying f.

$$x_0, \quad f(x_0), \quad f(f(x_0)), \ldots$$

i.e define a seq. $\{p_n\}$ as follows

$$p_0 = x_0, \quad p_{n+1} = f(p_n) \quad \forall\ n \geq 1$$

We will prove that $\{p_n\}$ converges to a fixed pt. of f.

First, we prove that $\{p_n\}$ is a Cauchy seq.

$$d(p_{n+1}, p_n) = d(f(p_n), f(p_{n-1})) \leqslant \alpha\, d(p_n, p_{n-1})$$

So, by ind^n, we have

$$d(p_{n+1}, p_n) \leqslant \alpha^n d(p_1, p_0) = c\alpha^n$$

By triangle ineq.,

$$d(p_m, p_n) \leqslant \sum_{k=n}^{m-1} d(p_{k+1}, p_k) \leqslant \sum_{k=n}^{m-1} c\alpha^k$$

$$= c\,\frac{\alpha^n - \alpha^m}{1-\alpha} < \frac{c\alpha^n}{1-\alpha}$$

Since $\alpha^n \to 0$ as $n \to \infty$, this ineq. shows that $\{p_n\}$ is a Cauchy seq.

$\because$ S is complete $\Rightarrow$ $\exists\, p \in S$ s.t $p_n \to p$

$$\therefore \quad f(p) = f(\lim_{n\to\infty} p_n) = \lim_{n\to\infty} f(p_n)$$

$$= \lim_{n\to\infty} p_{n+1} = p$$

as f is cont.

i.e p is a fixed pt. of f.

Uniqueness

Suppose $\exists$ 2 fixed pt. p & p' i.e $f(p) = p$ & $f(p') = p'$

$$d(f(p), f(p')) \leq \alpha d(p, p') \Rightarrow d(p, p') \leq \alpha d(p, p')$$

$$\Rightarrow d(p, p') = 0 \quad (\because \alpha < 1)$$

$$\Rightarrow p = p'$$

Derivative

If $f:(a,b)\to\mathbb{R}$, then for 2 distinct pts. x & c in (a,b), we can form the difference quotient

$$\frac{f(x)-f(c)}{x-c}$$

Then, f is said to be <u>differentiable</u> at c if the limit $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ exists.

The limit, denoted by $f'(c)$ is called the derivative of f at c.

This process defines a new fxn f', whose domain is the set of those pts. in (a,b) where f is diff. The fxn f' is called the first derivative of f.

Similarly, nth derivative of f, denoted by $f^{(n)}$ is defined to be the first derivative of $f^{(n-1)}$.

Note: To consider $f^{(n)}$, we require $f^{(n-1)}$ to be defined on an open interval.

Thm: If f is defined on (a,b) & diff. at $c \in (a,b)$, then $\exists$ a fn^n f^* which is cont. at c & satisfies

$$f(x) - f(c) = (x-c) f^*(x) \quad \forall\ x \in (a,b)$$

with $f'(c) = f^*(c)$

Conversely, if $\exists$ a fn^n f^*, cont. at c, which satisfies the above eq^n, then f is diff. & $f'(c) = f^*(c)$

Pf: If $f'(c)$ exists, def. $f^*(x) = \begin{cases} \dfrac{f(x)-f(c)}{x-c}, & x \neq c \\ f'(c), & x = c \end{cases}$

Then f^* is cont. at c & the eq^n holds $\forall\ x \in (a,b)$

Conversely, if $f(x) - f(c) = (x-c)\, f^*(x)$ for some f^* then divide by $x-c$ & let $x \to c$, we see that $f'(c)$ exists & equals $f^*(c)$.

$$f^*(x) = \frac{f(x) - f(c)}{x-c}$$

<u>Cor</u>: If f is diff. at c, then f is cont. at c.

<u>Pf</u>: Let $x \to c$ in $f(x) - f(c) = (x-c)\, f^*(x)$

Thm : Assume f & g are defined on (a,b) & diff. at c. Then $f+g$, $f-g$, fg are also diff. at c & if $g(c) \neq 0$, then f/g is also diff. at c.

Their derivatives are given by

$$(f \pm g)'(c) = f'(c) \pm g'(c)$$
$$(fg)'(c) = f'(c)g(c) + f(c)g'(c)$$
$$(f/g)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}, \quad g(c) \neq 0$$

Pf : We will prove it for fg.

By prev. thm,
$$f(x) = f(c) + (x-c) f^*(x)$$
$$g(x) = g(c) + (x-c) g^*(x)$$

$$\Rightarrow f(x)g(x) - f(c)g(c) = (x-c) \left\{ g(c) f^*(x) + f(c) g^*(x) + (x-c) f^*(x) g^*(x) \right\}$$

Divide by $(x-c)$ & let $x \to c$ to obtain the formula

Diff. & approximating by linear fn^n

f is diff. at $c \in (a,b)$, means $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists. & $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$

Rearrange, $0 = \lim_{x \to c} \frac{f(x)-f(c)-(x-c)f'(c)}{x-c}$

If we consider the linear fn^n $T: \mathbb{R} \to \mathbb{R}$ given by $T(v) = f'(c)v$ for $v \in \mathbb{R}$.

Then, $\lim_{x \to 0} \frac{f(x)-(f(c)+T(x-c))}{x-c} = 0$

Hence, $f(c) + T(x-c) = f(c) + f'(c)(x-c)$ is a linear approximation to $f(x)$ at c.

Thm : (Chain Rule)

Let $f : (a,b) \to \mathbb{R}$. Let g be defined on $f((a,b))$.
Let $c \in (a,b)$ s.t $f(c)$ is an interior pt. of $f((a,b))$.
If f is diff. at c & g is diff at $f(c)$,
then $g \circ f$ is diff. at c & we have

$$(g \circ f)'(c) = g'(f(c))\, f'(c)$$

Pf : We can write $f(x) - f(c) = (x-c)\, f^*(x) \quad \forall\, x \in (a,b)$
where f^* is a fn^n which is cont. at c,
& satisfying $f^*(c) = f'(c)$

Also, $g(y) - g(f(c)) = (y - f(c))\, g^*(y)$ where
g^* is cont. at $f(c)$ & satisfying $g^*(f(c)) = g'(f(c))$

Writing $y = f(x)$, we have
$$g(f(x)) - g(f(c)) = (f(x) - f(c))\, g^*(f(x))$$

By cont. for composite fn^ns
$$\lim_{x \to c} g^*(f(x)) = g^*(f(c)) = g'(f(c))$$

T'fore, $\lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x - c}$ exists

and equals $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \lim_{x \to c} g^*(f(x))$

$$= f'(c)\, g'(f(c))$$

Thm : Let $f : (a,b) \to \mathbb{R}$

Suppose for some $c \in (a,b)$, we have $f'(c) > 0$.
Then there is an interval $(c-\epsilon, c+\epsilon) \subseteq (a,b)$
on which $f(x) > f(c)$ if $x > c$ &
$f(x) < f(c)$ if $x < c$

Pf : We have $f(x) - f(c) = (x-c) f^*(x)$
where f^* is continuous & satisfies $f^*(c) = f'(c)$
(so $f^*(c) > 0$)

By a thm proved earlier, $\exists$ an interval
$(c-\epsilon, c+\epsilon) \subseteq (a,b)$ in which $f^*(x)$ has the
same sign as $f^*(c)$

This implies that $f(x) - f(c)$ has the same sign as $(x-c)$ on the interval $(c-\epsilon, c+\epsilon)$.

Note : Sim. result holds if $f'(c) < 0$ at some interior pt. c of (a, b)

Local Maximum & Minimum :

Let $f : (a, b) \to \mathbb{R}$ & $c \in (a, b)$. Then f is said to have a local maximum at c if $\exists$ an open interval $(c-\epsilon, c+\epsilon) \subseteq (a, b)$ s.t

$$f(x) \leq f(c) \quad \forall \; x \in (c-\epsilon, c+\epsilon)$$

If $f(x) \geq f(c) \;\; \forall \; x \in (c-\epsilon, c+\epsilon)$, then f is said to have local minimum.

Thm: Let $f:(a,b) \to \mathbb{R}$ & assume f has a local max or local min. at an interior pt. c of (a,b). If f is diff. at c, then $f'(c) = 0$.

Pf: By the prev. thm, if $f'(c) > 0$, then f cannot have a local max or local min at c. Sim. statement if $f'(c) < 0$.
T'fore we must have $f'(c) = 0$.

Note: 1. Converse of the thm is <u>not</u> true.

eg: $f(x) = x^3$, then $f'(0) = 0$ but 0 is neither a pt. of local max. or local min.

2. $f(x) = |x|$ has local min at $x = 0$, but f is <u>not</u> diff. at 0.

Thm : (Rolle's Thm)

Suppose $f : [a, b] \to \mathbb{R}$ is cont. & suppose that f is diff. at each pt. in (a, b). If $f(a) = f(b)$, there is at least one interior pt c at which $f'(c) = 0$

Pf: For the sake of contd^n, let us suppose that f' is never 0 in (a, b).

$\because$ f is cont. on the compact set $[a, b]$, it attains its max M & its min m somewhere $[a, b]$.

Neither of these extreme values can occur in (a, b), since f' would vanish at that pt. So, both max. & min are attained at the endpts.

$\because$ $f(a) = f(b)$, we must have $M = m$ and hence f is constant on $[a, b]$ $\to$ Contd^n

(f' is never 0 on (a, b))

T'fore $f'(c) = 0$ for some $c \in (a, b)$.

Thm: (Mean Value Thm)

Suppose $f:[a,b]\to\mathbb{R}$ is cont. & suppose that f is diff. at each pt. in (a,b).
Then $\exists\ c\in(a,b)$ s.t

$$f(b)-f(a) = f'(c)(b-a)$$

Pf: Let $h(x) = f(x)(b-a) - x(f(b)-f(a))$

Then h is cont. on $[a,b]$ & diff. on (a,b).

Also, $h(a) = f(a)(b-a) - a(f(b)-f(a))$
$\qquad = bf(a) - af(b)$

$h(b) = f(b)(b-a) - b(f(b)-f(a))$
$\qquad = bf(a) - af(b)$

T'fore, $h(a)=h(b)$, so by Rolle's thm, $h'(c)=0$ at some $c\in(a,b)$.
i.e $f(b)-f(a) = f'(c)(b-a)$ for some $c\in(a,b)$

<u>Thm</u>: Suppose $f : [a,b] \to \mathbb{R}$ is cont. & suppose that f is diff. at each pt. in (a,b).

<u>1.</u> If f' takes only +ve values on (a,b), then f is strictly increasing on $[a,b]$.

2. If f' takes only −ve values on (a,b), then f is strictly decreasing on $[a,b]$.

<u>Pf</u>: Let $x < y$ & apply the Mean value thm, to the subinterval $[x,y] \subseteq [a,b]$

$f(y) - f(x) = f'(c)(y-x)$ for some $c \in (x,y)$.

The statements follow.

Taylor Polynomial

Let $f:(a,b)\to\mathbb{R}$ be a fxn which is diff. upto rth order, i.e $f^{(r)}(x)$ exists for each $x\in(a,b)$

The rth order Taylor polynomial of f at x is

$$P(h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \dots \frac{f^{(r)}(x)}{r!}h^r$$

$$= \sum_{k=0}^{r} \frac{f^{(k)}(x)}{k!} h^k$$

Here, variable is h, & the coeffs. are $\frac{f^{(k)}(x)}{k!}$ are consts.

