Autumn 2025
IIT Bombay

MA 403 Real Analysis
Homework 1

1. Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

2. If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

3. Given any real number $x > 0$, prove that there is an irrational number between 0 and x.

4. Suppose x and y are real numbers and for each $\epsilon > 0$, $|x - y| \leqslant \epsilon$. Show that $x = y$.

5. Give an example of a bounded set S such that $\sup S$ is in S but $\inf S$ is not in S.

6. Suppose A and B are two subsets of $\mathbb{R}$ such that A is bounded from above and B is bounded from below. Show that the intersection $A \cap B$ is bounded from both above and below.

7. Let S be a (nonempty) set of real numbers such that $\sup S$ and $\inf S$ exist. Show that $\sup S$ and $\inf S$ are uniquely determined.

8. Let A and B be two sets of positive numbers which are bounded above, and let $a = \sup A$, $b = \sup B$. Let C be the set defined by

$$C = \{xy \,:\, x \in A \text{ and } y \in B\}$$

Prove that $ab = \sup C$.

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MA 403 Real Analysis
Homework 2

1. Find the sup and inf of the set S, where $S = \{x \,:\, 3x^2 - 10x + 3 < 0\}$.

2. Prove Lagrange's identity for real numbers:

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum_{1\leqslant k<j\leqslant n} (a_k b_j - a_j b_k)^2$$

Use this identity to deduce the Cauchy-Schwarz inequality.

3. Let $f : S \to T$ be a function. Prove that the following statements are equivalent:

(a) f is one-to-one on S.

(b) $f^{-1}\Big(f(A)\Big) = A$ for every subset A of S.

(c) For all subsets A and B of S with $B \subseteq A$, we have $f(A - B) = f(A) - f(B)$

4. Let S be the relation given by defining S to be the set of all pairs of real numbers (x, y) that satisfy the given equation (or inequality). Determine in each case, whether S is *reflexive*, or *symmetric*, or *transitive* (it may satisfy more than one of these conditions).

(a) $x \leqslant y$

(b) $x^2 + y^2 = 1$

5. Show that the following sets are countable:

(a) the set of circles in $\mathbb{R}^2$ having rational radii and centers with rational coordinates.

(b) any collection of disjoint intervals of positive length.

6. Is the set of all irrational real numbers countable? Explain your answer.

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MA 403 Real Analysis
Homework 3

1. Let $S \subset \mathbb{R}^n$. Prove that $\operatorname{int} S$ (the interior of S) is an open set.

2. Do S and $\bar{S}$ always have the same interiors? Do S and int$S$ always have the same closures?

3. Determine all accumulation points of the following subsets of the given space $\mathbb{R}^n$ and decide whether the sets are open or closed (or neither).

(a) $\mathbb{Z} \subset \mathbb{R}$

(b) $\mathbb{Q} \subset \mathbb{R}$

(c) $\{\frac{1}{n} + \frac{1}{m} : m, n \in \mathbb{Z}_+\} \subset \mathbb{R}$

(d) $\{(x, y) : x \geqslant 0\} \subset \mathbb{R}^2$

(e) $\{(x, y) : x^2 - y^2 < 1\} \subset \mathbb{R}^2$

(f) $\mathbb{Q}^n \subset \mathbb{R}^n$

4. Give an example of a bounded set of real numbers with exactly three accumulation points.

5. Given $S \subset \mathbb{R}^n$, prove that $\bar{S}$ is the intersection of all closed subsets of $\mathbb{R}^n$ containing S.

6. The collection F of open intervals of the form $\left(\frac{1}{n}, \frac{2}{n}\right)$, where $n \in \mathbb{Z}_+$, is an open over of the open interval $(0, 1)$. Prove that no finite subcollection of F covers $(0, 1)$.

EXTRA:

7. Prove that the set of open disks in the xy-plane with center at (x, x) and radius $x \in \mathbb{Q}_{>0}$, is a countable cover of the set $\{(x, y) : x > 0, y > 0\} \subset \mathbb{R}^2$.

