Assignment 1

1. Let H be a finite non-empty subset of a group G. If H is closed under multiplication then H is a subgroup of G.

2. Show that a subgroup of a cyclic group is cyclic.

3. Let G be a finite cyclic goup of order n. Show that for every divisor d of n, G has a unique subgroup of order d.

4. Compute:
   a) $Hom(\mathbb{Z}|_{4\mathbb{Z}}, \mathbb{Z}|_{6\mathbb{Z}})$.

   b) $Hom(\mathbb{Z}|_{4\mathbb{Z}}, \mathbb{Z}|_{7\mathbb{Z}})$

   c) $Hom(\mathbb{Z}|_{4\mathbb{Z}}, \mathbb{Z})$

5. Show that a group of order 4 is abelian.

Assignment 2

1. Show that every continuous homomorphism from $\mathbb{R}$ to itself is of the form $x \mapsto \alpha x$ for some fixed $\alpha \in \mathbb{R}$
2. Show that any finite subgroup of $\mathbb{C}^*$ is cyclic.
3. Let G be an abelian group which has elements of order m, n. Show that G has an element of order $\operatorname{lcm}(m, n)$.
4. In the group $G = \mathbb{C}^*$, find the cosets of the subgroup $H = \{z \in G \text{ such that } |z| = 1\}$ and describe them geometrically.
5. Let H be a subgroup of G such that $x^2 \in H$ for all $x \in G$. Show that H is a normal subgroup and G/H is abelian.
6. Let G be a finite group. Suppose for all $x \in G$, there exists $y \in G$ such that $y^2 = x$, then G has odd order, and conversely.

Assignment 3

1. Prove that the quotient group $\mathbb{R}/\mathbb{Z}$ is isomorphic to the circle (with respect to multiplication).

2. Let H and K be subgroups of a group G of finite index. Show that $H \cap K$ also has finite index.

3. Let G be a finite group and H and K are subgroups of G. Prove that :

$$|HaK| = \frac{|H||K|}{|H \cap aKa^{-1}|}$$

for all $a \in G$.

4. Suppose H is a subgroup of G such that whenever $Ha \neq Hb$, then $aH \neq bH$. Show that H is normal in G.

5. Let G be a finite abelian group such that the number of solutions of $x^n = e$ is atmost n for every positive integer n. Show that G is cyclic.

Assignment 4

1. Show that $Z_4 \oplus Z_6 \simeq Z_2 \oplus Z_{12}$
2. Show that if G is a finite group and H is a proper subgroup of G, then G cannot be written as a union of conjugates of H.
3. Show that any finite group with more than 2 elements has a non-trivial automorphism.
4. Prove that any finite group of even order contains an element of order 2.
   Hint: Show that $t(G) = \{g \in G : g^2 \neq e\}$ has an even number of elements.
5. Show that if $G/Z(G)$ is cyclic, then G is abelian.

Assignment 5

1. Consider the action of the group of upper triangular matrices in $GL_n(\mathbb{R})$ (non-zero entries on the diagonal allowed) acting on $\mathbb{R}^n - \{0\}$. Is this action :
a) Faithful ? b) Free ? c) Transitive ?

2. Show that if p is a prime and G is a group of order p^n, then for every $m < n$, G has a subgroup of order p^m.

3. Show that $\mathbb{Z}$ is not isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

4. Let G be a group of order pn where p is a prime number and $p > n$. Show that if H is a subgroup of order p then it is a normal subgroup of G.

5. Prove that a group of order 56 has a normal Sylow p-subgroup for some prime p dividing 56.

Assignment 6

1. Show that $Z_4 \oplus Z_6 \simeq Z_2 \oplus Z_{12}$. (This was remaining from the earlier assignment)
2. Compute :
   a) $\mathrm{Aut}(\mathbb{Z}/p\mathbb{Z})$ for a prime p.
   b) $\mathrm{Aut}(\mathbb{Z}/8\mathbb{Z})$.
3. Show that the dihedral group of order 8 is not isomorphic to the Quaternions.
4. Compute the center of the group D_{2n}.
5. Let G be a group of order p^n and let H be a proper subgroup of G. Show that there exists $x \in G - H$ such that $xHx^{-1} = H$.

Assignment 7

1. Show that the Quaternion group is not a semidirect product of two proper subgroups.
2. Show that if p is a prime, a group of order $2p$ is either cyclic or the dihedral group.
3. Describe the (unique) non-abelian group of order 21 using generators and relations.
4. Show that any group of order 75 is a semidirect product of two proper subgroups.
5. Show that product of two solvable groups is solvable.

Assignment 8

1. Show that if $\mathbb{Z}^r$ has an injective homomorphism to $\mathbb{Z}^s$, then $r \leq s$. Furthermore show that if such a homomorphism exists and $r = s$, then the image is a finite index subgroup of $\mathbb{Z}^s$.

2. Characterize those integers n for which all abelian groups of order n are cyclic.

3. List the elements of order 2 and 3 in $\mathbb{Z}_4 \oplus \mathbb{Z}_6$. Also find all the index 2 subgroups.

4. Prove that every finite abelian group is isomorphic to a direct product of cyclic groups of the form $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_r}$, where $n_i | n_{i+1}$ for $i = 1, 2, \cdots, r-1$. (Note that this was proved for p-groups in the lectures)

5. Show that $\mathbb{Q}$ can be written as an increasing union of subgroups, each of which is a free group.

Assignment 9

(Assume all rings are commutative with multiplicative identity)

1. Show that any ring automorphism of $\mathbb{R}$ is identity.
2. If R is an integral domain, compute the unit group of $R[X]$ and $R[[X]]$.
3. Let $f : R \to R'$ be a ring homomorphism. Show that if I is an ideal in R', then $f^{-1}(I)$ is an ideal in R. Also show that if I is a prime ideal, so is $f^{-1}(I)$. Is the same true for maximal ideals ?
4. Let X be a metric space and let $p \in X$ be a point. Let $\mathcal{C}(X)$ be the ring of all continuous real-valued continuous functions on X. Show that :

   a) $\mathfrak{m}_p = \{f \in \mathcal{C}(X) \mid f(p) = 0\}$ defines a maximal ideal in $\mathcal{C}(X)$.
   b) Show that if X is compact, every maximal ideal is of this type.
5. Give an example of a ring which is not a field but has a unique maximal ideal.

Assignment 10

(Assume all rings are commutative with multiplicative identity)

1. Let X be a metric space and let $p \in X$ be a point. Let $\mathcal{C}(X)$ be the ring of all continuous real-valued continuous functions on X. Show that :

   a) $\mathfrak{m}_p = \{f \in \mathcal{C}(X) \mid f(p) = 0\}$ defines a maximal ideal in $\mathcal{C}(X)$.
   b) Show that if X is compact, every maximal ideal is of this type.

   (This was pending from the previous tutorial)

2. Show that $\mathbb{Z}[X]$ is not a PID.

3. Given two rings R and S, show that every ideal in $R \times S$ is of the form $I \times J$ for some ideals $I \subset R$ and $J \subset S$. Which ones of these are prime ideals ?