SI427

Probability 1

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Attendance policy - Not compulsory

<u>Cauchy seq.</u> – $\{a_n\}$ $(\in \mathbb{Q})$ is Cauchy if

$\forall \epsilon > 0$ $(\in \mathbb{Q})$ $\exists N \in \mathbb{N}$ s.t $\forall n, m > N$

$$|a_n - a_m| < \epsilon$$

$\mathcal{C}$ – Set of all Cauchy seq. of rationals

<u>Eq. relⁿ</u> – $\{a_n\} \sim \{b_n\}$ if

$\forall \epsilon > 0$ $(\in \mathbb{Q})$ $\exists N \in \mathbb{N}$ s.t $\forall n > N$

$$|a_n - b_n| < \epsilon$$

$\mathcal{I}$ – Set of all null sequences

(seq. in $\mathbb{Q}$ that converge to 0)

Note $\mathcal{R}$ is a ring

- group under $+$
- $\times$ is associative
- 1 is id. for $\times$
- $a \times (b+c) = (a \times b) + (a \times c)$

$\mathcal{I}$ is an ideal
ie a subset of $\mathcal{R}$ which is a s.g s.t

$$r \cdot a \in \mathcal{I} \quad \forall\, r \in \mathcal{R},\ a \in \mathcal{I}$$

$\mathcal{R}/\mathcal{I}$ is a ring.
These are the real nos. $\mathbb{R}$

$\mathcal{C}/\sim$ - Set of eq. classes of $\mathcal{C}$.
These are the real nos. $\mathbb{R}$ as well

Easy to check that add^n & multip^n is well defined

<u>Ordering</u> - $\{x_n\} \leq \{y_n\}$ if
$\exists N \in \mathbb{N}$ s.t $\forall n > N$ $y_n > x_n$

$y > x$ if $y \geq x$ & $y \neq x$

<u>Thm</u> : $\mathbb{R}$ is complete

i.e Every Cauchy seq. in $\mathbb{R}$ converges.
$\Leftrightarrow$
Least Upper Bound axiom

i.e If $S \subset \mathbb{R}$ is a set bounded above ($\exists M$ s.t $\forall x \in S$, $x \leq M$), then it has a least upper bound

We say M_0 is lub (or supremum) of S if

- M_0 is an upper bound
- if M is an upper bound of S, then $M_0 \leq M$

Pf - Since S is bounded above, $\exists\, U \in \mathbb{Q}$ s.t U is u.b of S.

Let $L \in \mathbb{Q}$ be s.t $\exists\, s \in S$ s.t $L < s$

Def. 2 cauchy seq., with $u_0 = U$, $l_0 = L$ & $m_n = (l_{n-1} + u_{n-1})/2$

$$u_n = \begin{cases} m_n, & m_n \text{ is ub of } S \\ u_{n-1}, & \text{otherwise} \end{cases} \rightarrow \text{seq. of u.b}$$

$$l_n = \begin{cases} l_{n-1}, & m_n \text{ is ub of } S \\ m_n, & \text{otherwise} \end{cases} \rightarrow \text{seq. of l.b}$$

Also, $u_n - l_n$ is a null seq.

Hence, real no. u is the sup. of S.

Countable Set $-$ S (infinite) is countable if $\exists f : \mathbb{N} \leftrightarrow S$ (bij)

Cardinality $-$ Two sets have the same cardinality if $\exists$ bij. $f : X \leftrightarrow Y$

i.e $\quad |X| = |Y|$

eg $-$ 1. $\mathbb{Z}$ is countable

$$1 \to 1$$
$$2 \to 0$$
$$3 \to 2$$
$$4 \to -1$$
$$\vdots$$

$$f(n) = \begin{cases} \frac{n+1}{2}, & n \in \text{Odd} \\ 1 - n/2, & n \in \text{Even} \end{cases}$$

2. $\mathbb{Q}$ is countable

	1	2	3	4	...
1	1/1	2/1	3/1	4/1	
2	1/2	2/2	3/2	4/2	
3	1/3	2/3	3/3	4/3	
4	1/4	2/4	3/4	4/4	
⋮					

○ skip on repetition

Cardinality of a countably infinite set is denoted by $\aleph_0$ (aleph nought)

<u>Def^n</u>: If $\exists$ injection $f: A \hookrightarrow B$, then we say $|B| \geq |A|$

If $|B| \geq |A|$ & $\nexists$ surjection from A to B, we say $|B| > |A|$

Thm: (Schroeder – Bernstein)
Suppose $\exists\ f: A \hookrightarrow B$ & $g: B \hookrightarrow A$, the $\exists$ bij. from A to B

Pf: Let $a \in A$. Consider the sequence

$$a,\ g^{-1}(a),\ f^{-1}(g^{-1}(a)),\ g^{-1}(f^{-1}(g^{-1}(a))),\ \dots$$

if $a \in g(B)$ if $g^{-1}(a) \in f(A)$ ...

The sequence either:

1. ends in A (elem is A-ancestral) – S_A
2. ends in B (elem is B-ancestral) – S_B
3. never terminates (elem is dual ancestral) – S_C

A is the disjoint union of these 3 sets

$$A = S_A \sqcup S_B \sqcup S_C$$

We'll construct a bij. $\varphi: A \leftrightarrow B$

$$\varphi(a) = \begin{cases} f(a) & \text{if } a \in S_A \cup S_C \\ g^{-1}(a) & \text{if } a \in S_B \end{cases}$$

Claim : φ is bij.

1. inj : $\varphi(a_1) = \varphi(a_2)$

1.1 $a_1, a_2 \in S_A \cup S_C \Rightarrow f(a_1) = f(a_2)$

$\Rightarrow a_1 = a_2$ [$\because f$ is inj]

1.2 $a_1, a_2 \in S_B \Rightarrow g^{-1}(a_1) = g^{-1}(a_2)$

$\Rightarrow g(g^{-1}(a_1)) = g(g^{-1}(a_2))$

$\Rightarrow a_1 = a_2$

Hence φ is inj.

$\left\{ \begin{array}{l} \because \text{ seq. is} \\ g(b), b, f^{-1}(b), \dots \end{array} \right\}$

2. surj: consider $b \in B$

If $g(b) \in S_A \cup S_C \Rightarrow \exists\, a \in A$ s.t
$\varphi(a) = f(a) = b$

else, $g(b) \in S_B \Rightarrow \varphi(g(b)) = g^{-1}(g(b)) = b$

Hence, φ is surj.

Continuum Hypothesis : $\nexists\, S$ s.t $\aleph_0 < |S| < \aleph_1$

Godel (1940) – CH can't be disproved in ZFC
Cohen (1963) – CH can't be proved in ZFC

Random / Statistical Exp. : Exp. in which

1. all possible outcomes are known in advance.

2. any performance of the exp. results in an outcome that is not known in advance.

3. The exp. can be repeated under identical cond^{n}s.

Modelling Random Exp.

- Sample space (Ω) : set of all possible outcomes

- S (σ field / algebra) : non-empty set of subsets of Ω (with certain ppts)

- Event : element of S

Formally, (Ω, S) is the sample space.

Discrete Probability spaces (Ω, p)

Let Ω be finite or countable set.
Let $p : \Omega \to [0,1]$ be a fn^n s.t
$\sum_{\omega \in \Omega} p(\omega) = 1$

- Ω : Sample space
- $p(\omega)$: elementary probabilities

- $A \subset \Omega$: event (i.e $S = \mathcal{P}(\Omega)$)

- $P : S = \mathcal{P}(\Omega) \to [0,1]$
  $P(A) = \sum_{\omega \in A} p(\omega)$: probability of event A.

- $X : \Omega \to \mathbb{R}$: random variable

- $E[X] = \sum_{\omega \in \Omega} X(\omega)\, p(\omega)$: expected value of random var.

eg – 1. $\Omega = \{0,1\}$, $\underset{\substack{\uparrow \\ p(0)}}{p_0} = \underset{\substack{\uparrow \\ p(1)}}{p_1} = 1/2$

$A = \emptyset, \{0\}, \{1\}, \{0,1\}$

we can also change $p_0 = q$, $p_1 = 1-q$

$P(\emptyset) = 0$, $P(\{0\}) = q$, $P(\{1\}) = 1-q$, $P(\Omega) = 1$

2. $\Omega = \{0,1\}^n = \{\underline{\omega} = (\omega_1\ \omega_2 \cdots \omega_n) : \omega_k = 0,1,\ 1 \le k \le n\}$

$|\Omega| = 2^n$, $p(\underline{\omega}) = \dfrac{1}{2^n}$

No. of possible events $= 2^{2^n}$

$$A_k = \{ \underline{\omega} \in \Omega : \omega_1 + \omega_2 + \dots + \omega_n = k \}$$

For $k > n$, $|A_k| = 0$ $\quad (\because \Sigma \omega_i \leq n)$

$k \leq n$, $|A_k| = {}^nC_k$

$$P(A_k) = \sum_{\omega \in A_k} p(\underline{\omega}) = \frac{{}^nC_k}{2^k}$$

<u>Convention</u>: ${}^nC_k = 0$ if $k > n$.