Diff. P wrt h at $h=0$, gives

$$P(0) = f(x)$$
$$P'(0) = f'(x)$$
$$\vdots$$
$$P^{(r)}(0) = f^{(r)}(x)$$

We call $R(h) = f(x+h) - P(h)$ the Taylor remainder.

Thm: Let $f : (a,b) \to \mathbb{R}$ & let $x \in (a,b)$.
Suppose that f is r^{th}-order diff. at x.
Then

1. P approximates f to the order r at x in the following sense the Taylor remainder $R(h) = f(x+h) - f(x)$ satisfies $\lim_{h \to 0} \frac{R(h)}{h^r} = 0$

2. If, in add^n, f is $(r+1)^{th}$ order diff. on (a,b), then $\exists\ \theta \in (x, x+h)$ s.t

$$R(h) = \frac{f^{(r+1)}(\theta)}{(r+1)!} h^{r+1}$$

Rem: 1. Diff. at x,

$$\lim_{h \to 0} \frac{f(x+h) - (f(x) + f'(x)h)}{h} = 0$$

2. (2) generalizes mean value thm.

Pf: 1. We observed that, $P(0) = f(x)$

$$\vdots$$

$$P^{(r)}(0) = f^{(r)}(x)$$

Hence, first r derivatives of $R(h)$ exist & equal 0 at $h = 0$.

Now, if $h > 0$, we have

$$R(h) = R(h) - 0 = R(h) - R(0)$$

$\Rightarrow \quad R(h) - R(0) = R'(\theta_1)\, h \qquad [\text{MVT on } R : (0, h) \to \mathbb{R}]$

for some $\theta_1 \in (0, h)$

Further, $R'(0) = 0$

$\Rightarrow \quad R'(\theta_1) = R'(\theta_1) - R'(0) = R''(\theta_2)\,\theta_1 \quad [\text{MVT on } R' : (0, \theta_1) \to \mathbb{R}]$

for some $\theta_2 \in (0, \theta_1)$

$\Rightarrow \quad R(h) = R''(\theta_2)\,\theta_1\, h$

Proceeding in this way, by repeated applications of MVT, we have

$$R(h) = R^{(n-1)}(\theta_{n-1})\, \theta_{n-2} \cdots \theta_1 h$$

where $\quad 0 < \theta_{n-1} < \dots < \theta_1 < h$

T'fore, $$\left| \frac{R(h)}{h^n} \right| = \left| \frac{R^{(n-1)}(\theta_{n-1})\, \theta_{n-2} \cdots h}{h^n} \right|$$
$$\leqslant \left| \frac{R^{(n-1)}(\theta_{n-1}) - 0}{h} \right| \longrightarrow 0$$

as $h \to 0$ as $R^{(n)}$ exists at 0, and $R^{(n)}(0) = 0$.

If $h<0$, then we have a sim. argument with $\quad h < \theta_1 < \dots < \theta_{n-1} < 0$

2. Fix $h > 0$. Def $g(t) = f(x+t) - P(t) - \frac{R(h)}{h^{n+1}} t^{n+1}$

$$= R(t) - \frac{R(h)}{h^{n+1}} t^{n+1}$$

for $0 \leq t \leq h$

$\because$ $P(t)$ is a poly. of degree n,
$P^{(n+1)}(t) = 0 \quad \forall\ t$

Hence, $g^{(n+1)}(t) = f^{(n+1)}(x+t) - (n+1)!\, \frac{R(h)}{h^{n+1}}$

Also, note that,

$$g(0) = g'(0) = \ldots = g^{(n)}(0) = 0$$

&

$$g(h) = R(h) - R(h) = 0$$

By MVT applied to $g: (0, h) \to \mathbb{R}$, $\exists\ t_1 \in (0, h)$
s.t $g'(t_1) = \frac{g(h) - g(0)}{h} = \frac{0-0}{h} = 0$

By MVT applied to $g': (0, t_1) \to \mathbb{R}$, $\exists\ t_2 \in (0, t_1)$

s.t $\quad g'(t_2) = \dfrac{g'(t_1) - g'(0)}{t_1} = 0$

Continuing in this way, we get nos.

$h > t_1 > t_2 \dots > t_{r+1} > 0$ s.t $g^{(k)}(t_k) = 0 \quad \forall\ k = 1, \dots, r+1$.

The last eq^n, $g^{(r+1)}(t_{r+1})$ implies

$$f^{(r+1)}(x + t_{r+1}) - \frac{(r+1)!}{h^{(r+1)}} R(h) = 0$$

T'fore, if we set $\theta = x + t_{r+1}$, then we have

$$R(h) = \frac{f^{(r+1)}(\theta)\ h^{r+1}}{(r+1)!}$$

Discontinuities of real-valued fn^{n}s

Let $f : (a, b) \to \mathbb{R}$ & let $c \in [a, b)$. If $f(x) \to A$ as $x \to c$ through values greater than c, we say that A is the righthand limit of f at c, and we write $\lim_{x \to c^+} f(x) = A$

The righthand limit A is also denoted by $f(c^+)$

Another way of stating this is that $\forall\, \epsilon > 0$, $\exists\, \delta > 0$, s.t $|f(x) - f(c^+)| < \epsilon$ whenever $c < x < c + \delta < b$

<u>Note</u>: This def^n does <u>not</u> req. f to be defined at the pt. c. If f is defined at c & if $f(c^+) = f(c)$ we say that f is cont. from the right at c.

Lefthand limits & continuity from left at c are defined sim. if $c \in (a, b]$

If $a < c < b$, then f is cont. at c iff $f(c) = f(c^+) = f(c^-)$

We say that c is a discontinuity of f if f is <u>not</u> cont. at c. In this case, one of the following condⁿs is satisfied.

1. Either $f(c^+)$ or $f(c^-)$ does <u>not</u> exist

2. Both $f(c^+)$ & $f(c^-)$ exist but $f(c^+) \neq f(c^-)$

3. Both $f(c^+)$ & $f(c^-)$ exist but $f(c^+) = f(c^-) \neq f(c)$

In case of 3., the pt. c is called a <u>removable discont.</u> since the discont. can be removed by redifining f at c to have the value $f(c^+) = f(c^-)$.

In case of 1 & 2, we call c an <u>irremovable discont.</u>

Jump

Let $f : [a, b] \to \mathbb{R}$. If $f(c^+)$ & $f(c^-)$ both exist at some int. pt. c, then

1. $f(c) - f(c^-)$ is called the lefthand jump of f at c

2. $f(c^+) - f(c)$ is called the righthand jump of f at c

3. $f(c^+) - f(c^-)$ is called the jump of f at c.

If any one of these numbers is ± 0, the c is called a jump discont. of f.

Let $S \subseteq \mathbb{R}$ & let $f : S \to \mathbb{R}$. Then f is said to be <u>increasing (or non-decreasing)</u> on S if $\forall x, y \in S$, $x < y \Rightarrow f(x) \leq f(y)$

If $x < y \Rightarrow f(x) < f(y)$, then f is said to be <u>strictly increasing</u> on S.

Decreasing & strictly decreasing can be defined sim.

A fx^n is called <u>monotone</u> on S, if it is increasing on S or decreasing on S.

<u>Thm</u>: If f is inc. on $[a,b]$, then $f(c^+)$ & $f(c^-)$ both exist for each $c \in (a,b)$ & we have

$f(c^-) \leq f(c) \leq f(c^+)$. At the endpts, we have

$f(a) \leq f(a^+)$ & $f(b^-) \leq f(b)$

<u>Pf</u>: Let $A = \{ f(x) : a < x < c \}$

$\because$ f is inc., A is bounded above by $f(c)$.

Let $\alpha = \sup(A)$. Then $\alpha \leq f(c)$

We will prove $f(c^-)$ exists & equals α.

$\because$ $\alpha = \sup A$, $\forall \epsilon > 0$, $\exists$ $f(x_1) \in A$ s.t

$$\alpha - \epsilon < f(x_1) \leq \alpha$$

$\because$ f is inc., $\forall$ $x \in (x_1, c)$, we have

$$|f(x) - \alpha| < \epsilon$$

T'fore, setting $\delta = c - x_1$, we have that

$$c - \delta < x < c \Rightarrow |f(x) - \alpha| < \epsilon$$

which is saying that α is the lefthand limit of f at c.

The proofs for $f(c^+)$, $f(a^+)$, $f(b^-)$ are sim.

Fn^{n}s of bounded variation

Partition:

Let $[a, b]$ be compact interval. A (finite) set of pts.
$P = \{x_0, x_1, \ldots, x_n\}$ satisfying
$a = x_0 < x_1 < \ldots < x_n = b$ is called a partition of $[a, b]$

The interval $[x_{k-1}, x_k]$ is called the k^{th} subinterval of P.

We write $\Delta x_k = x_k - x_{k-1}$

Note that $\sum_{k=1}^{n} \Delta x_k = b - a$

We write $P[a, b]$ for the coll. of all possible partitions of $[a, b]$.

Let $f : [a,b] \to \mathbb{R}$. Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$. Write $\Delta f_k = f(x_k) - f(x_{k-1})$

Write $\Sigma(P) = \sum_{k=1}^{n} |\Delta f_k|$

If $\exists M \in \mathbb{Z}_{>0}$ s.t $\Sigma(P) \leq M$ $\forall$ partitions P of $[a,b]$, then f is to be of <u>bounded variation</u> on $[a,b]$

<u>Thm</u> : If f is monotone on $[a,b]$, then f is of bounded variation on $[a,b]$.

<u>Pf</u> : Suppose f is inc. & let P be any partition of $[a,b]$

Then $\Delta f_k = f(x_k) - f(x_{k-1}) \geqslant 0$

So, $|\Delta f_k| = \Delta f_k = f(x_k) - f(x_{k-1})$

$$\Rightarrow \Sigma(P) = \sum_{k=1}^{n} |\Delta f_k| = \sum_{k=1}^{n} f(x_k) - f(x_{k-1}) = f(x_n) - f(x_1)$$
$$= f(b) - f(a)$$

If we set $M = f(b) - f(a)$, we see that $\Sigma(P) \leq M$ $\forall$ partitions P.

<u>Thm</u>: If f is cont. on $[a,b]$ & if f' exists & $\exists\ A > 0$ s.t $|f'(x)| \leq A$ $\forall\ x \in (a,b)$, then f is of bounded var. on $[a,b]$.

<u>Pf</u>: By MVT,

$$f(x_k) - f(x_{k-1}) = f'(t_k)(x_k - x_{k-1}) \quad \text{for some } t_k \in (x_{k-1}, x_k)$$

$$\Rightarrow \sum_{k=1}^{n} |\Delta f_k| = \sum_{k=1}^{n} |f'(t_k)|(x_k - x_{k-1})$$

$$\leq \sum_{k=1}^{n} A(x_k - x_{k-1}) = A(b-a)$$

Thm: If f is of bounded var. on $[a,b]$, i.e $\Sigma(P) \leq M$ $\forall$ partitions P of $[a,b]$, then f is bounded on $[a,b]$ & in particular, $|f(x)| \leq |f(a)| + M$ $\forall\, x \in [a,b]$

Pf: Let $x \in [a,b]$. Consider the partition $P = \{a, x, b\}$.