8. Prove that a collection of disjoint open sets in $\mathbb{R}^n$ is necessarily countable. Give an example of a collection of disjoint closed sets which is not countable.

MA 403 Real Analysis
Homework 4

1. Which of the following subsets of $\mathbb{R}^2$ are compact?

(a) the set of all (x, y) such that $x^2 + y^2 = 1$

(b) the set of all (x, y) such that $x^2 + y^2 \geqslant 1$

(c) the set of all (x, y) such that $x, y \in \mathbb{Q}$ and $x^2 + y^2 \leqslant 1$

2. Give an example of a countable open cover F for $\mathbb{Z} \subset \mathbb{R}$ such that F has no finite subcover.

3. Which of the following functions defines a metric on $\mathbb{R}$? (Here $x, y \in \mathbb{R}$.)

(a) $d(x, y) = (x - y)^2$

(b) $\tilde{d}(x, y) = |x - 2y|$

(c) $d^*(x, y) = \dfrac{|x - y|}{1 + |x - y|}$

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Consider the following two functions on $\mathbb{R}^n \times \mathbb{R}^n$:

$$d_1(x, y) = \max_{1 \leqslant i \leqslant n} |x_i - y_i|, \qquad d_2(x, y) = \sum_{i=1}^{n} |x_i - y_i|$$

4. Prove that $(\mathbb{R}^n, d_1)$ is a metric space. Prove that $(\mathbb{R}^n, d_2)$ is a metric space.

5. In each of the following metric spaces prove that the ball $B(a; r)$ has the geometric appearance indicated:

(a) In $(\mathbb{R}^2, d_1)$, a square with sides parallel to the coordinate axes.

(b) In $(\mathbb{R}^2, d_2)$, a square with diagonals parallel to the coordinate axes.

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MA 403 Real Analysis
Homework 5

1. Consider the two metrics on $\mathbb{R}^n$ defined in HW 4:

$$d_1(x,y) = \max_{1\leqslant i\leqslant n} |x-y|, \qquad d_2(x,y) = \sum_{i=1}^{n} |x_i - y_i|.$$

Prove that d_1 and d_2 satisfy the following inequalities for all $x, y \in \mathbb{R}^n$:

$$d_1(x,y) \leqslant \|x-y\| \leqslant d_2(x,y) \qquad \text{and} \qquad d_2(x,y) \leqslant \sqrt{n}\,\|x-y\| \leqslant n\, d_1(x,y).$$

2. Let (M,d) be a metric space, and let S, T be subsets of M such that $S \subset T$. Prove that
(a) $\overline{S} \subset \overline{T}$. (b) $\operatorname{int}(S) \subset \operatorname{int}(T)$.

3. Let (M,d) be a metric space, and let A, B, C be subsets of M such that A is dense in B and B is dense in C. Prove that A is dense in C.

4. Let A and B denote arbitrary subsets of a metric space M.

(a) Give an example in which $\operatorname{int}(\partial A) = M$.

(b) Give an example in which $\operatorname{int} A = \operatorname{int} B = \emptyset$ but $\operatorname{int}(A \cup B) = M$.

5. Using the **definition of the limit**, prove that:

(a) $\frac{x^n}{n!} \to 0$ for all $x \in \mathbb{R}$.

(b) If $\{x_n\}$ is a sequence such that $x_n \geqslant 0$ for all $n \in \mathbb{Z}_+$ and $x_n \to a$, then $\sqrt{x_n} \to \sqrt{a}$.

6. In a metric space (S,d), suppose that $x_n \to x$ and $y_n \to y$. Prove that $d(x_n, y_n) \to d(x,y)$.

7. Let $f : [a,b] \to \mathbb{R}$ be a continuous function such that $f(x) = 0$ when x is rational. Prove that $f(x) = 0$ for every $x \in [a,b]$.