<u>Model</u>: Tossing n coins.

<u>3.</u> $\Omega = \{ \underline{\omega} = (\omega_1 \dots \omega_r) : 1 \leq \omega_k \leq m \}$

<u>Not^n</u>: $[n] = \{1, 2, \dots, n\}$

Clearly $\Omega = [m]^r$, $\quad |\Omega| = m^r$

$$p(\underline{\omega}) = \frac{1}{m^r}$$

No. of pos. events $= 2^{m^r}$

$A_1 = \{ \underline{\omega} \in \Omega : \quad \omega_r = 1 \}$

$A_2 = \{ \underline{\omega} \in \Omega : \quad \omega_k \neq 1 \, , \quad 1 \leq k \leq r \}$

$A_3 = \{ \underline{\omega} \in \Omega : \quad \omega_j \neq \omega_k \, , \quad j \neq k \, , \quad 1 \leq j, k \leq r \}$

$|A_1| = m^{r-1} \quad , \quad |A_2| = (m-1)^r \quad , \quad |A_3| = {}^m C_r \cdot r!$

$P(A_1) = \dfrac{m^{r-1}}{m^r} \qquad P(A_2) = \dfrac{(m-1)^r}{m^r} \qquad P(A_3) = \dfrac{{}^m C_r \cdot r!}{m^r}$

Model : - $m = 6$: Throwing dice r times

- r labelled balls in m labelled boxes
  ω_k = bin in which kth ball falls

- Birthday paradox : $m = 365$
  Choose r ppl and record their birthdays.
  ω_k = kth person's birthday.

4. $\Omega = \{0,1\}^n = \{ \underline{\omega} = (\omega_1 \ \omega_2 \cdots \omega_n) : \omega_k = 0, 1 \ \ 1 \leq k \leq n \}$

$|\Omega| = 2^n$,

Heads with prob. p & Tails with prob. $q = 1-p$

If $\underline{\omega}$ has k heads (0) & $n-k$ tails (1),

$P(\underline{\omega}) = {}^nC_k \, p^k q^{(n-k)}$

5. Countably inf. sample space

Heads with prob. p & Tails with prob. $q = 1-p$
Keep tossing the coin till heads appears.

Let 0^k1 be the seq. of k-tails followed by a heads

$\Omega = \{1, 0^1 1, 0^2 1, \ldots\} \cup \{0^*\}$
zero seq.

$p(0^k 1) = q^k p$
$p(0^*) = 0$

$A =$ at least n tails before heads appears

$$P(A) = \sum_{k=n}^{\infty} p(0^k 1) = q^n p + q^{n+1} p + \cdots$$
$$= pq^n (1 + q + \cdots)$$
$$= \frac{pq^n}{1-q} = q^n$$

- If Ω is count. inf.,
  what does $\sum_{\omega \in \Omega} p(\omega) = 1$ mean?

We have a bij $i \mapsto \omega_i \in \Omega$

$$\sum_{\omega \in \Omega} p(\omega) := \sum_{i=1}^{\infty} p(\omega_i)$$

Similarly $X : \Omega \to \mathbb{R}$, $E[X] = \sum_{\omega \in \Omega} X(\omega) p(\omega)$
$$= \sum_{n=1}^{\infty} X(\omega_n) p(\omega_n)$$

But, we need to check that this sum is indep. of the bij.

Lem: Let $\{a_n\}_{n=1}^{\infty}$ be a seq. of non-(-ve) nos. s.t $\sum_{n=1}^{\infty} a_n$ converges (say to S).

Let bij $\sigma : \mathbb{N} \leftrightarrow \mathbb{N}$.
Then $\sum_{n=1}^{\infty} a_{\sigma(n)}$ also converges & equals S.

eg – Why is fails for (-ve) nos.

$$\log(2) = 1 - \cancel{\tfrac{1}{2}} + \tfrac{1}{3} - \underline{\tfrac{1}{4}} + \tfrac{1}{5} - \cancel{\tfrac{1}{6}} \ \cdots$$
$$\tfrac{1}{2}\log(2) = \cancel{\tfrac{1}{2}} - \underline{\tfrac{1}{4}} + \cancel{\tfrac{1}{6}} \ \cdots$$

$$\frac{3}{2}\log(2) = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} \cdots = \log(2)$$

$$\Rightarrow \log(2) = 0 \quad \times$$

Thm: (Riemann Rearrangement)

Let $\{a_n\}_{n=1}^{\infty}$ be a seq. s.t $\sum_{n=1}^{\infty} a_n < \infty$

& $\sum_{n=1}^{\infty} |a_n| = \infty$

Then given any $M \in \mathbb{R}$, $\exists$ bij $\sigma : \mathbb{N} \hookleftarrow \mathbb{N}$

s.t $\sum_{n=1}^{\infty} a_{\sigma(n)} = M$

Absolute convergence : $\sum_{n=1}^{\infty} a_n$ is abs. conv.

if $\sum_{n=1}^{\infty} |a_n|$ conv.

Conditional covergence : $\sum_{n=1}^{\infty} a_n$ is cond. conv.

if $\sum_{n=1}^{\infty} |a_n|$ does not conv.

Thm: If $\sum_{n=1}^{\infty} a_n$ is abs. conv., then for any bij $\sigma : \mathbb{N} \hookleftarrow \mathbb{N}$, $\sum_{n=1}^{\infty} a_{\sigma(n)}$ conv.

& equals $\sum_{n=1}^{\infty} a_n$

Pf: (hem)

Let $\quad T_m = \sum_{n=1}^{m} a_{\sigma(n)}$

$\exists\ N_m \in \mathbb{N}$ s.t $\quad \sigma(n) \le N_m \quad \forall\ n \le m$

Let $\quad S_k = \sum_{n=1}^{k} a_n$

Clearly, $\quad T_m \le S_{N(m)} \le S \quad \forall\ m \in \mathbb{N}$

T_m – mono. $\uparrow$ & bound. above.

Hence, T_m conv. and $\lim_{m\to\infty} T_m \le S$

Reverse the roles of a_n & $a_{\sigma(n)}$ to see

$$S \le \lim_{m\to\infty} T_m$$

$$\Rightarrow \quad S = \lim_{m\to\infty} T_m$$

Suppose $a(n) \in \mathbb{R}$

$$a_+(n) = \max(a(n), 0)$$

$$a_-(n) = \max(-a(n), 0)$$

$$a(n) = a_+(n) - a_-(n)$$

Ex: The series $\sum_{n=1}^{\infty} a(n)$ is abs. conv. iff $\sum_{n=1}^{\infty} a_+(n)$ & $\sum_{n=1}^{\infty} a_-(n)$ are conv.

Further $\sum_{n=1}^{\infty} a(n) = \sum_{n=1}^{\infty} a_+(n) - \sum_{n=1}^{\infty} a_-(n)$

Rules for Probability

Let (Ω, P) be a prob. sp.

Then

1. For any event A, $0 \leq P(A) \leq 1$
and $P(\emptyset) = 0$, $P(\Omega) = 1$

2. Finite additivity
If $A_1, \dots, A_n$ are pairwise disjoint
$(A_i \cap A_j = \emptyset, \ i \neq j)$,

$$P(A_1 \cup A_2 \dots \cup A_n) = P(A_1) + \dots + P(A_n)$$

3. Countable additivity
If $A_1, A_2, \dots$ is a count. collection of sets,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)$$

Equality holds for pairwise disj. sets.

$\underline{Pp^n}$: $P(A') = 1 - P(A)$

$\underline{Ex}$: Suppose $\{A_n\}_{n=1}^{\infty}$ is a seq. of events which are non-decreasing, i.e $A_n \subseteq A_{n+1} \; \forall n$

Show that $\lim\limits_{n \to \infty} P(A_n) = P\left(\bigcup\limits_{n=1}^{\infty} A_n\right)$

$\underline{Cor}$: Suppose $\{A_n\}_{n=1}^{\infty}$ is a seq. of events which are non-increasing, i.e $A_n \supseteq A_{n+1} \; \forall n$

Show that $\lim\limits_{n \to \infty} P(A_n) = P\left(\bigcap\limits_{n=1}^{\infty} A_n\right)$

These two ppts are called continuity of prob. from below & above resp.

The Inclusion-Exclusion principle

For a discrete prob. sp. (Ω, P) & a collection of events $A_1, \dots, A_n$, let

$$S_k = \sum_{1 \leq i_1 < \dots < i_k \leq n} A_{i_1} \cap A_{i_2} \dots \cap A_{i_k} \quad , \quad 1 \leq k \leq n$$

Then, $$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n} (-1)^{k-1} P(S_k)$$

Rem: If $p(\omega) = \dfrac{1}{|\Omega|}$ $\forall\ \omega \in \Omega$, then

we get $$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k-1} |S_k|$$

which is the usual inclusion-exclusion for sets.