Then
$$\Sigma(P) = |f(x) - f(a)| + |f(b) - f(x)| \leq M$$
$$\Rightarrow |f(x) - f(a)| \leq M.$$

Then, Δ inequality implies
$$|f(x)| \leq |f(x) - f(a)| + |f(a)| \leq M + |f(a)|$$

eg: 1.
$$f(x) = \begin{cases} x\cos\left(\frac{\pi}{2x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

f is cont. on $[0,1]$. But if we take $P = \left\{0, \frac{1}{2n}, \frac{1}{2n-1}, \ldots, \frac{1}{2}, 1\right\}$, then

$$\Sigma(P) = \sum_{k=1}^{n} |\Delta f_k| = \frac{1}{2n} + \frac{1}{2n} + \frac{1}{2n-2} + \frac{1}{2n-2} + \cdots + \frac{1}{2} + \frac{1}{2}$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$\because \sum_{n=1}^{\infty} 1/n$ diverges

$\therefore \Sigma(P)$ is <u>not</u> bounded wrt n.

So, f is <u>not</u> of bounded var.

<u>2.</u> $f(x) = \begin{cases} x^2\cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$ on $[0,1]$

$f'(x) = \sin(1/x) + 2x\cos(1/x)$

$\because |f'(x)| \leq 3$ on $[0,1]$

$\therefore f$ is of bound. var. on $[0,1]$

<u>3.</u> $f(x) = x^{1/3}$ is monotonic, hence of bound. var. on every compact interval $[a,b]$.

However, $f'(x) \to \infty$ as $x \to 0$

Variation :

Let f be of bound. var. on $[a,b]$. The number $V_f(a,b) = \sup\{\Sigma(P) : P \in \mathcal{P}[a,b]\}$ is called the variation of f on the interval $[a,b]$

Note : Since f is of bound. var., so $V_f(a,b)$ is finite & $\geqslant 0$.

$V_f(a,b) = 0$ iff f is const. on $[a,b]$

Thm : Suppose f & g are of bound. var. on $[a,b]$. Then so are $f+g$, $f-g$, fg.

Also, $\quad V_{f\pm g} \leqslant V_f + V_g$ &

$V_{fg} \leqslant AV_f + BV_g$ where $A = \sup\{|g(x)| : x \in [a,b]\}$

$B = \sup\{|f(x)| : x \in [a,b]\}$

Pf: We will prove the statement for fg.

Let $h(x) = f(x)g(x)$. Let P be a partition of $[a,b]$.

Then, $$|\Delta h_k| = |f(x_k)g(x_k) - f(x_{k-1})g(x_{k-1})|$$
$$= |f(x_k)g(x_k) - f(x_{k-1})g(x_k)$$
$$+ f(x_{k-1})g(x_k) - f(x_{k-1})g(x_{k-1})|$$
$$\leq |g(x_k)|\,|f(x_k) - f(x_{k-1})| + |f(x_{k-1})|\,|g(x_k) - g(x_{k-1})|$$

$$\Rightarrow \sum_{fg}(P) \leq A \sum_{f}(P) + B\sum_{g}(P)$$
$$\leq A\, V_f(a,b) + B\, V_g(a,b)$$

Thm: (Additive ppt. of var.)

Let f be of bound. var. on $[a,b]$, & let $c \in [a,b]$. Then, f is of bound. var. on $[a,c]$ & $[c,b]$ & $V_f(a,b) = V_f(a,c) + V_f(c,b)$

Pf: Let P_1 be a partition of $[a,c]$ & let P_2 be a partition of $[c,b]$.

Then $P_0 = P_1 \cup P_2$ is a partition of $[a,b]$.

Then $\Sigma(P_1) + \Sigma(P_2) = \Sigma(P_0) \leq V_f(a,b)$

$\Rightarrow$ $\Sigma(P_1) \leq V_f(a,b)$ & $\Sigma(P_2) \leq V_f(a,b)$

So, f is of bound. var. on $[a,c]$ & $[c,b]$.

$\Sigma(P_1) + \Sigma(P_2) = \Sigma(P_0) \leq V_f(a,b)$ & the additive ppt. of supremum implies that $V_f(a,c) + V_f(c,b) \leq V_f(a,b)$

Now, to prove the reverse inequality.
Let $P = \{x_0, x_1, \ldots, x_n\} \in \mathcal{P}[a,b]$ & let $P_0 = P \cup \{c\}$

If $c \in [x_{k-1}, x_k]$, then
$$|f(x_k) - f(x_{k-1})| \leq |f(x_k) - f(c)| + |f(c) - f(x_{k-1})|$$

So, $\Sigma(P) \leq \Sigma(P_0)$

Consider, partition $P_1 = \{x_0, x_1, \ldots, x_{k-1}, c\}$ of $[a,c]$ & $P_2 = \{c, x_k, \ldots, x_n\}$ of $[c,b]$

$$\Rightarrow \Sigma(P) \leq \Sigma(P_0) = \Sigma(P_1) + \Sigma(P_2) \leq V_f(a,c) + V_f(c,b)$$

Therefore, $V_f(a,c) + V_f(c,b)$ is an upper bound of $\{\Sigma(P) : P \in \mathcal{P}[a,b]\}$. So, $V_f(a,b) \leq V_f(a,c) + V_f(c,b)$

$$\Rightarrow V_f(a,b) = V_f(a,c) + V_f(c,b)$$

<u>Thm</u> : Let f be of bounded var. on $[a,b]$. Let V be defined on $[a,b]$ as follows.

$$V(a) = 0$$
$$V(x) = V_f(a,x) \quad \text{if } x \in (a,b]$$

Then, 1. V is an inc. fxn on $[a,b]$

2. $V-f$ is an inc. fxn on $[a,b]$

Riemann Integration

Let $f : [a, b] \to \mathbb{R}$ be a b'nd fn^n.
Let $P = \{x_1, \ldots, x_n\}$ be a partition of $[a, b]$,
i.e $P \in \mathcal{P}[a, b]$.

We write $M_i = \sup\{ f(x) : x \in [x_{i-1}, x_i]\}$

$$m_i = \inf\{ f(x) : x \in [x_{i-1}, x_i]\}$$

We define the <u>upper sum</u> of f wrt partition P to be :

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$$

We define the <u>lower sum</u> of f wrt partition P to be :

$$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$$

$\because$ $f : [a, b] \to \mathbb{R}$ is assumed to be b'nd, there are two numbers m, M s.t

$$m \leq f(x) \leq M \quad \forall\ x \in [a, b]$$

Then, note that $\qquad U(P,f) \leq \sum_{i=1}^{n} M \Delta x_i = M(b-a)$

$$L(P,f) \geq \sum_{i=1}^{n} m \Delta x_i = m(b-a)$$

Also,

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i \leq \sum_{i=1}^{n} M_i \Delta x_i = U(P,f)$$

In short, we have

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$$

for all possible partitions P of $[a,b]$.

We define the <u>upper Riemann integral</u> of f over $[a,b]$ to be the number $\overline{\int_a^b} f(x)\,dx$ defined by

$$\overline{\int_a^b} f(x)\,dx = \inf \{ U(P,f) : P \in \mathcal{P}[a,b] \}$$

Sim, <u>lower Riemann integral</u> of f over $[a,b]$ is defined to be the number $\underline{\int_a^b} f(x)\,dx$ defined by

$$\underline{\int}_a^b f(x)\,dx = \sup\{L(P,f) : P \in \mathcal{P}[a,b]\}$$

If the upper & lower integrals are equal, we say that f is <u>Riemann-integrable</u> on $[a,b]$. If this happens, we denote the common value

$$\overline{\int}_a^b f(x)\,dx = \underline{\int}_a^b f(x)\,dx$$

by the symbol $\int_a^b f(x)\,dx$ & we call this the Riemann integral of f over $[a,b]$.

eg: Let $f : [a,b] \to \mathbb{R}$ def. by $f(x) = \begin{cases} 1, & x \in [a,b] \cap \mathbb{Q} \\ 0, & x \in [a,b] \cap \mathbb{Q}^c \end{cases}$

Let $P = \{x_1, \dots, x_n\}$ be a partition of $[a,b]$.
Then $m_i = 0$ for each i ($\because$ each $[x_{i-1}, x_i]$ contains an irrational no.) & $M_i = 1$ for each i ($\because$ each $[x_{i-1}, x_i]$ contains a rational no.)

T'fore $\quad L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i = 0$

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i = (b-a)$$

So, $\displaystyle\overline{\int_a^b} f(x)\,dx = \inf\{U(P,f) : P \in \mathcal{P}[a,b]\} = \inf\{b-a\}$
$= (b-a)$

$\displaystyle\underline{\int_a^b} f(x)\,dx = \sup\{L(P,f) : P \in \mathcal{P}[a,b]\} = \sup\{0\}$
$= 0$

$$\Rightarrow \quad \overline{\int_a^b} f(x)\,dx \neq \underline{\int_a^b} f(x)\,dx$$

& thus f is <u>not</u> Riemann-integrable.

<u>Refinement</u>: We say partition P^* is a refinement of the partition P if $P \subset P^*$

Given two partitions P_1 & P_2, we say P^* is their common refinement if $P^* = P_1 \cup P_2$

Thm: Let $f:[a,b]\to\mathbb{R}$ be a b'nd fn^n. If P^* is a refinement of P, then

$$L(P,f)\le L(P^*,f)$$
$$U(P,f)\ge U(P^*,f)$$

Pf: First, let $P=\{x_0,x_1,\dots,x_n\}$ & suppose that P^* contains just one more pt. than P. Let x^* be this extra pt. & suppose $x_{i-1}<x^*<x_i$.

Let
$$w_1=\inf\{f(x): x\in[x_{i-1},x_i^*]\}$$
$$w_2=\inf\{f(x): x\in[x_{i-1},x_i^*]\}$$

Then $w_1\ge m_i$ & $w_2\ge m_i$ where
$m_i=\inf\{f(x): x\in[x_{i-1},x_i]\}$.

Hence,
$$\begin{aligned}L(P^*,f)-L(P,f) &= w_1(x^*-x_{i-1})+w_2(x_i-x^*)\\ &\quad -m_i(x_i-x_{i-1})\\ &= (w_1-m_i)(x^*-x_{i-1})\\ &\quad +(w_2-m_i)(x_i-x^*)\\ &\ge 0\end{aligned}$$

ie $L(P^*,f)\ge L(P,f)$

If P^* contains k more pts. than P, repeat the procedure k times.

Sim., we can prove $U(P^*, f) \leq U(P, f)$.

<u>Thm</u>: Let $f : [a,b] \to \mathbb{R}$ be b'nd. Then

$$\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx$$

<u>Pf</u>: Let P_1 & P_2 be any two partitions of $[a,b]$, & let P^* be their common refinement.

Then, by the thm proved above,

$$L(P_1, f) \leq L(P^*, f) \leq U(P^*, f) \leq U(P_1, f)$$

Keep P_2 fixed & take the supremum over all P_1

we get $\underline{\int_a^b} f(x)\,dx \leq U(P_2, f)$

Now, taking the infimum as P_2 ranges over all partitions of $[a,b]$, we get

$$\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx$$

Thm: Let $f:[a,b]\to\mathbb{R}$ be b'nd. Then the following two statements are equivalent:

1. f is Riemann-integrable
2. $\forall\, \epsilon > 0$, $\exists$ a partition P s.t $U(P,f) - L(P,f) < \epsilon$

Pf: ($\Rightarrow$) Suppose f is Riemann-integrable. Let $\epsilon > 0$.