8. Let f, g be defined on $[0,1]$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational.} \end{cases} \qquad g(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ x & \text{is } x \text{ is rational.} \end{cases}$$

Prove that f is not continuous anywhere in $[0,1]$, and that g is continuous only at $x = 0$.

MA 403 Real Analysis
Homework 6

1. Let f be defined and continuous on a closed set $S \subset \mathbb{R}$. Let $A = \{x \in S : f(x) = 0\}$. Prove that A is a closed subset of $\mathbb{R}$.

2. Let f be continuous on a compact interval $[a, b]$. Suppose that f has a local maximum at x_1 and a local maximum at x_2. Show that there must be a third point between x_1 and x_2 where f has a local minimum.

3. In each case, give an example of a function f, continuous on S and such that $f(S) = T$, or else explain why there can be no such f:

(a) $S = (0, 1)$, $\quad T = (0, 1]$.

(b) $S = (0, 1)$, $\quad T = (0, 1) \cup (1, 2)$.

(c) $S = \mathbb{R}$, $\qquad T = \mathbb{Q}$.

(d) $S = [0, 1] \times [0, 1]$, $\quad T = \mathbb{R}^2$.

(e) $S = (0, 1) \times (0, 1)$, $\quad T = \mathbb{R}^2$.

4. Let $f : (S, d_S) \to (T, d_T)$ be a function between two metric spaces. Prove that f is continuous on S if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for every subset A of S.

5. Prove that a metric space S is connected if and only if the only subsets of S which are both open and closed in S are the empty set and S itself.

6. Prove that if S is connected and if $S \subset T \subset \overline{S}$, then T is connected. In particular, this implies that the closure of a set is connected.

7. If $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^2$, prove that f is not uniformly continuous on $\mathbb{R}$.

8. Assume $f : (S, d_S) \to (T, d_T)$ is uniformly continuous on S. If $\{x_n\}$ is any Cauchy sequence in S, prove that $\{f(x_n)\}$ is a Cauchy sequence in T.

EXTRA:

9. Prove that the only connected subsets of $\mathbb{R}$ are (a) the empty set, (b) sets consisting of a single point, (c) intervals (open, closed, half-open, or infinite).

MA 403 Real Analysis
Homework 7

1. Let $f : \mathbb{R} \to \mathbb{R}$, and suppose that $|f(x) - f(y)| \leqslant (x-y)^2 \quad$ for all $x, y \in \mathbb{R}$. Prove that f is constant.

2. Define $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} e^{-(1/x^2)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that:

(a) f is continuous for all x.

(b) $f^{(n)}$ exists and is continuous for all x, and $f^{(n)}(0) = 0$, $n = 1, 2, \cdots$.

3. We say a function $g : (a, b) \to \mathbb{R}$ is of class C^k if the k^{th} derivative $g^{(k)}$ exists and is continuous on (a, b).

Let $f_n(x) = \begin{cases} x^n \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

Prove that f_1 is continuous but not differentiable at 0. Prove that f_2 is differentiable but not of class C^1. In general what can you say about f_n?

4. Let $f : [a, b] \to \mathbb{R}$ be continuous and assume that f is differentiable at all points in (a, b), and suppose $f'(x) = 0$ for all $x \in (a, b)$. Prove that f is a constant function.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f'(x)$ exists for all $x \neq 0$ and such that $\lim\limits_{x \to 0} f'(x) = 3$. Prove that $f'(0)$ exists and $f'(0) = 3$.

Hint: Mean value theorem.

6. Assume f has a finite derivative in (a, b) and is continuous on $[a, b]$, with $a \leqslant f(x) \leqslant b$ for all $x \in [a, b]$ and $|f'(x)| \leqslant \alpha < 1$ for all $x \in (a, b)$. Prove that f has a unique fixed point in $[a, b]$.

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Homework 8

1. In this exercise we show that $f'(c) > 0$ at some point c is not sufficient to guarantee the existence of an open interval containing c on which f is increasing. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} x + 2x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

(a) Compute $f'(0)$ and $f'(x)$ for $x \neq 0$.