Pf: Let $A = \bigcup_{i=1}^{n} A_i$

$$\text{LHS} = P(A) = P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{\omega \in \Omega} \mathbb{1}_A(\omega)\, p(\omega)$$

where $\mathbb{1}_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \text{otherwise} \end{cases}$

$$P(S_k) = P(A_{i_1} \cap \dots \cap A_{i_k})$$
$$= \sum_{\omega \in A_{i_1} \cap \dots \cap A_{i_k}} p(\omega) = \sum_{\omega \in \Omega} p(\omega) \prod_{j=1}^{k} \mathbb{1}_{A_{i_j}}$$

$$\text{RHS} = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \dots < i_k \le n} \sum_{\omega \in \Omega} p(\omega) \prod_{j=1}^{k} \mathbb{1}_{A_{i_j}}$$
$$= \sum_{\omega \in \Omega} \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \dots < i_k \le n} p(\omega) \prod_{j=1}^{k} \mathbb{1}_{A_{i_j}}(\omega)$$

(reording Σ)

$$= -\sum_{\omega \in \Omega} p(\omega) \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \dots < i_k \le n} \prod_{j=1}^{k} \left(-\mathbb{1}_{A_{i_j}}(\omega)\right)$$

The exp. above resembles

$$(1-x_1)(1-x_2)\dots(1-x_k)$$
$$= 1-\Sigma x_i + \Sigma x_i x_j - \dots$$

$$\Rightarrow -\sum_{\omega\in\Omega} p(\omega) \underbrace{\left(\prod_{i=1}^{n}(1-\mathbb{1}_{A_i}(\omega))-1\right)}_{I}$$

If $\omega \in A$, $\omega \in A_i$ for at least one i, $1 \le i \le n$,
$I = 1$

If $\omega \notin A$, $I = 0$

Hence, $LHS = RHS$

Ex: Calc. prob. that at least m of the events A_i, $1 \leq i \leq n$ occur.

$$B_m = \bigcup_{1 \leq i_1 < \dots < i_m \leq n} A_{i_1} \cap \dots \cap A_{i_m}$$

i.e calc. $P(B_m)$.

For exactly m events, $P(B_m \setminus B_{m+1})$

eg : 1. Placing r labelled balls in m labelled bins.

$$\Omega = [m]^r \quad ([m] = \{1, 2, \ldots, m\})$$

$p(\omega) = m^{-r}$, ω_i : label of the bin in which i-th ball is put.

A : event that some bin is empty

$$A = \bigcup_{l=1}^{m} A_l \quad , \quad A_l = \underbrace{\{ \omega \in \Omega : \omega_i \neq l \ \forall\, 1 \leq i \leq r \}}_{\text{bin } l \text{ is empty}}$$

$$P(A_{i_1} \cap \ldots \cap A_{i_k}) = \frac{(m-k)^r}{m^r}$$

For $k = m$, $P(A_{i_1} \cap \ldots \cap A_{i_m}) = 0$

$\because$ since all bins can't be empty

$$P(A) = \sum_{k=1}^{m} (-1)^{k-1} \, {}^{m}C_k \left(1 - \frac{k}{m}\right)^{r}$$

$\because$ ${}^{m}C_k$ events of the form $P(A_{i_1} \cap \dots \cap A_{i_k})$

2. $\Omega = S_m$, $p(\omega) = \frac{1}{|S_m|} = \frac{1}{m!}$

$\sigma : [m] \to [m]$ is a permutation ($\sigma \in S_m$)

Derangement is σ s.t $\sigma(i) \neq i \;\; \forall \; 1 \leq i \leq m$

A_ℓ : event s.t $\sigma(\ell) = \ell$

A : subset of derangements of S_n.

Note, $A' = \bigcup_{\ell=1}^{m} A_\ell$

$P(A_{i_1} \cap \dots \cap A_{i_k}) = \frac{(m-k)!}{m!}$ ($\because$ $\sigma(i_j) = i_j$, $j = 1, \dots, k$ rest are free to map to any value)

By inclusion-exclusion,

$$P(A') = \sum_{k=1}^{m} (-1)^{k-1} \, {}^{m}C_k \, \frac{(m-k)!}{m!}$$

$$= \sum_{k=1}^{m} (-1)^{k-1} \frac{1}{k!} = s_m$$

s_m : m^{th} partial sum of $1-e^{-1}$

$s_m \sim 1-e^{-1}$

$P(A') \sim 1-e^{-1}$

$P(A) \sim e^{-1}$ $\qquad \because \lim_{m\to\infty} 1-s_m = e^{-1}$

Bonferroni's Inequality

Thm : $A = \bigcup_{i=1}^{m} A_i$

$$P(A) \leqslant \sum_{k=1}^{n} (-1)^{k-1} S_k \quad \text{if } n \text{ is odd}$$

$$P(A) \geqslant \sum_{k=1}^{n} (-1)^{k-1} S_k \quad \text{if } n \text{ is even}$$

eg : In the r balls & m bins example

$$P(A) = \sum_{k=1}^{m} (-1)^{k-1} \, {}^{m}C_k \left(1 - \frac{k}{m}\right)^{r}$$

Take $n = 1, 2$

$${}^{m}C_1 \left(1 - \frac{1}{m}\right)^{r} - {}^{m}C_2 \left(1 - \frac{2}{m}\right)^{r} \leqslant P(A) \leqslant {}^{m}C_1 \left(1 - \frac{1}{m}\right)^{r}$$

<u>Independent Events</u> : A & B are said to be indep. if $P(A \cap B) = P(A)P(B)$

<u>eg</u> : <u>1.</u> n fair coin tosses

$$\Omega = \{ \omega = (\omega_1, \dots, \omega_n) : \omega_i \in \{0,1\} \}$$

$$p(\omega) = 2^{-n}$$

$$A = \{ \omega \in \Omega : \omega_1 = 0 \}$$
$$B = \{ \omega \in \Omega : \omega_2 = 0 \}$$
$$A \cap B = \{ \omega \in \Omega : \omega_1 = \omega_2 = 0 \}$$

$$|A| = 2^{n-1}, \quad |B| = 2^{n-1}, \quad |A \cap B| = 2^{n-2}$$

$$P(A \cap B) = \frac{2^{n-2}}{2^n} = \frac{1}{4} = \frac{2^{n-1}}{2^n} \cdot \frac{2^{n-1}}{2^n} = P(A)P(B)$$

2. Throwing a pair of dice

$\Omega = \{(i,j) : 1 \leq i,j \leq 6\}$

$p(\omega) = 1/36$

$A = \{(i,j) : i \text{ is odd}\}$

$B = \{(i,j) : j = 1, 6\}$

$C = \{(i,j) : i+j = 4\}$

$P(A) = 1/2$, $P(B) = 1/3$, $P(C) = 1/12$

$P(A \cap B) = 1/6 = P(A)P(B)$

$P(A \cap C) = 1/18 \neq P(A)P(C)$

Pr^n : If A & B are indep., the following pair of events are also indep. -

(A', B), (A, B'), (A', B')

For a discrete prob. sp. (Ω, P), the events $A_1, \dots, A_n$ are mutually indep. if given any subcollection $A_{i_1}, \dots, A_{i_k}$, $1 \leq i_1 < \dots < i_k \leq n$, $1 \leq k \leq n$

$$P(A_{i_1} \cap \dots \cap A_{i_k}) = P(A_{i_1}) \dots P(A_{i_k})$$

Ex: Let $A_1, \dots, A_n$ be (mutually) indep. events. Suppose $P(A_k) = p_k$

Show that the probability of m or more events occuring simultaneously is

$$\leq \frac{(p_1 + \dots + p_n)^m}{m!}$$

Conditional Probability

for dps. (Ω, P) & events A, B, suppose $P(B) \neq 0$.

$$P(A|B) := \frac{P(A \cap B)}{P(B)}$$

prob. of A occ.
given B has already occ.

eg: We have 2 urns

Urn 1: 1 red ball (R_{11}), 2 black balls (B_{11}, B_{12})
Urn 2: 2 red balls (R_{21}, R_{22}), 1 black ball (B_{21})

Toss a coin: If H, choose ball from urn 1, otherwise choose ball from urn 2.

A: event that chosen ball is black.

$$\Omega = \{(H, R_{11}), (H, B_{11}), (H, B_{12}), (T, R_{21}), (T, R_{22}), (T, B_{21})\}$$

$$P(A) = 3/6 = 1/2 = 1/2(1/3) + 1/2(2/3) = P(H)P(A|H) + P(T)P(A|T)$$

Bayes Thm

$$P(A \cap B) = P(A)\,P(B|A) = P(B)\,P(A|B)$$

$$\Rightarrow \quad P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

eg: Monty Hall Problem

Let A : event that car is behind the initially chosen door D_1

B : Monty Hall opens door D_2 i.e D_2 has goat behind it.