$\because$ $\inf\{U(P,f) : P\in\mathcal{P}[a,b]\} = \int_a^b f(x)\,dx$, $\exists\, P_1\in\mathcal{P}[a,b]$ s.t $U(P_1,f) - \int_a^b f(x)\,dx < \epsilon/2$

$\because$ $\sup\{L(P,f) : P\in\mathcal{P}[a,b]\} = \int_a^b f(x)\,dx$, $\exists\, P_2\in\mathcal{P}[a,b]$ s.t $\int_a^b f(x)\,dx - L(P_2,f) < \epsilon/2$

Let P be the common refinement of P_1 & P_2.
Then,

$$U(P,f) \leq U(P_1,f) < \int_a^b f(x)\,dx + \epsilon/2 < L(P_2,f) + \epsilon/2 + \epsilon/2 \leq L(P,f) + \epsilon$$

So, P is a partition of $[a,b]$ which satisfies

$$U(P,f) - L(P,f) < \epsilon$$

($\Leftarrow$) Let $\epsilon > 0$ & P be a partition of $[a,b]$ s.t

$$U(P,f) - L(P,f) < \epsilon$$

We have, $L(P,f) \leq \underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx \leq U(P,f)$

This implies, $0 \leq \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \leq \epsilon$

This holds $\forall \epsilon > 0 \Rightarrow \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$

So, f is Riemann integrable.

eg: Let $f:[0,1]\to\mathbb{R}$ be given by

$$f(x) = \begin{cases} 1, & x = 1/2 \\ 0, & x \neq 1/2 \end{cases}$$

Let $\epsilon > 0$. Consider the partition $\{0, \frac{1}{2}-\frac{\epsilon}{2}, \frac{1}{2}+\frac{\epsilon}{2}, 1\}$

Then, $U(P,f) = 0\cdot\left(\frac{1}{2}-\frac{\epsilon}{2}\right) + 1\cdot(\epsilon) + 0\cdot\left(\frac{1}{2}-\frac{\epsilon}{2}\right)$

$= \epsilon$

$L(P,f) = 0\cdot\left(\frac{1}{2}-\frac{\epsilon}{2}\right) + 0\cdot(\epsilon) + 0\cdot\left(\frac{1}{2}-\frac{\epsilon}{2}\right)$

$= 0$

So, for this partition, $U(P,f) - L(P,f) = \epsilon$

Hence, f is Riemann-integrable.

Thm: Let $f : [a,b] \to \mathbb{R}$ be cont. Then f is Riemann integrable.

Pf: Let $\epsilon > 0$. f is uniformly cont. on $[a,b]$, ($\because$ compact)
so $\exists\, \delta > 0$ s.t $|f(x) - f(y)| < \dfrac{\epsilon}{b-a}$

whenever $|x-y| < \delta$

If P is any partition of $[a,b]$ s.t $\Delta x_i < \delta$ $\forall\, i = 1, \dots, n$ then $M_i - m_i \leq \dfrac{\epsilon}{b-a}$

Therefore,
$$U(P,f) - L(P,f) = \sum_{i=1}^{n} (M_i - m_i)\,\Delta x_i$$
$$\leq \frac{\epsilon}{b-a} \underbrace{\sum_{i=1}^{n} \Delta x_i}_{(b-a)} = \epsilon$$

Hence, f is Riemann int'ble on $[a,b]$.

Thm: Let $f : [a,b] \to \mathbb{R}$ be monotonic. Then f is RI. on $[a,b]$

Pf: Let $\epsilon > 0$. For any $n \in \mathbb{Z}_{>0}$, consider the partition P for which $\Delta x_i = \frac{b-a}{n}$ for each $i = 1, \dots, n$

We will assume that f is increasing. Then $M_i = f(x_i)$, $m_i = f(x_{i-1})$

T'fore
$$\begin{aligned} U(P,f) - L(P,f) &= \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \\ &= \sum_{i=1}^{n} \left(f(x_i) - f(x_{i-1})\right)\left(\frac{b-a}{n}\right) \\ &= \frac{(b-a)}{n}\left(f(b) - f(a)\right) \end{aligned}$$

So, by choosing n large enough s.t $\frac{(b-a)}{n}(f(b) - f(a)) < \epsilon$, we have obtained a partition P which satisfies $U(P,f) - L(P,f) < \epsilon$

This proves that f is RI.

Thm: Let $f:[a,b] \to \mathbb{R}$ be b'nd & suppose that f has only finitely many pts. of discontinuity on $[a,b]$. Then f is RI.

Pf: Let $\epsilon > 0$. Let $M = \sup\{|f(x)| : x \in [a,b]\}$
Let $E \subset [a,b]$ be the set of pts. at which f is discont.

$\because$ E is finite, we can cover E by finitely many intervals $[u_j, v_j] \subset [a,b]$ s.t

1. $\sum_{j=1}^{m} v_j - u_j < \epsilon$

2. Each pt. of E lies in (u_j, v_j) for some $j = 1, \dots, m$

Def. K to be the set $K = [a,b] \setminus \left(\bigcup_{j=1}^{m} (u_j, v_j) \right)$

Then K is a compact set.
Hence, f is uniformly cont. on K.

So, $\exists \delta > 0$ s.t $|f(x) - f(y)| < \epsilon$ whenever $x, y \in K$ with $|x-y| < \delta$

Now, let P be a partition of $[a,b]$ s.t

1. P contains each u_j
2. P contains each v_j
3. no pt. of any (u_j, v_j) occurs in P
4. If x_{i-1} is <u>not</u> one of the u_j's then $\Delta x_i < \delta$

Now, observe that for this partition, $M_i - m_i \leq 2M$ for each i & $M_i - m_i \leq \epsilon$ unless x_{i-1} is one of the u_js

Hence, $U(P,f) - L(P,f) \leq \underbrace{(b-a)\epsilon}_{\substack{\text{on } K \\ M_i - m_i \leq \epsilon}} + \underbrace{2M\epsilon}_{\substack{\text{on } [a,b]\setminus K \\ \sum \Delta x_i < \epsilon}}$

$= (b-a+2M)\epsilon$

This is true for any $\epsilon > 0$, so it shows that f is RI.

Thm: Suppose $f:[a,b]\to\mathbb{R}$ is RI. Suppose $m \le f(x) \le M \ \forall\, x\in[a,b]$. Suppose $\varphi:[m,M]\to\mathbb{R}$ is cont. Let $h(x)=\varphi(f(x))$ on $[a,b]$. Then h is RI.

Pf: $[a,b] \xrightarrow{f} [m,M] \xrightarrow{\varphi} \mathbb{R}$

Let $\epsilon>0$. $\because$ φ is cont. on $[m,M]$, hence $\exists\,\delta>0$ s.t $\delta<\epsilon$ & $|\varphi(s)-\varphi(t)|<\epsilon$ if $s,t\in[m,M]$ with $|s-t|<\delta$.

$\because$ f is RI on $[a,b]$, there is a partition $P=\{x_0,x_1,\dots,x_n\}$ of $[a,b]$ s.t $U(P,f)-L(P,f)<\delta^2$

Let
$$M_i = \sup\{f(x) : x\in[x_{i-1},x_i]\}$$
$$m_i = \inf\{f(x) : x\in[x_{i-1},x_i]\}$$
$$M_i^* = \sup\{h(x) : x\in[x_{i-1},x_i]\}$$
$$m_i^* = \inf\{h(x) : x\in[x_{i-1},x_i]\}$$

Divide the nos. $1,\dots,n$ into two gps. A & B:
$i\in A$ if $M_i-m_i<\delta$, $\quad i\in B$ if $M_i-m_i\geqslant\delta$

for $i \in A$, $\quad M_i^* - m_i^* \leq \epsilon \quad \left(\begin{array}{l}\text{by uniform cont.}\\ \text{of } \varphi : [m, M] \to \mathbb{R}\end{array}\right)$

for $i \in B$, $\quad M_i^* - m_i^* \leq 2K$, $\quad$ where $K = \sup\{|\varphi(t)| : t \in [m, M]\}$

We have,
$$\delta \sum_{i \in B} \Delta x_i \leq \sum_{i \in B} (M_i - m_i) \Delta x_i$$
$$< \sum_{i=1}^{n} (M_i - m_i) \Delta x_i$$
$$= U(P, f) - L(P, f) < \delta^2$$

$$\Rightarrow \quad \sum_{i \in B} \Delta x_i < \delta$$

T'fore,
$$U(P, h) - L(P, h) = \sum_{i=1}^{n} (M_i^* - m_i^*) \Delta x_i$$
$$= \sum_{i \in A} (M_i^* - m_i^*) \Delta x_i + \sum_{i \in B} (M_i^* - m_i^*) \Delta x_i$$
$$< \delta(b-a) + 2K\delta = (b - a + 2K)\delta$$
$$< (b - a + 2K)\epsilon$$

$\because$ $\epsilon > 0$ was arbitrary, this shows h is RI.

Thm : (Properties of integral)

1. If $f, g : [a,b] \to \mathbb{R}$ are RI, then $f+g$ is RI & cf is RI for each $c \in \mathbb{R}$, and

$$\int_a^b f(x) + g(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

$$\int_a^b cf(x)\, dx = c \int_a^b f(x)\, dx$$

2. If $f(x) \leq g(x)$ on $[a,b]$, then $\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$

3. If $f : [a,b] \to \mathbb{R}$ is RI & $c \in (a,b)$, then f is RI on $[a,c]$ & $[c,b]$, and

$$\int_a^c f(x)\, dx + \int_c^b f(x)\, dx = \int_a^b f(x)\, dx$$

Thm : Let $f, g : [a, b] \to \mathbb{R}$ be RI. Then

1. fg is RI
2. $|f|$ is RI. & $\int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$

Pf : 1. If we take $\varphi(t) = t^2$, we can conclude that since $f+g$ & $f-g$ are RI (prev. thm), t'fore $(f+g)^2$ & $(f-g)^2$ are RI.

Then, $fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$

So, fg is RI.

2. If we take $\varphi(t) = |t|$, we see that $|f|$ is RI.

Choose $c = \pm 1$ in such a way that

$$\left| \int_a^b f(x)\,dx \right| = c \int_a^b f(x)\,dx = \int_a^b c f(x)\,dx$$
$$\leq \int_a^b |f(x)|\,dx$$

So, $\int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$

Integration & Differentiation

If $F : [a,b] \to \mathbb{R}$, then we say f is diff at a if $\lim_{x \to a^+} \frac{F(x) - F(a)}{x - a}$ exists and if it exists we set

$$F'(a) = \lim_{x \to a^+} \frac{F(x) - F(a)}{x - a}$$

Sim, we will say that f is diff at b if $\lim_{x \to b^-} \frac{F(x) - F(b)}{x - b}$ exists and if it exists, we set

$$F'(b) = \lim_{x \to b^-} \frac{F(x) - F(b)}{x - b}$$

Thm: Let $f : [a,b] \to \mathbb{R}$ be RI. For $a \le x \le b$, def.

$$F(x) = \int_a^x f(t)\, dt$$

Then F is cont. on $[a,b]$.