(b) Prove that there exists a sequence of points $\{x_n\}$ with $x_n \neq 0$, $x_n \to 0$ and $f'(x_n) < 0$.

2. In class, we proved that if $f : (a, b) \to \mathbb{R}$ is r^{th} order differentiable at x, and $P(h) := \displaystyle\sum_{k=0}^{r} \frac{f^{(k)}(x)}{k!} h^k$, then $R(h) := f(x+h) - P(h)$ satisfies the property: $\displaystyle\lim_{h\to 0} \frac{R(h)}{h^r} = 0.$

Prove that P is the unique polynomial of degree $\leqslant r$ with this property.

3. (a) Suppose f is defined in an open interval containing a, and suppose $f''(a)$ exists. Show that

$$f''(a) = \lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

Hint: Taylor's theorem.

(b) Give an example where the limit of the quotient in part (a) exists but where $f''(a)$ does not exist.

4. Locate and classify the points of discontinuity of the following functions f defined on $\mathbb{R}$:

(a) $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

(b) $f(x) = \begin{cases} e^{\frac{1}{x}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

5. Let f be an increasing function defined on $[a, b]$ and let $x_1, \cdots, x_n$ be n points in (a, b) such that $a < x_1 < x_2 < \cdots < x_n < b$.

(a) Show that $\displaystyle\sum_{k=1}^{n} [f(x_k+) - f(x_k-)] \leqslant f(b) - f(a)$.

(b) For each $m \in \mathbb{Z}_+$, let S_m be the set of points in $[a, b]$ where the jump of f is greater than $\frac{1}{m}$. Use part (a) to show that S_m is a finite set.

(c) Show that the set of discontinuities of f is countable.

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MA 403 Real Analysis
Homework 9

1. Determine which of the following functions are of bounded variation on $[0,1]$.

(a) $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$ (b) $f(x) = \begin{cases} \sqrt{x} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

2. A function f, defined on $[a,b]$, is said to satisfy a uniform Lipschitz condition of order $\alpha > 0$ on $[a,b]$ if there exists a constant $M > 0$ such that

$$|f(x) - f(y)| < M|x-y|^\alpha \quad \text{for all } x \text{ and } y \text{ in } [a,b].$$

If f is such a function, show that $\alpha > 1$ implies f is constant on $[a,b]$, whereas $\alpha = 1$ implies f is of bounded variation on $[a,b]$.

3. A function $f : [a,b] \to \mathbb{R}$ is said to be *absolutely continuous* on $[a,b]$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon$$

for every n disjoint open subintervals (a_k, b_k) of $[a,b]$, $n = 1, 2, \ldots$ satisfying $\displaystyle\sum_{k=1}^{n}(b_k - a_k) < \delta$.

Prove that every absolutely continuous function on $[a,b]$ is continuous and of bounded variation on $[a,b]$.

4. Let $f : [a,b] \to \mathbb{R}$ be a function which is Riemann-integrable, and let c be any real number. Show that the function cf is Riemann-integrable on $[a,b]$ and $\displaystyle\int_a^b cf(x)dx = c\int_a^b f(x)dx$.

5. Let $f, g : [a,b] \to \mathbb{R}$ be Riemann-integrable functions. Show that the function $f+g$ is Riemann-integrable on $[a,b]$ and $\displaystyle\int_a^b \big(f(x) + g(x)\big)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.

6. Let $f, g : [a,b] \to \mathbb{R}$ be Riemann-integrable functions such that $f \geqslant g$. Show that $\displaystyle\int_a^b f(x)dx \geqslant \int_a^b g(x)dx$.

7. Let $f : [a,b] \to \mathbb{R}$ be a non-negative continuous function satisfying $\displaystyle\int_a^b f(x)dx = 0$. Prove that $f = 0$.

MA 403 Real Analysis
Homework 10

1. Let $f : [a, b] \to \mathbb{R}$ be continuous, and suppose $\int_a^b f(x)dx = 0$. Prove that there exists a point $c \in [a, b]$ such that $f(c) = 0$.