$P(A) = 1/3$

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}, \qquad P(B|A) = 1/2$$

$$= \frac{1/2 \cdot 1/3}{1/2}$$

$$= 1/3$$

Car behind D_1 Opening D_2

$$P(B) = 1/3 \cdot 1/2 + 1/3 \cdot 0 + 1/3 \cdot 1$$

$$= 1/2$$

Car behind D_2 Opening D_2

Thm : Let $\{A_n\}_{n\in\mathbb{N}}$ be a countable coll. of pairwise disjoint events which are mutually exhaustive (i.e $\bigcup_{n=1}^{\infty} A_n = \Omega$), for any event B,

1. $$P(B) = \sum_{n=1}^{\infty} P(A_n)\, P(B|A_n)$$

2. If $P(B) > 0$, $$P(A_k|B) = \frac{P(B|A_k)\, P(A_k)}{\sum_{n=1}^{\infty} P(A_n)\, P(B|A_n)}$$

Ppn : Let $A_1, \ldots, A_n$ be a coll. of events s.t $P(\bigcap_{k=1}^{n} A_k) \neq 0$, then

$$P\left(\bigcap_{k=1}^{n} A_k\right) = P(A_1)\, P(A_2|A_1)\, P(A_3|A_1 \cap A_2) \ldots P\left(A_n \,\middle|\, \bigcap_{k=1}^{n-1} A_k\right)$$

Consider dps. (Ω, p) & random var. $X : \Omega \to \mathbb{R}$

Assume range of $\mathbb{R}$ is countable.

Probability Mass Fxn : $f_X : \mathbb{R} \to \mathbb{R}$

$$f_X(t) = \begin{cases} P(\{w \in \Omega : X(w) = t\}) , & \text{if } t \in \text{Range}(X) \\ 0 , & \text{otherwise} \end{cases}$$

$$f_X(t) = P(X^{-1}(t))$$

Note, $\sum_{t \in \mathbb{R}} f_X(t) = 1$

Let range of X be $\{t_1, t_2, \dots\}$

$$\because \quad \sum_{t \in \mathbb{R}} f_X(t) = \sum_{n=1}^{\infty} P(X^{-1}(t_n)) = \sum_{n=1}^{\infty} \sum_{w \in X^{-1}(t_n)} p(w)$$

$$= \sum_{w \in \Omega} p(w) = 1$$

<u>Cumulative distribution fn</u> : $F_X : \mathbb{R} \to [0,1]$

$$F_X(t) = P(\{\omega \in \Omega : X(\omega) \leq t\})$$

$$F_X(t) = \sum_{u \leq t} f_X(u)$$

eg : $\Omega = \{(i,j) : 1 \leq i,j \leq 6\}$, $p(\omega) = 1/36$

$$X : \Omega \to \mathbb{R}$$
$$(i,j) \mapsto i+j$$

t	$f_X(t)$	$F_X(t)$	
<2	0	0	
2	1/36	1/36	$t \in [2,3)$
3	2/36	3/36	$t \in [3,4)$
4	3/36	6/36	$t \in [4,5)$
5	4/36	10/36	$t \in [5,6)$
6	5/36	15/36	$t \in [6,7)$
7	6/36	21/36	$t \in [7,8)$
8	5/36	26/36	$t \in [8,9)$
9	4/36	30/36	$t \in [9,10)$
10	3/36	33/36	$t \in [10,11)$
11	2/36	35/36	$t \in [11,12)$
12	1/36	1	

Ppts of CDF

1. F is an increasing fx^n on $\mathbb{R}$.

2. $\lim\limits_{t \to \infty} F(t) = 1$, $\lim\limits_{t \to -\infty} F(t) = 0$

3. F is right cont.

$$\lim_{h \to 0^+} F(t+h) = F(t) \quad \forall\, t \in \mathbb{R}$$

4. F is a step fx^n. It will jump exactly at the pts. where it is not left continuous.

Pf: 1. Let $t < s$.

Consider $A = \{\omega \in \Omega : X(\omega) \leq t\}$

$B = \{\omega \in \Omega : X(\omega) \leq s\}$

Clearly, $A \subseteq B \Rightarrow P(A) \leq P(B) \Rightarrow F_X(t) \leq F_X(s)$

2. Let $A_n = \{ \omega \in \Omega : X(\omega) \leq n \}$

Clearly, $A_n \subseteq A_{n+1} \quad \forall n$

Note $\Omega = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} X^{-1}(n)$

$$\underbrace{P(\Omega)}_{1} = \lim_{n\to\infty} P\left(\bigcup_{k=1}^{n} A_k\right) = \lim_{n\to\infty} P(A_n) = \lim_{t\to\infty} F_X(t)$$

$$\Rightarrow \lim_{t\to\infty} F_X(t) = 1$$

Sim. for $\varphi = \bigcap_{n=-1}^{-\infty} A_n$,

$$\underbrace{P(\varphi)}_{0} = \lim_{n\to -\infty} P\left(\bigcap_{k=-1}^{n} A_k\right) = \lim_{n\to -\infty} P(A_n) = \lim_{t\to -\infty} F_X(t)$$

$$\Rightarrow \lim_{t\to -\infty} F_X(t) = 0$$

3. $A_n = \{ \omega \in \Omega : X(\omega) \leq t + 1/n \}$

$$F_X(t) = \lim_{n\to\infty} P\left(\bigcap_{k=1}^{n} A_k\right) = \lim_{h\to 0^+} F_X(t+h)$$

eg: 1. <u>Bernoulli distr.</u>

$$f(t) = \begin{cases} p, & \text{if } t=1 \\ q, & \text{if } t=0 \end{cases}$$

$$F(t) = \begin{cases} 1, & \text{if } t \geq 1 \\ q, & \text{if } t \in [0,1) \\ 0, & \text{if } t < 0 \end{cases}$$

A random var. having this PMF or CDF is said to have Bernoulli distr. with parameter p & we write $X \sim Ber(p)$

$\Omega = [10]$, $\quad p(\omega) = 1/10$

$X(\omega) = \mathbb{1}_{\omega \leq 3}$, $\quad$ then $\quad X \sim Ber(0.3)$

2. <u>Binomial distr</u>

with parameter n & p $\quad$ $Bin(n, p)$

For $n \geq 1$ & $p \in [0,1]$

$$f(k) = {}^{n}C_k \, p^k q^{n-k}$$

$$\Omega = \{0,1\}^n \qquad p(\omega) = p^{\sum_i \omega_i} q^{n - \sum_i \omega_i}$$

$$X(\omega) = \omega_1 + \dots + \omega_n$$

$X \sim Bin(n, p)$

3. <u>Geometric distr</u>

$p \in [0,1]$, $\quad f(k) = q^{k-1} p$, $\quad k \in \mathbb{N}$

$$F(t) = \begin{cases} 0 & , \text{ if } t < 1 \\ 1 - q^k & , \text{ if } k \leq t < k+1 \end{cases}$$

$X \sim Geo(p)$

4. Poisson distr.

Fix $\lambda > 0$. $\quad f(k) = \dfrac{e^{-\lambda}\lambda^k}{k!}$

$X \sim \text{Pois}(\lambda)$

5. Hypergeometric distr

Fix $b, w, m \in \mathbb{Z}_{>0}$ with $m \leq b+w$

$$f(k) = \frac{{}^{b}C_k \; {}^{w}C_{m-k}}{{}^{b+w}C_m}$$

Consider a pop. with b men & w women. The number of men in a random sample (w/o replacement) of size m is a random var $X \sim \text{Hypergeo}(b, w, m)$

Expectation values

$X : \Omega \to \mathbb{R}$

$$E[X] = \sum_{\omega \in \Omega} X(\omega) p(\omega)$$

Claim : $E[X] = \sum_{t \in \mathbb{R}} t f_X(t)$

$\text{Range}(X) = \{x_1, x_2, \ldots\}$
$A_k = \{\omega \in \Omega : X(\omega) = x_k\}$

Note, $A_k = X^{-1}(x_k)$ are mutually exclusive & exhaustive events.

$$E[X] = \sum_{\omega \in \Omega} X(\omega) p(\omega) = \sum_k \sum_{\omega \in A_k} X(\omega) p(\omega)$$

$$= \sum_k x_k P(A_k) = \sum_k x_k f(x_k)$$

Sim. $E[X^2] = \sum_k x_k^2 f(x_k)$

$$E[h(X)] = \sum_k h(x_k) f(x_k), \quad h: \mathbb{R} \to \mathbb{R}$$

Ex: $X \sim Bin(n, p)$. Find $E[X]$

$$E[X] = \sum_{k=1}^{n} k\, {}^nC_k\, p^k q^{n-k}$$

$$= \sum_{k=1}^{n} n\, {}^{n-1}C_{k-1}\, p^k q^{n-k} = np \sum_{k=1}^{n} {}^{(n-1)}C_{k-1}\, p^{k-1} q^{(n-1)-(k-1)}$$

$$= np\,(p+q)^{n-1}$$

$$= np$$

$$E[X^2] = \sum_{k=1}^{n} k^2\, {}^nC_k\, p^k q^{n-1} = \sum_{k=1}^{n} k(k-1)\, {}^nC_k\, p^k q^{n-1}$$

$$+ \underbrace{\sum_{k=1}^{n} k\, {}^nC_k\, p^k q^{n-1}}_{np}$$

$$= \sum_{k=1}^{n} n(n-1)\, {}^{(n-2)}C_{(k-2)}\, p^k q^{n-k} + np$$

$$= p^2 n(n-1) + np$$

<u>Rem</u> : - In all the examples, $X(\Omega) \subseteq \mathbb{Z}_{\geq 0}$

Since $X(\Omega)$ is discrete in $\mathbb{R}$.,

$F(t)$ is const. in $[t_i, t_j)$.