Additionally, if f is cont. at some pt. $x_0 \in [a,b]$, then F is diff. at x_0, and $F'(x_0) = f(x_0)$

Pf: Suppose $|f(t)| \le M$ for $a \le x \le b$.
If $a \le x < y \le b$, then $|F(y) - F(x)| = \left| \int_x^y f(t)\, dt \right|$

$$\le M(y-x)$$

Given $\epsilon > 0$, we see that if $|x-y| < \delta = \epsilon/M$, then $|F(x) - F(y)| < \epsilon$

So, we conclude that F is cont.
(infact, F is uniformly cont.)

Now, to investigate diff. first, observe that for $x \in [a,b]$, $x \neq x_0$, we have

$$\frac{F(x)-F(x_0)}{x-x_0} = \frac{1}{x-x_0}\int_{x_0}^{x} f(t)-f(x_0)\, dt$$

Now, suppose that f is cont. at x_0.

Given $\epsilon>0$, choose $\delta>0$ s.t $|f(t)-f(x_0)|<\epsilon$ whenever $|t-x_0|<\delta$

Hence, if $x\in(x_0-\delta, x_0+\delta)$, $x\neq x_0$, then

$$\begin{aligned}\left|\frac{F(x)-F(x_0)}{x-x_0} - f(x_0)\right| &= \left|\frac{1}{x-x_0}\int_{x_0}^{x} f(t)-f(x_0)\, dt\right| \\ &\leq \frac{1}{|x-x_0|}\int_{x_0}^{x} |f(t)-f(x_0)|\, dt \\ &\leq \frac{1}{|x-x_0|}\int_{x_0}^{x} \epsilon\, dt = \epsilon\end{aligned}$$

It follows that $\displaystyle\lim_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0}$ exists & equals $f(x_0)$

<u>Note</u>: If $f:[a,b]\to\mathbb{R}$ is RI, we def.

$$\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$$

Thm : (Fundamental thm of calculus)

Suppose $f:[a,b]\to\mathbb{R}$ is RI. Suppose $F:[a,b]\to\mathbb{R}$ is diff. fn st $F'=f$. Then

$$\int_a^b f(x)\,dx = F(b)-F(a)$$

Pf: Let $\epsilon>0$. Choose a partition $P=\{x_0,x_1,\dots,x_n\}$ of $[a,b]$ s.t $U(P,f)-L(P,f)<\epsilon$

By the mean-value thm, $\exists$ pts. $t_i\in(x_{i-1},x_i)$ s.t

$$\begin{aligned} F(x_i)-F(x_{i-1}) &= F'(t_i)\Delta x_i \\ &= f(t_i)\Delta x_i \quad \text{for } i=1,\dots,n \end{aligned}$$

T'fore, $$\sum_{i=1}^n f(t_i)\Delta x_i = \sum_{i=1}^n F(x_i)-F(x_{i-1})$$

$$= F(b)-F(a)$$

Now, note

$$L(P,f)\le\sum_{i=1}^n f(t_i)\Delta x_i\le U(P,f) \qquad [\because f(t_i)\in[m,M]]$$

Also, $\quad L(P,f) \leq \int_a^b f(x)\,dx \leq U(P,f)$

$$\Rightarrow \left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f(t_i)\Delta x_i \right| < \epsilon$$

T'fore, $\quad 0 = \left| \int_a^b f(x)\,dx - (F(b)-F(a)) \right| < \epsilon$

This is true $\forall\ \epsilon > 0$, so we conclude that

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Thm : (Integration by parts)

Suppose F & G are diff. fn^{n}s on $[a,b]$.

Suppose $F' = f$, $G' = g$, and suppose f, g are RI.

Then, $$\int_a^b F(x)\, g(x)\, dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)\, G(x)\, dx$$

Pf : Let $H(x) = F(x)\, G(x)$ & apply the fundamental thm of calculus to H and

$$H'(x) = F(x)\, g(x) + f(x)\, G(x)$$

Note that H' is RI so assumptions of FTC are satisfied.

Thm: (Change of variable)

Suppose $g : [c, d] \to [a, b]$ is a fn^n of class C^1 s.t $a = g(c)$, $b = g(d)$. If $f : [a, b] \to \mathbb{R}$ is a cont. fn^n, then

$$\int_a^b f(x)\, dx = \int_c^d f(g(t))\, g'(t)\, dt$$

Pf: Let $F(x) = \int_a^x f(u)\, du$

Then F is diff. & $F' = f$. T'fore the composition $H(t) = F(g(t))$ is a fn^n of class C^1, on interval $[c, d]$

Also, by the chain rule,

$$H'(t) = F'(g(t))\, g'(t)$$
$$= f(g(t))\, g'(t)$$

T'fore, $\int_c^d f(g(t))\, g'(t)\, dt = \int_c^d H'(t)\, dt$

$$= H(d) - H(c)$$
$$= F(g(d)) - F(g(c))$$
$$= F(b) - F(a) = \int_a^b f(x)\, dx$$

Improper Integral

Suppose $f: [a,b] \to \mathbb{R}$ is RI when restricted to any closed subinterval $[a,c]$ where $c<b$. Here we will allow the cases where $f(x)$ becomes unbounded as $x \to b$, or the case where $b = \infty$.

If the lim. of $\int_a^c f(x)\,dx$ exists as $c \to b$ then we define the improper Riemann integral

$$\int_a^b f(x)\,dx = \lim_{c \to b} \int_a^c f(x)\,dx$$

provided the limit exists.

Sim, we can def. the improper integral

$$\int_a^b f(x)\,dx = \lim_{t \to a} \int_t^b f(x)\,dx$$

(if the limit exists) when f is RI on $[t,b]$ for each $t \in (a,b]$

For a two-sided improper integral, fix some $m \in (a, b)$ & require that both $\int_m^b f(x)\,dx$ & $\int_a^m f(x)\,dx$ exist.

Rectifiable paths & arc length

A path in $\mathbb{R}^k$ is a cont. map $\gamma : [a, b] \to \mathbb{R}^k$

For $P = \{t_0, t_1, \ldots, t_n\}$ a partition of $[a, b]$, the pts. $\gamma(t_0), \ldots, \gamma(t_n)$ are the vertices of an inscribed polygon.

The length of this polygon is denoted by $\Lambda(P, \gamma)$ & is defined to be

$$\Lambda(P, \gamma) = \sum_{i=1}^{n} \| \gamma(t_i) - \gamma(t_{i-1}) \|$$

If the set of numbers $\Lambda(P, \gamma)$ is bound. $\forall$ partitions P of $[a, b]$, then the path of γ is said to be <u>rectifiable</u> and its length, denoted by $\Lambda(\gamma)$ is def. by

$$\Lambda(\gamma) = \sup \{ \Lambda(P, \gamma) : P \in \mathcal{P}[a, b] \}$$

If the set of numbers $\Lambda(P,\gamma)$ is unbounded, then γ is said to be non-rectifiable.

The next result relates rectifiable paths & fn^n of bound. var.

Thm: Let $\gamma = (f_1, \dots, f_k) : [a,b] \to \mathbb{R}^k$ be a path. Then, γ is rectifiable iff each component $f: [a,b] \to \mathbb{R}$, is of bound. var. on $[a,b]$

eg: The graph of the fn^n $f(x) = \begin{cases} x\cos\left(\frac{\pi}{2x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

is a non-rectifiable curve.

Thm: If $\gamma: [a,b] \to \mathbb{R}^k$ is a path s.t γ' is cont. on $[a,b]$, then γ is rectifiable, and

$$\Lambda(\gamma) = \int_a^b \|\gamma'(t)\| \, dt$$

Recall some stuff about seq. of real nos.

- A seq. $\{a_n\}$ of pts. in $\mathbb{R}$ is said to converge if $\exists$ a pt $p \in \mathbb{R}$ s.t $\forall \epsilon > 0$, $\exists N \in \mathbb{Z}_{>0}$ s.t $|a_n - p| < \epsilon$ whenever $n > N$

- A seq. $\{a_n\}$ is said to be Cauchy if $\forall \epsilon > 0$, $\exists N \in \mathbb{Z}_{>0}$ s.t $|a_m - a_n| < \epsilon$ whenever $n, m \geqslant N$

$\because \mathbb{R}$ is complete, a seq. $\{a_n\}$ in $\mathbb{R}$ is convergent iff it is Cauchy.

- A seq. $\{a_n\}$ is said to diverge to $+\infty$ if $\forall M > 0$, $\exists N \in \mathbb{Z}_{>0}$ s.t $a_n > M$ whenever $n \geqslant N$

In this case, we write $\lim\limits_{n \to \infty} a_n = +\infty$

- If $\lim\limits_{n \to \infty} (-a_n) = +\infty$, we write $\lim\limits_{n \to \infty} a_n = -\infty$ & we say $\{a_n\}$ diverges to $-\infty$.

Limit superior & limit inferior

Let $\{a_n\}$ be a seq. of real nos. Suppose there is a real no. U satisfying the two cond^{n}s :

1. $\forall \epsilon > 0$, $\exists N \in \mathbb{Z}_{>0}$ s.t $n > N \Rightarrow a_n < U + \epsilon$

2. $\forall \epsilon > 0$ & given $m > 0$, $\exists$ integer $n > m$ s.t $a_n > U - \epsilon$

Then U is called the limit superior of $\{a\}$, and we write

$$U = \limsup_{n\to\infty} a_n$$

If the set $\{a_1, a_2, \ldots\}$ is <u>not</u> bounded above, we define $\limsup_{n\to\infty} a_n = +\infty$

If the set $\{a_1, a_2, \ldots\}$ is bounded above, but <u>not</u> bounded below, and if the seq. $\{a_n\}$ has no finite lim sup, then we say that $\limsup_{n\to\infty} a_n = -\infty$

The limit inferior of $\{a_n\}$ is defined as

$$\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty} b_n \quad , \quad \text{where } b_n = -a_n$$

for $n = 1, 2, \ldots$

<u>Rem</u>: The defn of $\limsup_{n\to\infty} a_n$ & $\liminf_{n\to\infty} a_n$ is equivalent to the following:

Let E be the set of nos. x (including $\pm\infty$) s.t $a_{n_k} \to x$ for some subseq. $\{a_{n_k}\}$. Then

$$\limsup_{n\to\infty} a_n = \sup E$$

$$\liminf_{n\to\infty} a_n = \inf E$$

<u>Ex</u>: If $\{a_n\}$ is a convergent seq., with $\lim_{n\to\infty} a_n = a$, then

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = a$$

eg: 1. $a_n = (-1)^n (1 + 1/n)$

$\liminf_{n\to\infty} a_n = -1$ $\quad$ $\limsup_{n\to\infty} a_n = 1$

Note that $\lim_{k\to\infty} a_{2k+1} = -1$, $\lim_{k\to\infty} a_{2k} = 1$

2. $a_n = (-1)^n$

$\liminf_{n\to\infty} a_n = -1$ $\quad$ $\limsup_{n\to\infty} a_n = 1$

3. $a_n = (-1)^n n$

$\liminf_{n\to\infty} a_n = -\infty$ $\quad$ $\limsup_{n\to\infty} a_n = +\infty$

4. $a_n = n^2 \sin\left(\frac{n\pi}{2}\right)$

$\liminf_{n\to\infty} a_n = -\infty$ $\quad$ $\limsup_{n\to\infty} a_n = +\infty$

5. $a_n = n^2 \sin^2\left(\frac{n\pi}{2}\right)$

$\liminf_{n\to\infty} a_n = 0$ $\quad$ $\limsup_{n\to\infty} a_n = +\infty$

Infinite series

Given a seq. $\{a_n\}$ in $\mathbb{R}$, we form a new seq. $\{s_n\}$ as follows

$$s_n = a_1 + \dots + a_n = \sum_{k=1}^{n} a_k$$

The formal sum $\sum_{k=1}^{\infty} a_k$ is called an infinite series. The no. s_n is called the n^{th} partial sum of the series.