Hint: Use Problem 7 from HW 9.

2. Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove that there exists $c \in [a, b]$ such that

$$\int_a^b f(x)dx = (b - a)f(c).$$

Hint: Use Problem 1.

3. Suppose g is Riemann-integrable on $[a, b]$, and $f : [a, b] \to \mathbb{R}$ is a function such that $f(x) = g(x)$ except at a finite number of points x. Prove that f is Riemann-integrable and

$$\int_a^b f(x)dx = \int_a^b g(x)dx.$$

4. Let $f : [0, 1] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational or zero} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms.} \end{cases}$$

(Here the second description means: if $x \in \mathbb{Q}$, $x \neq 0$, and $x = \frac{p}{q}$ where $p, q \in \mathbb{Z}_{>0}$ such that p and q have no common factors other than 1.)

Prove that f is Riemann-integrable on $[0, 1]$ and has integral zero.

5. A function $f : [a, b] \to \mathbb{R}$ is called *piecewise-monotone* if there is a partition P of $[a, b]$, on each subinterval of which f is either increasing or decreasing. Prove that every piecewise monotone function is Riemann-integrable.

6. Give an example of a bounded function $f : [a, b] \to \mathbb{R}$ such that $|f|$ is Riemann-integrable but for which $\int_a^b f(x)dx$ does not exist.

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MA 403 Real Analysis
Homework 11

1. Let $f : (0, 1] \to \mathbb{R}$ be a function, and suppose that f is Riemann-integrable on $[c, 1]$ for each $c > 0$. Recall the definition of the improper integral:

$$\int_0^1 f(x)dx := \lim_{c \to 0} \int_c^1 f(x)dx$$

if this limit exists and is finite.

(a) If f is Riemann-integrable on $[0, 1]$, show that this definition agrees with the old one.

(b) Construct a function f such that the above limit exists, but it fails to exist with $|f|$ in place of f.

2. Let $\gamma_1 : [a, b] \to \mathbb{R}^k$ be a path. Let $\phi : [c, d] \to [a, b]$ be a continuous, 1-1, onto map such that $\phi(c) = a$. Define $\gamma_2(s) = \gamma_1(\phi(s))$.

(a) Prove that γ_2 is a rectifiable curve if and only if γ_1 is a rectifiable curve.

(b) Prove that γ_2 and γ_1 have the same length.

3. For any two real sequences $\{a_n\}$ and $\{b_n\}$ which are bounded below, show that

$$\limsup_{n \to \infty} (a_n + b_n) \leqslant \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n.$$

4. Let $\{a_n\}$ be a sequence of real numbers. Prove that:

(a) $\liminf_{n \to \infty} a_n \leqslant \limsup_{n \to \infty} a_n$.

(b) The sequence $\{a_n\}$ converges if and only if $\limsup_{n \to \infty} a_n$ and $\liminf_{n \to \infty} a_n$ are both finite and equal, in which case, $\lim_{n \to \infty} a_n = \limsup_{n \to \infty} a_n = \liminf_{n \to \infty} a_n$.

5. Assume that $\{a_n\}$ and $\{b_n\}$ are two sequences such that $a_n \leqslant b_n$ for each n. Prove that $\limsup_{n \to \infty} a_n \leqslant \limsup_{n \to \infty} b_n$ and $\liminf_{n \to \infty} a_n \leqslant \liminf_{n \to \infty} b_n$.

6. In each case, test for convergence of $\sum_{n=1}^{\infty} a_n$.

(a) $a_n = \sqrt{n+1} - \sqrt{n}$ (b) $a_n = \dfrac{\sqrt{n+1} - \sqrt{n}}{n}$ (c) $a_n = (\sqrt[n]{n} - 1)^n$

MA 403 Real Analysis
Homework 12

1. Find the radius of convergence of each of the following power series:

(a) $\sum n^3 x^n$ (b) $\sum \frac{2^n}{n!} x^n$

2. Given that the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence 2, find the radius of convergence of each of the following power series:

(a) $\sum_{n=0}^{\infty} a_n^k x^n$, where k is a fixed positive integer. (b) $\sum_{n=0}^{\infty} a_n x^{n^2}$.