This is <u>not</u> true if $X(\Omega)$ is dense such as $\mathbb{Q}$

- By using random var., we convert the prob. sp. (Ω, p) to $(\mathbb{R}, F(t))$

If we want to talk about Ω uncountable & $X : \Omega \to \mathbb{R}$ s.t $X(\Omega)$ is <u>not</u> necessarily countable, we need to restrict ourselves to measurable fn's.

CDF : A cumulative distr. fxn is a fxn
$F : \mathbb{R} \to \mathbb{R}$ satisfying

1. non-decreasing
2. $\lim_{t \to \infty} F(t) = 1$, $\lim_{t \to -\infty} F(t) = 0$
3. right continuity

Ppn : If X is a rand. var., then $F_X(t)$ is a CDF.

PDF :
Suppose $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a piecewise cont. fxn
s.t $\int_{-\infty}^{\infty} f(u)\, du = 1$

Such a fxn is called a probability density fxn.

Let
$$F(t) = \int_{-\infty}^{t} f(u)\, du$$

Then F is a CDF.

Infact, $F(t)$ is cont. & diff. at all pts where f is cont.

eg: 1. <u>Uniform distr. on $[a,b]$</u>

$$f(t) = \begin{cases} \frac{1}{b-a}, & t \in [a,b] \\ 0, & \text{otherwise} \end{cases}$$

$$F(t) = \begin{cases} 0, & t \leqslant a \\ \frac{t-a}{b-a}, & t \in (a,b) \\ 1, & t \geqslant b \end{cases}$$

2. <u>Exponential distr. with para. λ</u>

$$f(t) = \begin{cases} 0, & t \leq 0 \\ \lambda e^{-\lambda t}, & t > 0 \end{cases}$$

$$F(t) = \begin{cases} 0, & t \leq 0 \\ 1 - e^{-\lambda t}, & t > 0 \end{cases}$$

3. <u>Normal distr. $N(\mu, \sigma)$</u>

$$\varphi_{\mu,\sigma}(t) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-(t-\mu)^2}{2\sigma^2}}$$

$$\varphi(t) = \varphi_{0,1}(t)$$

$$F(t) = \int_{-\infty}^{\infty} \varphi_{\mu,\sigma}(t)\, dt$$

<u>Check</u>: $\lim_{t\to\infty} F(t) = 1$

By change of variables, it suffices to show this for $\varphi(t)$ i.e

$$\int_{-\infty}^{\infty} e^{-u^2}\, du = \sqrt{2\pi}$$

Let $\quad I = \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du$

$$\Rightarrow \quad I^2 = \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du \int_{-\infty}^{\infty} e^{-\frac{v^2}{2}} dv = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{(u^2+v^2)}{2}} du\, dv$$

$$= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2/2}\, r\, dr\, d\theta$$

$$= \int_{0}^{2\pi} \underbrace{\left[-e^{-r^2/2}\right]_0^{\infty}}_{1} d\theta$$

$$= 2\pi$$

$$\Rightarrow \quad I = \sqrt{2\pi}$$

Hence, F is a CDF.

<u>Gamma fx^n</u> : $\Gamma(s) = \int_0^\infty \underbrace{e^{-t} t^{s-1}}_{f(t)} dt$, $Re(s) > 0$

$f(t) = e^{-t} e^{(s-1)\log(t)}$

$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges $\forall z \in \mathbb{C}$

$e^{(s-1)\log(t)} = e^{(\sigma-1)\log(t)} e^{i\tau \log(t)}$, $s = \sigma + it$

$\Gamma(s) = \int_0^\infty e^{-t} t^s d^\times t$, $d^\times t = dt/t$

$\Gamma(s/2) = \int_0^\infty e^{-t} t^{s/2-1} dt = \int_0^\infty e^{-x^2} (x^2)^{s/2-1} 2x \, dx$

$= \int_{-\infty}^{\infty} e^{-x^2} |x|^s d^\times x$

e^{-x^2} is a Schwartz fn^n in $\mathbb{R}$ i.e it is in $\mathcal{C}^\infty(\mathbb{R})$
& for any polynomial $\lim_{x\to\infty} p(x)\, e^{-x^2} = 0$

Restrict $f(x)$ to $\mathbb{R}^* = \mathbb{R}\setminus\{0\}$

$$Mf(x) = \int_{-\infty}^{\infty} f(x)\, |x|^s\, d^\times x$$

Mellin transform of f.

So, $\Gamma(s)$ is Mellin transform of e^{-x^2}.

$$\Gamma(1+s) = \int_0^\infty e^{-t} t^s\, dt = \left[t^s - e^{-t}\right]_0^\infty + s\int_0^\infty e^{-t} t^{s-1}\, dt$$

$$= s\,\Gamma(s)$$

$$\Rightarrow \Gamma(s) = \frac{\Gamma(s+1)}{s}$$

Functional eq^n for Γ-fn^n

$$\Gamma(s) = \underbrace{\int_0^1 e^{-t}\, t^{s-1}\, dt}_{I_1} + \underbrace{\int_1^\infty e^{-t}\, t^{s-1}\, dt}_{I_2}$$

$$\frac{t^{s-1}}{e^t} = \frac{t^{s-1}}{1+t+\frac{t^2}{2!}+\dots+\frac{t^n}{n!}\dots} \leq n!\; t^{s-1-n}$$

If $n > s$, $\qquad s-1-n < -1$

$$I_2 \leq \int_1^\infty n!\; t^{s-n-1}\, dt = n! \left[\frac{t^{s-n}}{s-n}\right]_1^\infty = \frac{n!}{s-n}$$

If $s > 0 \quad (s \in \mathbb{R})$

$$I_1 = \int_0^1 e^{-t}\, t^{s-1}\, dt \leq \int_0^1 t^{s-1}\, dt = \left[\frac{t^s}{s}\right]_0^1 = \frac{1}{s}$$

So, $\Gamma(s)$ is well-defined for $s > 0$

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}$$

RHS is def. if $s > -1$, $s \neq 0$

$$\Gamma(s+2) = (s+1)\Gamma(s+1) = (s+1)s\Gamma(s)$$

$$\Gamma(s) = \frac{\Gamma(s+1)}{s(s+1)}$$

RHS is def. if $s > -2$, $s \neq 0, -1$

Sim. we can define Γ $\forall\ s \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$

If $n \in \mathbb{Z}_{>0}$, $\quad \Gamma(n+1) = n\Gamma(n)$

$$= n!$$

<u>Sterling's Formula</u>

$$\Gamma(s) = \sqrt{2\pi}\, s^{s-1/2} e^{-s}\, O(1+1/s)$$

$g(x) \in \mathbb{R}_{\geq 0}$ $\forall\ x$, $\quad f(x) = O(g(x))$ as $x \to \infty$

means $|f(x)| \leq c g(x)$ for some const. c

4. <u>Gamma distr. with shape para. ν & scalar para. λ</u>

$$f(t) = \begin{cases} 0 , & t \leq 0 \\ \dfrac{\lambda^{\nu}}{\Gamma(\nu)} t^{\nu-1} e^{-\lambda t} , & t > 0 \end{cases}$$

$$F(t) = \begin{cases} 0 , & t \leq 0 \\ \displaystyle\int_0^t f(u)\, du , & t > 0 \end{cases}$$

For $\nu = 1$, this becomes exponential distr.

<u>Ex</u>: Check F is a CDF

<u>Ex</u>: Find an exp. for F when $\nu \in \mathbb{N}$

5. Beta distr. with para. a & b

$$f(t) = \begin{cases} 0 , & t \notin (0,1) \\ \dfrac{t^{a-1}(1-t)^{b-1}}{B(a,b)} , & t \in (0,1) \end{cases}$$

$$F(t) = \begin{cases} 0 , & t \leq 0 \\ \int_0^t f(u)\,du , & t \in (0,1) \\ 1 , & t \geq 1 \end{cases}$$

Beta fx^n

$$B(a,b) = \int_0^1 t^{(a-1)}(1-t)^{(b-1)}\,dt$$

Putting $u = 1-t$ shows $B(a,b) = B(b,a)$

Ppt: $B(a,b) = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

Pf: $\Gamma(a)\Gamma(b) = \int_0^\infty e^{-u} u^{a-1} du \int_0^\infty e^{-v} v^{b-1} dv$

$$= \int_0^\infty \int_0^\infty e^{-(u+v)} u^{a-1} v^{b-1} \, du \, dv$$

Let $u = st \Rightarrow s = u+v$

$v = s(1-t) \qquad t = u/(u+v)$

$$J = \det\begin{pmatrix} u_s & v_s \\ u_t & v_t \end{pmatrix} = \det\begin{pmatrix} t & (1-t) \\ s & -s \end{pmatrix} = -s$$

$|J| = s$

$$\Rightarrow \Gamma(a)\Gamma(b) = \int_0^\infty \int_0^1 e^{-s} s^{a-1} t^{a-1} s^{b-1} (1-t)^{(b-1)} s \, ds \, dt$$

$$= \int_0^\infty e^{-s} s^{(a+b)-1} ds \int_0^1 t^{a-1} (1-t)^{b-1} dt$$

$$= \Gamma(a+b) \, B(a,b)$$

$$\Rightarrow \quad B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

6. Cauchy distr. with para. $\lambda > 0$, $a \in \mathbb{R}$

$$f(t) = \frac{\lambda}{\pi(\lambda^2 + (t-a)^2)}$$

$F(t) =$ Ex?