The series $\sum_{k=1}^{\infty} a_k$ is said to converge or diverge, accordingly as the seq. $\{s_n\}$ converges or diverges.

If $\{s_n\}$ converges to s, we say that the series converges & write $\sum_{k=1}^{\infty} a_k = s$

eg: If $0 < x < 1$, then the geometric series $\sum_{k=0}^{\infty} x^k$ converges to $\frac{1}{1-x}$. If $x \geq 1$, the series diverges.

Pf: If $x \neq 1$, then $s_n = \sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$

T'fore, the result follows by letting $n \to \infty$.
If $x = 1$, then $s_n = n$ & $\{s_n\}$ diverges & so the series diverges.

Thm: (Cauchy cond^n for series)
The series $\sum a_n$ converges iff $\forall \epsilon > 0$, $\exists N \in \mathbb{Z}_{>0}$ s.t $\forall$ $m \geq n > N$,

$$\left| \sum_{k=n}^{m} a_k \right| < \epsilon$$

Pf: Let $s_n = \sum_{k=1}^{n} a_k$. Observe that $s_m - s_{n-1} = \sum_{k=n}^{m} a_k$, and use the fact that a seq. of real nos. is conv. iff it is Cauchy.

By taking $n=m$, we have $|a_n| < \epsilon$ whenever $n \geq N$. Hence we have

Cor: If $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$

Rem: $\lim_{n\to\infty} a_n = 0$ is a necessary but not sufficient condⁿ for conv. of $\sum a_n$

eg: $a_n = 1/n$
If we take $n = 2^p+1$ & $m = 2^p + 2^p$, then

$$\sum_{k=n}^{m} a_n = \frac{1}{2^p+1} + \dots + \frac{1}{2^p+2^p} \geq \frac{2^p}{2^p+2^p} = \frac{1}{2}$$

Hence, the Cauchy condⁿ cannot be satisfied when $\epsilon < 1/2$.
T'fore the series diverges

Thm: A series of non-negative terms conv. iff its partial sums form a bounded seq.

Pf: Let $s_n = \sum_{k=1}^{n} a_k$. Note that $\{s_n\}$ is a monotone seq. i.e. $s_k \leq s_{k+1}$ $\forall\, k = 1, 2, \ldots$

If $\sum a_n$ conv., it means $s_n \to s$ for some $s \in \mathbb{R}$ & $s_n \leq s$ $\forall\, n \in \mathbb{Z}_{>0}$. So $\{s_1, s_2, \ldots\}$ is bounded above by s (& bounded below by s_1)

For the converse, if the set $A = \{s_1, s_2, \ldots\}$ is b'nd from above, and so A has a supremum, say $\sup A = s$.

Then using ppts. of sup., it can be seen that $\lim_{n\to\infty} s_n = s$, i.e the series $\sum_{k=1}^{\infty} a_k$ converges to s.

Thm: (Comparison Test)

If $a_n > 0$, $b_n > 0$, for $n = 1, 2, \ldots$ & if $\exists N \in \mathbb{Z}_{>0}$ s.t $a_n \le b_n$ for $n \geq N$, then convergence of $\sum b_n$ implies convergence of $\sum a_n$

Pf: If the partial sums of $\sum b_n$ are b'nd, then the partial sums of $\sum a_n$ are also b'nd.
Apply the prev. thm.

Thm: If $\{a_n\}$ is a decreasing seq. conv. to 0, the alternating series $\sum (-1)^{n+1} a_n$ converges, say to s.
We have $0 < (-1)^n (s - s_n) < a_{n+1}$ for $n = 1, 2, \ldots$

A series $\sum a_n$ is called absolutely conv. if $\sum |a_n|$ conv. It is called conditionally conv. if $\sum a_n$ conv. but $\sum |a_n|$ diverges

Thm: Absolute conv. of $\sum a_n$ implies conv.

Pf: Apply the Cauchy cond^n to the ineq.

$$|a_1 + \dots + a_m| \leq |a_1| + \dots + |a_m|$$

eg: The converse is not true.
$\sum_{n=1}^{\infty} (-1)^{n+1}/n$ converges, but does not absolutely conv.

Thm: (Root Test)

Given a series $\sum a_n$, let $\rho = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$.
Then,

a) If $\rho<1$, $\sum a_n$ converges
b) If $\rho>1$, $\sum a_n$ diverges
c) If $\rho=1$, test gives no information

Pf: a) If $\rho<1$, choose x s.t $\rho<x<1$.
By def^n of ρ, $\exists N\in\mathbb{Z}_{>0}$ s.t $|a_n|<x^n$ $\forall$ $n\geqslant N$
T'fore ($\because \sum x^n$ conv.), the comparison test tells us that $\sum|a_n|$ conv. Hence, $\sum a_n$ conv. as well.

b) If $\rho>1$, then there are infinitely many $n\in\mathbb{Z}_{>0}$ for which $|a_n|>1$. Hence we cannot have $\lim_{n\to\infty} a_n=0$, so $\sum a_n$ diverges

c) Consider the series $\sum\frac{1}{n}$, $\sum\frac{1}{n^2}$. In both cases $\rho=1$ & first series diverges but the second converges (proof by comparison with $\sum\frac{1}{n(n-1)}$)

Thm: (Ratio test)

The series $\sum a_n$

a) converges if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

b) diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1 \ \forall\, n \geq n_0$ where n_0 is some fixed integer.

Pf: If $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then we can find a number $x<1$ & $N \in \mathbb{Z}_{>0}$ s.t $\left|\frac{a_{n+1}}{a_n}\right| < x \ \forall\, n \geq N$

Then,
$$|a_{N+1}| < x|a_N|$$
$$|a_{N+2}| < x|a_{N+1}| < x^2|a_N|$$
$$\vdots$$
$$|a_{N+p}| < x^p|a_N|$$

Hence, $|a_n| < |a_N| x^{-N} x^n$ for $n \geq N$

T'fore, the result follows by comparison test ($\because \sum x^n$ conv.)

b) If $|a_{n+1}| \geq |a_n|$ $\forall\, n \geq n_0$, we see that $\lim_{n\to\infty} a_n = 0$ does not hold, so result follows.

Thm: For any seq. $\{a_n\}$ of +ve nos., we have

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} \sqrt[n]{a_n}$$

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} \geq \limsup_{n\to\infty} \sqrt[n]{a_n}$$

Pf: We will prove the second ineq., the first can be done similarly.

Let $\alpha = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$. If $\alpha = +\infty$, there is nothing to prove. So let us assume α is finite.

Choose $\beta > \alpha$. Hence, by defn of lim. sup, there is an $N \in \mathbb{Z}_{>0}$ s.t $\frac{a_{n+1}}{a_n} < \beta$ for $n \geq N$.

$$\Rightarrow \quad a_{N+p} \leq \beta^p a_N \quad \forall\, p > 0$$

i.e $\quad a_n \leq a_N \beta^{-N} \cdot \beta^n \quad \forall\, n \geq N$

$$\Rightarrow \quad \sqrt[n]{a_n} \le \sqrt[n]{a_N \beta^{-N}} \cdot \beta \quad \forall n \ge N$$

$$\Rightarrow \quad \lim_{n\to\infty} \sqrt[n]{a_n} \le \beta$$

(We need that $\forall\, p > 0$, $\lim_{n\to\infty} \sqrt[n]{p} = 1$)

This is true $\forall\, \beta > \alpha$, hence $\lim_{n\to\infty} \sqrt[n]{a_n} \le \alpha$

Power series:

An infinite series of the form $\Sigma a_n x^n$ is called a power series in the variable x. The nos. a_n are called the coeffs. of the series.

In general, the series could converge or diverge depending on the value of x.

Thm: Given the power series $\sum a_n x^n$, let $\rho = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$ & define $R = 1/\rho$. Then, $\sum a_n x^n$ conv. if $|x| < R$ & diverges if $|x| > R$.

Pf: Apply the root test to the series $\sum b_n$, where $b_n = a_n x^n$

$$\limsup_{n\to\infty} \sqrt[n]{|b_n|} = |x| \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{|x|}{R}$$

Note: R is called the radius of convergence of the power series $\sum a_n x^n$

eg:

1. The series $\sum n^n x^n$ has $R = 0$

2. The series $\sum \frac{x^n}{n!}$ has $R = +\infty$

3. The series $\sum \frac{x^n}{n}$ has $R = 1$

It diverges if $x = 1$ & converges if $x = -1$.

4. The series $\sum \frac{x^n}{n^2}$ has $R=1$

It conv. when $|x|=1$

If $\sum a_n x^n$ is a power series with radius of convergence R, the fn f def. by

$$f(x) = \sum_{n=1}^{\infty} a_n x^n$$

makes sense for $x \in (-R, R)$

Is f cont.? diff.? Riemann integrable?

Sequences & series of functions

Let $\{f_n\}$ be a seq. of fn's def. on a set E, and suppose that the seq. nos. $\{f_n(x)\}$ conv. $\forall x \in E$

The fn f defined by

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad (x \in E)$$

is called the limit fn of seq. $\{f_n\}$, and we say that $\{f_n\}$ conv. pointwise to f on the set E.

Are properties of functions (like continuity, differentiability, integrability) preserved under the limit?

Note: f is cont. at a lim. pt. x means

$$\lim_{t\to x} f(t) = f(x)$$

So, if we ask whether the lim. of a seq. of cont. fn's is cont., we are asking whether

$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t)$$

is true i.e can we change the order of the limits ?

In general, no.

eg: Consider the double seq. $s_{m,n} = \dfrac{m}{m+n}$ for $m, n \in \mathbb{Z}_{>0}$

for every fixed n, $\lim_{m \to \infty} s_{m,n} = 1$

So, $\lim_{n \to \infty} \lim_{m \to \infty} s_{m,n} = \lim_{n \to \infty} 1 = 1$

for every fixed m, $\lim_{n \to \infty} s_{m,n} = 0$

So, $\lim_{m \to \infty} \lim_{n \to \infty} s_{m,n} = \lim_{m \to \infty} 0 = 0$

$\Rightarrow \lim_{n \to \infty} \lim_{m \to \infty} s_{m,n} \neq \lim_{m \to \infty} \lim_{n \to \infty} s_{m,n}$

eg: A seq. of cont. fxn's with discontinuous lim.

Let $f_n(x) = \dfrac{x^{2n}}{1+x^{2n}}$ for $x \in \mathbb{R}$.

Then, $\lim_{n \to \infty} f_n(x)$ exists for each $x \in \mathbb{R}$.

The limit fxn f is given by

$$f(x) = \begin{cases} 0, & \text{if } |x|<1 \\ 1/2, & \text{if } |x|=1 \\ 1, & \text{if } |x|>1 \end{cases}$$

Each f_n is cont. on $\mathbb{R}$, but f is discont. at $x = \pm 1$

eg: A seq. of fxⁿ for which

$$\lim_{n\to\infty} \int_0^1 f(x)\, dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)\, dx$$

Let $f_n(x) = n^2 x(1-x)^n$ if $x \in \mathbb{R}$, $n = 1, 2, \ldots$

If $0 \leq x \leq 1$, the limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists & equals 0 i.e $f \equiv 0$.