3. Prove that every uniformly convergent sequence of bounded functions $\{f_n\}$ is uniformly bounded. That is, show that there is an $M > 0$ such that $|f_n(x)| \leqslant M$ for all x in the common domain S of the functions f_n and for all $n \in \mathbb{Z}_+$.

4. Construct a sequence $\{f_n\}$ of functions on $\mathbb{R}$ which converge pointwise to 0, but such that none of the functions f_n is bounded.

5. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.

6. Construct sequences $\{f_n\}$, $\{g_n\}$ which converge uniformly on some set E, such that $\{f_n g_n\}$ converges pointwise on E, but such that $\{f_n g_n\}$ does not converge uniformly on E.

7. (a) Let $f_n(x) = \dfrac{1}{nx+1}$ for $x \in (0,1)$ and $n = 1, 2, \ldots$ Prove that $\{f_n\}$ converges pointwise but not uniformly on $(0,1)$.

(b) Let $g_n(x) = \dfrac{x}{nx+1}$ for $x \in (0,1)$ and $n = 1, 2, \ldots$ Prove that g_n converges to 0 uniformly on $(0,1)$.

MA 403 Real Analysis
Homework 13

Consider the set $\mathcal{C}([0,2\pi])$ of all continuous functions $f : [0,2\pi] \to \mathbb{R}$.

1. Prove that $\mathcal{C}([0,2\pi])$ is a vector space over $\mathbb{R}$.

2. Consider the function $\|\cdot\|_{\sup} : \mathcal{C}([0,2\pi]) \to \mathbb{R}$, given by

$$\|f\|_{\sup} := \sup\{f(x) \ : \ x \in [0,2\pi]\}.$$

Prove that $\|\cdot\|_{\sup}$ defines a norm on the vector space $\mathcal{C}([0,2\pi])$.

3. Consider the metric $d_{\sup}$ on $\mathcal{C}([0,2\pi])$ defined by

$$d_{\sup}(f,g) := \|f-g\|_{\sup}.$$

If $f, g \in \mathcal{C}([0,2\pi])$ are defined by $f(x) = \sin(x)$, $g(x) = \cos(x)$, find the values of:

(a) $\|f\|_{\sup}$ (b) $\|g\|_{\sup}$ (c) $d_{\sup}(f,g)$.

4. Let $\{f_n\}$ be a sequence of functions defined on an interval I. Let $x \in I$ be a point and suppose that each f_n is continuous at x. Suppose that $\{f_n\}$ converges uniformly to f on I. Show that if $\{x_n\}$ is a sequence of points in I satisfying $x_n \to x$ as $n \to \infty$, then $f_n(x_n) \to f(x)$ as $n \to \infty$.
Is the conclusion still true if the convergence $f_n \to f$ is not uniform?

5. In class we proved that if $\{f_n\}$ is a sequence of functions which are Riemann-integrable on a compact interval $[a,b]$, and $f_n \to f$ uniformly on $[a,b]$, then $\lim_{n\to\infty} \int_a^b f_n(t)dt = \int_a^b f(t)dt$.

Show that the theorem fails for improper integrals, by producing a sequence $\{f_n\}$ of continuous functions on $[0,\infty)$, such that each f_n vanishes outside a bounded interval, and such that $\int_a^b f_n(t)dt = 1$ for each n, but for which $\{f_n\}$ converges uniformly to 0 on $[0,\infty]$.

6. Consider the sequence of functions $\{f_n\}$, where $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$.

(a) Identify the pointwise limit function f.

(b) Prove that the sequence $\{f_n\}$ converges uniformly to the limit function f.

(c) Is f differentiable? What goes wrong? Does $\{f_n'\}$ converge uniformly?