Given a CDF, does it arise from a PDF?

<u>Necessary cond^n</u>: CDF must be continuous

<u>Necessary & Sufficient cond^n</u>: CDF is cont. & diff. except at finitely many pts.
(by Fundamental Thm of Calculus)

Field: A subset $M \subseteq \mathcal{P}(\Omega)$ is called a field/algebra (on Ω) if

1. $\Omega \in M$
2. $A_1, A_2 \in M \Rightarrow A_1 \cup A_2 \in M$
3. $A_1, A_2 \in M \Rightarrow A_1 \cap A_2 \in M$
4. $A \in M \Rightarrow A^c \in M$

Rem: 1. 1 & 4. $\Rightarrow \emptyset \in M$

2. $A, B \in M \Rightarrow A \setminus B \in M$

σ-Algebra: A subset $M \subseteq \mathcal{P}(\Omega)$ is called a σ-algebra if it is an algebra & given $\{A_n\}_{n \in \mathbb{N}}$, $A_n \in M$, $A = \bigcup_{n=1}^{\infty} A_m \in M$

Rem: $A_n \in M$, $n \in M \Rightarrow A = \bigcap_{n=1}^{\infty} A_m \in M$

Given any $A \subseteq \mathcal{P}(\Omega)$, we can define

$$M_A = \bigcap_{\substack{\Sigma_\alpha : \sigma\text{-alg.} \\ \text{containing } A}} \Sigma_\alpha$$

as the smallest σ-alg. containing A.
($\because$ the intersection of σ-alg. is a σ-alg.)

Ex: If M is σ-alg., show that $|M| \neq \aleph_0$

eg: $\Omega = \mathbb{R}$

$U \subseteq \mathbb{R}$ is open $\Rightarrow$ U is a countable disjoint union of open intervals

T: collection of all open sets in $\mathbb{R}$.

$\mathcal{B}$: smallest σ-alg. containing T (Borel σ-alg.)

(works for $\mathbb{R}^n$. Infact, any topological space.)

<u>Measurable space</u>: (Ω, M) is called a measurable sp.

Elem. in M are called measurable sets.

For $\Omega = \mathbb{R}$, $M = \mathcal{B}$, the elems. of M are called Borel measurable sets.

Finitely Additive Probability Measure:

Let F be an alg. on Ω.

A finitely additive probability measure is a fn $P : F \longrightarrow \mathbb{R}_{\geq 0}$ s.t

$$P(A \cup B) = P(A) + P(B) \quad \text{if } A \cap B = \emptyset$$

In particular, $P(A^C) = P(\Omega) - P(A)$

If we define $P'(A) = \dfrac{P(A)}{P(\Omega)}$, $A \in F$,

we see that $P'(\Omega) = 1$

Clearly, P' is also a finitely additive probability measure.

Hence, we will always assume that $P(\Omega) = 1$ for any fin. add. prob. measure.

<u>Rem</u>: $P(\emptyset) = P(\Omega) - P(\Omega) = 1 - 1 = 0$

More generally, a pre-measure (or a finitely additive measure) μ on alg. F is a fn^n s.t

$$\mu : F \longrightarrow [0,\infty) \cup \{-\infty\}$$

$$\mu(A \cup B) = \mu(A) + \mu(B) \quad \forall\, A, B \in F, \quad A \cap B = \emptyset$$

Convention: $\quad a + \infty = \infty \quad \forall\, a \in [0,\infty] = [0,\infty) \cup \{-\infty\}$

Monotone class: A coll. of subsets M of Ω is called a monotone class if given any sequence of non-increasing (or non-decreasing) sets A_n in M, $\bigcap_{n=1}^{\infty} A_n \in M$ $\left(\text{or } \bigcup_{n=1}^{\infty} A_n \in M \right)$

Ex: The smallest monotone class generated by an algebra is a σ-algebra.

$$I_{a,b} = \begin{cases} (a, b] , & b \neq \infty \\ (a, \infty) , & b = \infty \end{cases} \qquad a \in \mathbb{R} \cup \{-\infty\} , \quad b \in \mathbb{R} \cup \{\infty\}$$

$$\mathcal{G} = \{ I_{a,b} \}$$

Let $\mathcal{F}$: coll. of finite disjoint union of sets in $\mathcal{G}$
& $I_{a,b}^c \cup \{\emptyset\}$

Further, $\mathcal{B}$ is generated by $\mathcal{F}$.

$$\mu(I_{a,b}) = b - a , \quad b \neq \infty$$
$$= \infty , \quad b = \infty$$

A (fin. add) prob. measure on a field/alg. F is said to be <u>countably additive</u> if given any decreasing seq. $A_n \in F$, $n \in \mathbb{N}$ s.t $\bigcap_{n=1}^{\infty} A_n = \emptyset$

then
$$\lim_{n \to \infty} P(A_n) = 0$$

A count. add. prob. measure on a σ-alg. M is a fx^n $P : M \to \mathbb{R}_{\geq 0}$ s.t $A_n \in M$ is a seq. of pairwise disjoint sets then

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

Same definition works for any measure μ.

Ex: Let $\{\omega_n\}_{n=1}^{\infty}$ be a seq. of pts. in Ω.
Let $p_n \in [0,1]$ s.t $\sum_{n=1}^{\infty} p_n = 1$. Def. $P(A) = \sum_{n \mid \omega_n \in A} p_n$
Then P is count. add. prob. measure on $\mathcal{P}(\Omega)$

Ex: Let F be an alg. on Ω. Then F is a σ-alg. iff it is a monotone class i.e. if $A_n \in F$ & A_n is increasing (or decreasing) then $\bigcup_{n=1}^{\infty} A_n \in F$ (or $\bigcap_{n=1}^{\infty} A_n \in F$)

Ex: If $A, B \in F$ a field & P is a fin. add. prob. measure, show that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Ex: For a field F, fin. add. prob. measure P, if $A, B \in F$

$$|P(A) - P(B)| \leq P(A \Delta B) \qquad [A \Delta B = (A \setminus B) \cup (B \setminus A)]$$

If $B \subseteq A$, $\quad 0 \leq P(A) - P(B) = P(A \cap B^c) \leq P(B^c)$

Ex: If P is count. add. on σ-alg. F,

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n)$$

Continuous fin. add. prob. measure :

We will say a fin. add. prob. measure P on an alg. F is continuous at $\emptyset$ if given any non-increasing sequence $A_n \in F$ s.t $\bigcap_{n=1}^{\infty} A_n = \emptyset$ $\left(\lim_{n\to\infty} A_n = \emptyset \right)$ then $\lim_{n\to\infty} P(A_n) = 0$

Continuity of P at $\emptyset$ $\Leftrightarrow$

Suppose A_n is a seq. of pairwise disjoint open sets in F s.t $\bigcup_{n=1}^{\infty} A_n \in F$, then $P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$

Thm: (Caratheodory Extension Thm)
Let P be count. add. prob. measure on an alg. F on Ω. If P is cont. at $\emptyset$, it extends uniquely as a count. add. prob. measure to the smallest σ-alg. containing F. (σ-alg. generated by F)

To construct a cont. add. fin. prob. measure on $\mathcal{B}$, it suffices to construct a fin. add. prob. measure on $\mathcal{I}$.

Where to find fapm?
From CDFs!

$F : \mathbb{R} \to [0,1]$ non-decreasing & $\lim_{x \to \infty} F(x) = 0$
& $\lim_{x \to \infty} F(x) = 1$

$$P_F(I_{a,b}) = F(b) - F(a)$$

$$E = \coprod_{j=1}^{n} I_{a_j, b_j} , \qquad P_F(E) = \sum_{j=1}^{n} P(I_{a_j, b_j})$$

Continuity of P_F at $\emptyset$ follows from right-continuity of F.

So, CDFs give capm.

<u>Thm</u> : (Lebesgue)
The fapm P_F is cont. at $\emptyset$ iff F is a CDF.
Thus every CDF gives rise to a capm on $\mathcal{B}$.
Conversely every capm P on $\mathcal{B}$ arises from some CDF.

Measurable sp : $(\Omega, \mathcal{M})$ ← σ-alg. on Ω

Measure sp : Let $\mu : \mathcal{M} \to [0, \infty]$ be a measure on $\mathcal{M}$ $\left\{ \text{i.e for } \{A_n\}_{n=1}^{\infty} \text{ pairwise disjoint} \quad \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \right\}$

Then, $(\Omega, \mathcal{M}, \mu)$ called a measure sp.

If $\mu(\Omega) = 1$, $(\Omega, \mathcal{M}, \mu)$ is called a probability measure sp.