Hence, $\int_0^1 f(x)\, dx = 0$

But, $\int_0^1 f_n(x)\, dx = \dfrac{n^2}{(n+1)(n+2)}$

So, $\lim_{n\to\infty} \int_0^1 f_n(x) = 1$

eg: A seq. of diff. fn^ns $\{f_n\}$ with lim 0 for which $\{f_n'\}$ diverges

Let $f_n(x) = \dfrac{\sin(nx)}{\sqrt{n}}$ if $x \in \mathbb{R}$, $n = 1, 2, \ldots$

Then, $\lim_{n\to\infty} f_n(x) = 0$ for each x.

But, $f_n'(x) = \sqrt{n}\cos(nx)$

So, $\lim_{n\to\infty} f_n'(x)$ does not exist for any x.

Uniform convergence

A seq. of fn^ns $\{f_n\}$ is said to converge uniformly to f on a set S, if $\forall\, \epsilon > 0$, $\exists\, N \in \mathbb{Z}_{>0}$ s.t $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon$ $\forall\, x \in S$

Note: In pointwise convergence, N depends on both ϵ & x whereas in uniform convergence, n depends only on ϵ.

We say that the series Σf_n converges uniformly on the set S, if the seq. $\{s_n\}$ of partial sums

$$s_n(x) = \sum_{k=1}^{n} f_k(x)$$

converges uniformly on S.

Thm: (Cauchy criterion for uniform convergence)
A seq. $\{f_n\}$ of fn^{n}s defined on a set S converges uniformly on S iff $\forall \epsilon > 0$, $\exists N \in \mathbb{Z}_{>0}$ s.t $m, n \geq N$ implies $|f_n(x) - f_m(x)| < \epsilon$ $\forall x \in S$

Pf: Suppose that $\{f_n\}$ converges uniformly on S, to a fn^n f. Then $\forall \epsilon > 0$, $\exists N \in \mathbb{Z}_{>0}$ s.t $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon/2$ $\forall x \in S$.

T'fore, if $m, n \geq N$, then $\forall x \in S$

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n(x) - f(x)| + |f_m(x) - f(x)| \\ &< \epsilon/2 + \epsilon/2 \\ &< \epsilon \end{aligned}$$

So, $\{f_n\}$ satisfies the Cauchy condⁿ.

Conversely, suppose that $\{f_n\}$ satisfies the Cauchy condⁿ.

Then, for each $x \in S$, the seq. of real nos. $\{f_n(x)\}$ is a Cauchy seq. Since $\mathbb{R}$ is complete, $\{f_n(x)\}$ converges to some real no. Define $f(x)$ to be this no. Then we have obtained a fxⁿ $f: S \to \mathbb{R}$ s.t f is the pt. wise limit of the seq. $\{f_n\}$

We still need to show that $\{f_n\}$ converges uniformly to f.

Given any $\epsilon > 0$, $\exists N \in \mathbb{Z}_{>0}$ s.t $m, n \geqslant N$ implies that $|f_n(x) - f_m(x)| < \epsilon/2 \; \forall \, x \in S$.

In the expression, keep $n \geqslant N$ fixed and let m tend to ∞.

Then we have:

$$|f_n(x) - f(x)| \leq \epsilon/2 < \epsilon \quad \forall \, x \in S \quad \forall \, n \geqslant N$$

Thm: (Weierstrass M test)

Let $\{M_k\}$ be a seq. of non-ve nos. s.t $\sum M_k$ converges, and let $\{f_k\}$ be a seq. of fn^n s defined on a set S, s.t $0 \leq |f_k(x)| \leq M_k \quad \forall k = 1, 2, \dots, \ \forall x \in S$

Then $\sum f_k$ converges uniformly on the set S.

Pf: Given $\epsilon > 0$, the Cauchy criterion for conv. of $\sum M_k$ implies that $\exists N \in \mathbb{Z}_{>0}$ s.t $m \geq n \geq N$ implies $\sum_{k=n}^{m} M_k < \epsilon$

Then $\left| \sum_{k=n}^{m} f_k(x) \right| \leq \sum_{k=n}^{m} M_k < \epsilon \quad \forall x \in S \ \& \ \forall m, n \geq N$

Then, the Cauchy conv. of fn^n implies that $\sum f_k$ conv. uniformly.

Thm: (Uniform convergence & continuity)

Assume that $f_n \to f$ uniformly on S. If each f_n is cont. at a pt. $c \in S$, then the lim. f_n f is also cont. at c.

Pf: Suppose that c is a lim. pt. of S.

For every $\epsilon > 0$, there is an $N \in \mathbb{Z}_{>0}$ s.t $n \geq N$ implies

$$|f_n(x) - f(x)| < \epsilon/3 \quad \forall\, x \in S$$

$\because$ f_N is cont. at c, there is a ball $B(c, r)$ s.t

$$|f_N(x) - f_N(c)| < \epsilon/3 \quad \forall\, x \in B(c, r) \cap S$$

Then, for $x \in B(c, r) \cap S$, we have

$$\begin{aligned} |f(x) - f(c)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| \\ &\quad + |f_N(c) - f(c)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \end{aligned}$$

i.e f is cont. at c.

eg: Consider the seq. of fn"s $\{f_n\}$ where

$f_n : [0,1] \to \mathbb{R}$ is given by $f_n(x) = x^n$

$\{f_n\}$ converges pt. wise to f where

$$f(x) = \begin{cases} 1 & , \quad 0 \le x < 1 \\ 0 & , \quad x = 1 \end{cases}$$

Here, f is <u>not</u> cont. & the convergence is <u>not</u> uniform.

Function space

Let $\mathcal{C}([a,b])$ denote the set of all cont. fn^{n}s from $[a,b] \to \mathbb{R}$. Note that $\mathcal{C}([a,b])$ has the structure of a vector sp. over $\mathbb{R}$.

Def. the sup. norm on $\mathcal{C}([a,b])$ as

$$\|f\| = \sup\{|f(x)| : x \in [a,b]\}$$

The sup norm satisfies the norm axioms.

- $\|f\| \geqslant 0$ & $\|f\| = 0$ iff $f = 0$
- $\|cf\| = |c|\,\|f\| \quad \forall\, c \in \mathbb{R}$
- $\|f+g\| \leqslant \|f\| + \|g\|$

If we define the dist. b/w $f \in \mathcal{C}([a,b])$ & $g \in \mathcal{C}([a,b])$ to be

$$\begin{aligned} d(f,g) &= \|f-g\| \\ &= \sup\{|f(x)-g(x)| : x \in [a,b]\} \end{aligned}$$

Note: $f : [a,b] \to \mathbb{R}$ is cont. & $[a,b]$ is compact, hence f is bounded i.e $\exists\, M > 0$ s.t $|f(x)| \leqslant M \;\forall\, x \in [a,b]$

Then $\|f\| \leqslant M$, in particular it is $< \infty$

So, $(\mathcal{C}([a,b]), d)$ is a metric sp.

A seq. $\{f_n\}$ conv. uniformly to f on $[a,b]$ iff $f_n \to f$ wrt to the metric of $\mathcal{C}([a,b])$

Why?

$\{f\}$ conv. to f uniformly means $\forall \epsilon > 0, \exists N \in \mathbb{Z}_{>0}$ s.t $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon \quad \forall x \in [a,b]$

$$\Leftrightarrow |(f_n - f)(x)| < \epsilon$$
$$\Leftrightarrow \|f_n - f\| \leq \epsilon$$
$$\Leftrightarrow d(f_n, f) \leq \epsilon$$

Theorem: $(\mathcal{C}([a,b], d)$ is a complete metric sp.

Pf: Let $\{f_n\}$ be a Cauchy seq. in $(\mathcal{C}[a,b], d)$.
That is $\forall \epsilon > 0, \exists N \in \mathbb{Z}_{>0}$ s.t $n, m \geq N$ implies $d(f_n, f_m) < \epsilon$ (i.e $\|f_n - f_m\| < \epsilon$)

That is, $|f_n(x) - f_m(x)| < \epsilon \quad \forall x \in [a,b] \quad \forall m, n \geq N$

Then, by the Cauchy criterion, $\exists f : [a,b] \to \mathbb{R}$ s.t $\{f_n\}$ conv. to f uniformly.

We still need to show that f is cont.

But this follows since $\{f_n\}$ are cont. & conv. uniformly to f

Hence, $f \in \mathcal{C}[a,b]$ & since $f_n \to f$ uniformly on $[a,b]$, we have $\|f_n - f\| \to 0$ as $n \to \infty$.
(i.e. $d(f_n, f) \to 0$ as $n \to \infty$).

Thm: (Uniform conv. & integration)
Suppose $f_n : [a,b] \to \mathbb{R}$ is Riemann-integrable fx^n for $n = 1, 2, \ldots$ Suppose $f_n \to f$ uniformly on $[a,b]$, then f is Riemann-integrable on $[a,b]$ &

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

Pf: Given $\epsilon > 0$, let $N \in \mathbb{Z}_{>0}$ s.t $|f_n(x) - f(x)| < \epsilon$ $\forall\, x \in [a,b]$, $\forall\, n \geq N$.

Then, $$f_N(x) - \epsilon < f(x) < f_N(x) + \epsilon \quad \forall\, x \in [a,b]$$

T'fore,
$$\int_a^b f_N(x) - \epsilon \, dx \leq \underline{\int_a^b} f(x)\, dx \leq \overline{\int_a^b} f(x)\, dx \leq \int_a^b f_N(x) + \epsilon \, dx$$

$$\Rightarrow \quad 0 \leq \overline{\int_a^b} f(x)\, dx - \underline{\int_a^b} f(x)\, dx \leq 2\epsilon(b-a)$$

This is true $\forall\, \epsilon > 0$, hence $\underline{\int_a^b} f\, dx = \overline{\int_a^b} f\, dx$

so f is Riemann-integrable.

Similarly, we see that,
$$\left| \int_a^b f_n(x)\, dx - \int_a^b f(x)\, dx \right| < \epsilon(b-a)$$
whenever $n \geq N$.

Hence, $$\lim_{n \to \infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx$$

Thm: (Term by term integration)

A uniformly convergent series of Riemann-integrable fn's Σf_k can be integrated term by term.

$$\int_a^b \sum_{k=0}^{\infty} f_k(x)\, dx = \sum_{k=0}^{\infty} \int_a^b f_k(x)\, dx$$

Pf: The seq. of partial sums $\{s_n\}$ uniformly converges to Σf_k

Each s_n is Riemann-integrable since it is a sum of finitely many Riemann-integrable fn's.

By the prev. thm.

$$\sum_{k=0}^{n} \int_a^b f_k(x)\, dx = \int_a^b \sum_{k=0}^{n} f_k(x)\, dx = \int_a^b s_n(x)\, dx$$

$$\to \int_a^b \Sigma f(x)\, dx$$

as $n \to \infty$

Thm: (Uniform convergence & differentiation)

Suppose $\{f_n\}$ is a seq. of fn^{n}s which are diff. on (a,b) & suppose that $[f_n(x_0)]$ converges for $x_0 \in (a,b)$

Suppose that $\{f'_n\}$ converges uniformly to some fn^n g on (a,b).

Then $\{f_n\}$ converges uniformly to some fn^n f on (a,b) & $f'(x) = g(x)$ for each $x \in (a,b)$.

Pf: Let $c \in (a,b)$.

Define
$$g_n(x) = \begin{cases} \dfrac{f_n(x) - f_n(c)}{x-c}, & x \neq c \\ f_n'(c), & x = c \end{cases}$$

Then, $g_n(c) = f_n'(c)$, so $\{g_n(c)\}$ converges.

Step 1: We will prove that $\{g_n\}$ converges uniformly on (a,b).

Let $h(x) = f_n(x) - f_m(x)$

By assumption, $h'(x)$ exists $\forall\, x \in (a,b)$ & equals $f_n'(x) - f_m'(x)$. If $x \neq c$, we have

$$g_n(x) - g_m(x) = \frac{h(x) - h(c)}{x - c}$$

By Mean-Value thm applied to h, we have

$g_n(x) - g_m(x) = h'(x_1) = f_n'(x_1) - f_m'(x_1)$ for some x_1 b/w x & c.

Since $\{f'\}$ converges uniformly on (a,b), the Cauchy criterion implies that $\{g_n\}$ converges uniformly on (a,b).

Step 2: We will prove that $\{f_n\}$ converges uniformly on (a,b).

Consider the sequence $\{g_n\}$ formed by using $c = x_0$.
Then, by def^n of g_n, $f_n(x) = f_n(x_0) + (x - x_0)\, g_n(x)$ $\forall\, x \in (a,b)$ $\forall\, n$.

Hence, we have

$$f_n(x) - f_m(x) = f_n(x_0) - f_m(x_0) + (x - x_0)(g_n(x) - g_m(x))$$

Using this eq^n along with the conv. of $\{f_n(x_0)\}$ & the uniform conv. of $\{g_n\}$, the Cauchy criterion shows that $\{f_n\}$ conv. uniformly on (a,b) to some fx^n f.

Step 1: We will prove that $f'(x)$ exists $\forall x \in (a,b)$ &

$f'(x) = g(x)$.

Consider the seq. $\{g_n\}$ defined using an arbitrary pt. $c \in (a,b)$. Def. $G(x) = \lim_{n\to\infty} g_n(x)$

$\because f_n'$ is assumed to exist, we have $\lim_{x\to c} g_n(x) = g_n(c)$

So, each g_n is cont. at c.

Since $g_n \to G$ uniformly on (a,b), therefore G is also cont. at c.

T'fore, $\lim_{x\to c} G(x)$ exists, and equals $G(c)$.

For $x \neq c$,

$$G(x) = \lim_{n\to\infty} \frac{f_n(x) - f_n(c)}{x-c} = \frac{f(x) - f(c)}{x-c}$$

Hence, $G(c) = \lim_{x\to c} G(x)$ implies $f'(c)$ exists, and equals $G(c)$.

But $g(c) = \lim_{n\to\infty} g_n(c) = \lim_{n\to\infty} f_n'(c)$

Hence, $f'(c) = g(c)$

$\because$ c was arbitrary, this proves that f is diff at all pts. of (a,b), & $f' = g$ on (a,b).

Thm: (Term-by-term diff.)

A uniformly conv. series of diff. fn's can be differentiated term-by-term, provided that the derivative series converges uniformly:

$$\left(\sum_{k=0}^{\infty} f_k(x)\,dx\right)' = \sum_{k=0}^{\infty} f_k'(x)\,dx$$

Pf: Apply the prev. thm to seq. of partial sums.

Recall, the power series $\sum a_k x^k$ has radius of conv. R, given by $$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$$

Its interval of conv. is $(-R, R)$.

<u>Thm</u>: If $r < R$, then the power series $\sum a_k x^k$ conv. uniformly on the interval $[-r, r]$.

<u>Pf</u>: Pick β s.t $r < \beta < R$

Then, $\exists\, N \in \mathbb{Z}_{>0}$ s.t $\forall\, n \geq N$, we have

$$\sqrt[n]{|a_n|} < \frac{1}{\beta}$$

So, if $|x| \leq r$, then $|a_n x^n| \leq \left(\frac{r}{\beta}\right)^n$

Note, that $\sum \left(\frac{r}{\beta}\right)^n$ is a conv. geometric series.
Hence, by the M-test, $\sum a_k x^k$ conv. uniformly when $x \in [-r, r]$.

Thm: A power series can be integrated & diff. term-by-term on its interval of conv. i.e for $f(x) = \sum a_k x^k$ & $|x| < R$, we have

$$\int_0^x f(t)\,dt = \sum_{k=0}^{\infty} \frac{a_k x^{k+1}}{k+1} \quad \& \quad f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

Moreover, the integral series & the derivative series have the same radius of conv. as the original series.

Pf: By the prev. thm, the series conv. uniformly on any interval $[-r, r] \subset (-R, R)$.

Hence, term-by-term integration is valid.

Also, $\displaystyle \limsup_{n\to\infty} \sqrt[n]{\left|\frac{a_{n-1}}{n}\right|} = \limsup_{n\to\infty} \left(|a_{n-1}|^{\frac{1}{n-1}}\right)^{\left(\frac{n-1}{n}\right)} \cdot \left(\frac{1}{n}\right)^{\frac{1}{n}}$

$\because$ $\dfrac{n-1}{n} \to 1$ as $n \to \infty$ and $\left(\dfrac{1}{n}\right)^{\frac{1}{n}} \to 1$ as $n \to \infty$.

we see that the integral series has the same radius of conv. R as the original series.

A similar calc. for the derivative series shows that it too has the same radius of conv. R.
T'fore the derivative series also conv. uniformly on any $[-r, r] \subset (-R, R)$

Hence, term-by-term diff. is valid.

If $f(x) = \sum a_k x^k$, then f can be diff. & int. term as many times as we want for $x \in (-R, R)$

$$\therefore \quad f^{(n)}(x) = \sum_{k=n}^{\infty} k(k-1)\ldots(k-n+1)\, x^{k-n}$$

& hence, $f^{(n)}(0) = n!\, a_n$

So, we can write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \quad \text{for } x \in (-R, R)$$

Generally, if

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

with interval of convergence $(x_0 - R, x_0 + R)$

then $$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \quad \text{for } x \in (x_0 - R, x_0 + R)$$

<u>Analytic fx^n</u>: A fx^n f is said to be analytic if it can be expressed locally as a conv. power series.

That is, $f:(a,b) \to \mathbb{R}$ is said to be analytic if $\forall\, x_0 \in (a,b)$, $\exists$ a power series $\sum a_k (x-x_0)^k$ & $\delta > 0$ s.t if $|x-x_0| < \delta$, then the series conv. &

$$f(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k$$

<u>Note</u>: If f is analytic, then f is C^∞ (i.e f is in C^k for $k=0,1,2,\ldots$ – also called smooth fx^ns)

eg: (non-analytic smooth fxⁿ)

Def. $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

One can check that $f^{(k)}(0) = 0$ for each k.

Hence, the power series $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)\, x^k}{k!}$ conv. to the 0 fxⁿ.

But $f \neq 0$, so f is <u>not</u> analytic in a neighbourhood of $x = 0$.

Thm: Let $f \in C^\infty$ on (a,b). Let $x_0 \in (a,b)$. Suppose $\exists\, \delta > 0$ & a const. $M > 0$ s.t $|f^{(n)}(x)| \leq M^n$ for each $x \in (x_0-\delta, x_0+\delta)$ & $n = 1, 2, \ldots$

Then, for each $x \in (x_0-\delta, x_0+\delta)$, we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k$$

Pf: By Taylor's thm, $\forall\, x \in (a,b)$ & $\forall\, n = 1, 2, \ldots$

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k + \frac{f^{(n)}(x_1)}{n!} (x-x_0)^n$$

for some x_1 b/w x & x_0

Then, $$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k \right| \leq \frac{|f^{(n)}(x_0)|\, |x-x_0|^n}{n!}$$

$$\leq \frac{|M(x-x_0)|^n}{n!} \to 0$$

as $n \to \infty$

Thm: There exists a cont. fn^n $f: \mathbb{R} \to \mathbb{R}$ which is nowhere diff.

Pf: Def. $\varphi_0(x) = |x|$ for $-1 < x < 1$

Extend φ_0 to a fn^n def. on all of $\mathbb{R}$ by requiring it to be periodic with period 2: $\varphi_0(x+2) = \varphi_0(x)$ $\forall x \in \mathbb{R}$

The fn^n φ_k ($k \in \mathbb{Z}_{\geq 0}$) def. by

$$\varphi_k(x) = \left(\frac{3}{4}\right)^k \varphi_0(4^k x)$$

has a period $2/4^k$

That is, if $t = x + 2m/4^k$ for $m \in \mathbb{Z}$, then

$$\varphi_k(t) = \varphi_k(x)$$

By the M-test, the series $\Sigma \varphi_k(x)$ conv. uniformly to a limit fn^n f,

$$f(x) = \sum_{k=0}^{\infty} \varphi_k(x)$$

f is cont. since it is uniform limit of cont. fn^{n}s.

We claim that f is nowhere diff.

Fix a real no. x.

We will show that $f'(x)$ does not exist.

Def. $\delta_n = \pm \frac{1}{2} 4^{-n}$ with sign chosen s.t

no integer lies b/w $4^n x$ & $4^n(x+\delta_n)$

(can do this since the length $4^n|\delta_n|$ of this interval is $1/2$)

We will show that

$$\frac{\Delta f}{\Delta x} = \frac{f(x+\delta_n) - f(x)}{\delta_n}$$

does not converge to a limit as $\delta_n \to 0$ (i.e $n \to \infty$), hence $f'(x)$ does not exist.

We have, $$\frac{\Delta f}{\Delta x} = \sum_{k=0}^{\infty} \frac{\varphi_k(x+\delta_n) - \varphi_k(x)}{\delta_n}$$

There are 3 types of terms in the series –
$k>n$, $k=n$, $k<n$.

If $k>n$, we have $\delta_n = \frac{1}{2} \cdot 4^{-n} = 4^{k-(n+1)} \cdot \frac{2}{4^k}$

So, δ_n is an integer multiple of the period φ_k,
and hence $\varphi_k(x+\delta_n) - \varphi_k(x) = 0$.

Hence, the infinite series expression for $\Delta f/\Delta x$ reduces to

$$\frac{\Delta f}{\Delta x} = \frac{\varphi_n(x+\delta_n) - \varphi_n(x)}{\delta_n} + \sum_{k=0}^{n-1} \frac{\varphi_k(x+\delta_n) - \varphi_k(x)}{\delta_n}$$

Observe, $$\frac{\varphi_n(x+\delta_n) - \varphi_n(x)}{\delta_n} = 3^n$$

Also, $$\left| \frac{\varphi_k(x+\delta_n) - \varphi_k(x)}{\delta_n} \right| \leq 3^k \quad \text{for } k<n$$

Hence, we have

$$\left|\frac{\Delta f}{\Delta x}\right| \geqslant 3^n - (3^{n-1} + \dots + 1)$$

$$= 3^n - \left(\frac{3^n - 1}{3-1}\right) = \frac{1}{2}(3^n + 1)$$

which tends to ∞ as $\delta_n \to 0$ $(n \to \infty)$

Hence, $f'(x)$ does <u>not</u> exist.