σ-finite measure : A measure μ on $\mathcal{M}$ is said to be σ-finite if ∃ a seq. $\Omega_n \in \mathcal{M}$ s.t $\mu(\Omega_n) < \infty$ & $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$

Measurable fx^n :

Let $(\Omega, \mathcal{M})$ be a measurable sp.

A fx^n $X : \Omega \to \mathbb{R}$ is said to be (Borel) measurable if $X^{-1}(B) \in \mathcal{M}$ $\forall$ $B \in \mathcal{B}$

If $(\Omega, \mathcal{M}, \mu)$ is a prob. sp., a measurable fx^n is called random variable.

Rem : $X : \Omega \to \mathbb{R}$ is measurable iff $X^{-1}(U) \in \mathcal{M}$ $\forall$ open set $U \in \mathbb{R}$

If $\Omega = \mathbb{R}$ & $\mathcal{M} = \mathcal{B}$, if X is continuous then X is measurable

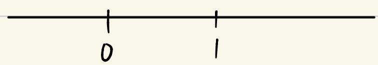
($\because$ pre-image of open sets is open)

Let $A = \mathbb{R} \setminus \mathbb{Q}$, $\mathbb{1}_A = \begin{cases} 1 , & x \in \mathbb{R}\setminus\mathbb{Q} \\ 0 , & \text{otherwise} \end{cases}$

If $E \subseteq \mathcal{B}$, $\mathbb{1}_E$ is measurable

$$\mathbb{1}_E = \begin{cases} 1 , & x \in E \\ 0 , & x \notin E \end{cases}$$

$\because$ for $I_{a,b}$, if



$0, 1 \notin I_{a,b}$, $\mathbb{1}_E^{-1}(I_{a,b}) = \emptyset$

$0, 1 \in I_{a,b}$, $\mathbb{1}_E^{-1}(I_{a,b}) = \mathbb{R}$

$0 \notin I_{a,b}$, $1 \in I_{a,b}$, $\mathbb{1}_E^{-1}(I_{a,b}) = E$

$0 \in I_{a,b}$, $1 \notin I_{a,b}$, $\mathbb{1}_E^{-1}(I_{a,b}) = E'$

So, for $X : \mathbb{R} \longrightarrow \mathbb{R}$

- If X is cont., X is m'ble
- If $X = \mathbb{1}_E$, $E \in \mathcal{B}$, then X is m'ble

Let $f_n : [0,1] \to \mathbb{R}$ be a seq. of fn's.

eg: $f_n(x) = x^n$, $n \in \mathbb{N}$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0 , & x \neq 1 \\ 1 , & x = 1 \end{cases}$$

So, lim. of cont. fn's is <u>not</u> necessarily cont.

However, lim of m'ble fn's is m'ble.

<u>Uniform covengence</u> :

$X \subseteq \mathbb{R}$. $f_n : \mathbb{R} \to \mathbb{R}$.

The seq. f_n is said to be uniformly convergent on X to a fn $f(x)$ if $\forall\, \epsilon > 0$ & $x \in X$, $\exists\, N \in \mathbb{N}$ s.t $|f_n(x) - f(x)| < \epsilon$ whenever $n > N$

<u>Thm</u> : If $f_n \to f$ uniformly on X, and the f_n are continuous, then f is continuous on X.

$$|f(x)-f(y)| = |f(x)-f_n(x)+f_n(x)-f_n(y)+f_n(y)-f(y)|$$
$$\leq \underbrace{|f(x)-f_n(x)|}_{<\epsilon/3} + \underbrace{|f_n(x)-f_n(y)|}_{\epsilon/3} + \underbrace{|f_n(y)-f(y)|}_{\epsilon/3}$$
$$< \epsilon$$

Instead of $f : \Omega \to \mathbb{R}$, we can look at $f : \Omega \to \mathbb{R}^2$

Any open set in $\mathbb{R}^2$ is a countable union of open rectangles : $R = (a,b) \times (c,d)$

So, $f : \Omega \to \mathbb{R}^2$ is measurable iff $f^{-1}(R) \in \mathcal{M}$ $\forall$ open rectangles.

1. For $\Omega = \mathbb{R}$, $\mathcal{M} = \mathcal{B}$, any cont. fxn $f : \mathbb{R} \to \mathbb{R}$ is measurable

2. For $\Omega = \mathbb{R}$, $\mathcal{M} = \mathcal{B}$, $h : \mathbb{R} \to \mathbb{R}$ cont. (or even m'ble)
Suppose $f : \mathbb{R} \to \mathbb{R}$ is m'ble, then $h \circ f$ is m'ble

3. Suppose $f, g : \Omega \to \mathbb{R}$ m'ble.
Then $\varphi : \Omega \to \mathbb{R}^2$ given by $\varphi = (f, g)$ is m'ble

Enough to check $\varphi^{-1}(R) \in \mathcal{M}$ for open rectangles R i.e
need $(f, g)^{-1}((a, b) \times (c, d))$

If $\omega \in f^{-1}((a, b))$, $\omega \in g^{-1}((c, d)) \Rightarrow \varphi(\omega) \in R$

Then $\varphi^{-1}(R) = \underbrace{f^{-1}((a, b))}_{\in \mathcal{M}} \cap \underbrace{g^{-1}((c, d))}_{\in \mathcal{M}} \Rightarrow \varphi$ is m'ble

4. If $\varphi : \Omega \to \mathbb{R}^2$ is m'ble & $h : \mathbb{R}^2 \to \mathbb{R}$ is cont.,
then $h \circ \varphi : \Omega \to \mathbb{R}$ is m'ble

5. If $f, g : \Omega \to \mathbb{R}$ are m'ble, so are $f+g$, $f-g$, $f \cdot g$

$$\Omega \xrightarrow{\varphi} \mathbb{R}^2 \xrightarrow{h} \mathbb{R}$$

$$(x,y) \longmapsto x \pm y, \; xy \qquad (h \text{ is cont.})$$

6. If $u, v : \Omega \to \mathbb{R}$ are m'ble, then so is $f : \Omega \to \mathbb{C}$ given by $u + iv$.

$$\omega \mapsto u(\omega) + iv(\omega)$$

Hence, if $f, g : \Omega \to \mathbb{C}$ are (complex) m'ble fn^{n}s so are $f+g$, $f-g$, $f \cdot g$, $|f|$

7. If $f, g : \Omega \to \mathbb{R}$ are m'ble, so are $\max\{f, g\}$ & $\min\{f, g\}$

$$\max\{f, g\}(x) = \max\{f(x), g(x)\}$$
$$\min\{f, g\}(x) = \min\{f(x), g(x)\}$$

$\because$ $\max\{f, g\} = \dfrac{f+g}{2} + \dfrac{|f-g|}{2}$

8. Suppose $\left.\begin{array}{l} f^+ = \max\{f, 0\} \\ f^- = -\min\{f, 0\} \end{array}\right\} \to$ Both non-negative fxⁿs

If f is m'ble, f^+ & f^- are m'ble (by 7)

Note: $f = f^+ - f^-$
$|f| = f^+ + f^-$

9. If $f_n : \Omega \to \mathbb{R}$ is a seq. of m'ble fxⁿs, then $\sup f_n$ & $\inf f_n$ are m'ble

Let $h = \sup f_n$

$$\begin{aligned} h^{-1}((a,\infty)) &= \{x : h(x) > a\} \\ &= \{x : f_n(x) > a\} \text{ for some } n \in \mathbb{N} \\ &= f_n^{-1}((a,\infty)) \text{ for some } n \\ &= \underbrace{\bigcup f_n^{-1}((a,\infty))}_{\in \mathcal{M}} \end{aligned}$$

$\Rightarrow h^{-1}((a,\infty)) \in \mathcal{M}$

Sim. for $h = \inf f_n$, $h^{-1}((a,\infty)) = \bigcap f_n^{-1}((a,\infty)) \in \mathcal{M}$

lim sup f_n & lim inf f_n are also m'ble.

Hence, $\lim_{n\to\infty} f_n$ (if exists) is m'ble

<u>Simple fx^ns</u> : $(\Omega, \mathcal{M})$ = m'ble sp.

A fx^n $f : \Omega \to \mathbb{R}$ is called a simple fx^n if the range of f is finite.

If $\text{Range}(f) = \{c_1, \dots, c_n\}$

$A_i = f^{-1}(c_i)$

Then, $f = \sum_{i=1}^{n} c_i \mathbb{1}_{A_i}$

We will further assume that f is m'ble, so the A_i will be m'ble set.

$\Rightarrow$ $\mathbb{1}_{A_i}$ are m'ble fx^n

Notice, the A_is are pairwise disjoint.

Suppose $\Omega \neq \coprod_{i=1}^{n} A_i$, let $A_{n+1} = \Omega \setminus \coprod_{i=1}^{n} A_i$

& $c_{n+1} = 0$

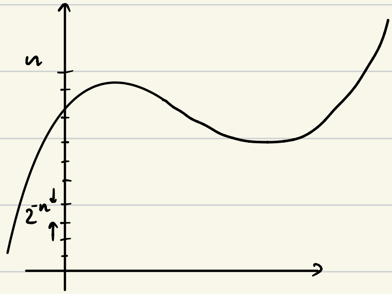
$$\Rightarrow \sum_{i=1}^{n+1} c_i \mathbb{1}_{A_i} = f = \sum_{i=1}^{n} c_i \mathbb{1}_{A_i}$$

So, there is no loss of generality in assuming that $\coprod_{i=1}^{n} A_i = \Omega$

$\underline{\text{Pp}^n}$: Let $f : \Omega \to [0, \infty)$ be a m'ble fx^n
Then $\exists$ a seq. $0 \leq s_1 \leq s_2 \ldots \leq s_n \leq \ldots \leq f$ of simple fx^ns s.t $\lim_{n \to \infty} s_n(x) = f(x)$

$\underline{\text{Pf}}$: Each $y \in [0, n)$ lies in an interval

$$[k_{n,y} \cdot 2^{-n}, (k_{n,y}+1) \cdot 2^{-n})$$



$0 \leq k_{n,y} < 2^n \cdot n$
(integer)

$$s_n(x) = \begin{cases} k_{n,y} \cdot 2^{-n}, & \text{if } f(x) = y,\ y < n \\ n, & \text{if } f(x) = y \geq n \end{cases}$$

Notice, $\quad s_n(x) \leq s_{n+1}(x) \quad \forall\, n \in \mathbb{N}$

If $f(x) = y$, then $\forall\, n > y$,

$$|f(x) - s_n(x)| \leq 2^{-n}$$

This shows $\quad \lim_{n \to \infty} s_n(x) = f(x)$

<u>Rem</u>: If f is bounded, then $s_n(x) \to f$ uniformly.

Let $s = \sum_{i=1}^{n} c_i \mathbb{1}_{A_i}$ be a simple fx^n,

$$\int_\Omega s\ \mu = \int_\Omega s\, d\mu := \sum_{i=1}^{n} c_i\, \mu(A_i)$$

$$\int_E s\, d\mu = \sum_{i=1}^{n} c_i\, \mu(E \cap A_i)$$

If $f : \Omega \to [0, \infty)$ is m'ble, def.

$$\int_E f \, d\mu = \sup_{s \leq f} \int_E s \, d\mu$$

This is the Lebesgue integral of f on E wrt the measure μ.

<u>Rem</u> : $$\int_E f \, d\mu = \int_\Omega \mathbb{1}_E f \, d\mu$$

Thm : (Lebesgue Monotone Convergence Thm)

Suppose $f_n : \Omega \to [0, \infty)$ be a seq. of m'ble fn's s.t

1. $0 \leq f_1(x) \leq f_2(x) \ldots \leq f_n(x) \quad \forall\, x \in \Omega$
2. $\lim\limits_{n\to\infty} f_n(x)$ exists $\forall\, x \in \Omega$

   Call this lim $f(x)$

Then $$\lim_{n\to\infty} \int_\Omega f_n \, d\mu = \int_\Omega f \, d\mu$$

If $f : \Omega \to \mathbb{R}$ is m'ble, write $f = f^+ - f^-$

Def. $$\int_E f = \int_E f^+ - \int_E f^-$$

If $f = u + iv$ is a complex m'ble fn, we def

$$\int_E f \, d\mu := \int_E u \, d\mu + i \int_E v \, d\mu$$

Integrable fn^n:

We will say that a m'ble fn^n $f : \Omega \to \mathbb{C}$ is in $\underline{L^1(\Omega, \mu)}$ if $\int_\Omega |f| \, d\mu < \infty$

Such fn^{n}s are also called Lebesgue integrable fn^{n}s.

If $f \in L^1(\Omega, \mu)$, $\int f \, d\mu$ is defined as before (& is a 'good' definition: $\left| \int_\Omega f \, d\mu \right| \leq \int_\Omega |f| \, d\mu$)

Thm: If $f, g \in L^1(\Omega, \mu)$ & $a, b \in \mathbb{C}$, then

$$\int_\Omega af + bg \, d\mu = a \int_\Omega f \, d\mu + b \int_\Omega g \, d\mu$$

Rem: $f \to \int_\Omega f \, d\mu$ is a linear functional on $L^1(\Omega, \mu)$

<u>Thm</u> : (Lebesgue Dominated Convergence Thm)

Suppose $f_n : \Omega \to \mathbb{C}$ be a seq. of m'ble fn's s.t $\exists\ g \in L^1(\Omega, \mu)$ so that $|f_n(x)| \leq g(x)$ $\forall\ n \in \mathbb{Z}_{>0}$, $x \in \Omega$ and $\lim_{n\to\infty} f_n(x) = f(x)$ exists, then $\lim_{n\to\infty} \int_\Omega f_n \, d\mu = \int_\Omega f \, d\mu$

Fn^{n}s of (continuous) random variable

(Ω, M, P) : prob. measure sp.

$X : \Omega \to \mathbb{R}$ random var.

$g : \mathbb{R} \to \mathbb{R}$ cont. (or even m'ble)

$Y = g \circ X$ is a random var.

Can we determine the CDF/PDF of Y if we know them for X?

Continuous random var:

We will say that a random var X with CDF F_X is of cont. type, if F_X is absolutely cont.

i.e $\exists$ a non-negative fn^n $f(t)$ $[= f_X(t)]$ s.t

$$F(x) = \int_{-\infty}^{t} f(t)\, dt$$

Rem: Often $f_X(t)$ will be cont., so $F_X(t)$ will not only be cont. (not just right-cont.) but diff

Given f_X, determine f_Y where $Y = g \circ X$

Thm: Let X be a ran. var. with a cont. PDF f_X. Let $g: \mathbb{R} \to \mathbb{R}$ be a diff. fn^n with $g'(x) > 0 \ \forall x \in \mathbb{R}$ or $g'(x) < 0 \ \forall x \in \mathbb{R}$.

Let $a = \min\{g(-\infty), g(\infty)\}$ (possibly $-\infty$)
$b = \max\{g(-\infty), g(\infty)\}$ (possibly ∞)

Assume $f_X(g^{-1}(a)) = f_X(g^{-1}(b)) = 0$ if $a, b \in \mathbb{R}$

Then
$$f_Y = \begin{cases} f_X(g^{-1}(y)) \, |[g^{-1}(y)]'| , & y \in (a,b) \\ 0 , & y \notin (a,b) \end{cases}$$

$\because$ $g'(x) > 0$ or $g'(x) < 0$, it is strictly inc. or strictly dec., hence $g : \mathbb{R} \to (a, b)$ is bij. map.

$$F_Y(y) = P(Y^{-1}(-\infty, y]) = \int_a^y f_Y(t)\, dt$$

Suppose $g'(x) > 0 \ \forall x \in \mathbb{R}$.

$$F_Y(y) = \int_a^y f_Y(t)\ dt = \int_{-\infty}^{g^{-1}(y)} f_X(u)\ du = F_X(g^{-1}(y))$$

$$f_Y(y) = F_Y'(y) = f_X(g^{-1}(y))\,[g^{-1}(y)]'$$

Suppose $g'(x) < 0 \ \forall x \in \mathbb{R}$

$$\begin{aligned} F_Y(y) = P(Y^{-1}(-\infty, y)) &= P(X^{-1}(g^{-1}(y), \infty)) \\ &= 1 - P(X^{-1}(-\infty, g^{-1}(y))) \\ &= 1 - F_X(g^{-1}(y)) \end{aligned}$$

$$f_Y(y) = -f_X(g^{-1}(y))\,[g^{-1}(y)]' = f_X(g^{-1}(y))\,\left|[g^{-1}(y)]'\right|$$

$y \notin [a,b]$, $g^{-1}(y) = \emptyset$ so $Y^{-1}(y) = X^{-1}(g^{-1}(y)) = \emptyset$

$P(Y^{-1}(y)) = 0 \Rightarrow f_Y(y) = 0$

<u>Rem</u>: If f_X vanishes outside $[c,d]$, we can take $a = \min\{g(c), g(d)\}$, $b = \max\{g(c), g(d)\}$.
Enough to assume condⁿs on g in the interval (c,d).

<u>eg</u>: <u>1.</u> $X \sim \mathrm{Uni}(0,1)$, $Y = e^X$ $\quad (g(x) = e^x)$

$X = \mathbb{1}_{(0,1)}$ & f_X is the uniform distribution in $(0,1)$

$g^{-1}(x) = \log(x)$, $\quad c = 0,\ d = 1$

$\qquad a = 1,\ b = e$

$$f_Y(y) = \underbrace{f_X(g^{-1}(y))}_{1}\ \underbrace{|[g^{-1}(y)]'|}_{\frac{1}{|y|}} = \begin{cases} \left|\frac{1}{y}\right| , & y \in (1,e) \\ 0 , & \text{otherwise} \end{cases}$$

2. $X \sim N(0,1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad , \qquad g(x) = x^2$$

$$Y = X^2$$

$$P^{-1}(Y^{-1}(-\infty, y)) = P(x : X(x) \in [-\sqrt{y}, \sqrt{y}])$$
$$= F(\sqrt{y}) - F(-\sqrt{y})$$

Diff. ,

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-y/2} \quad , & y > 0 \\ 0 \quad , & y \leq 0 \end{cases}$$

Let $(\Omega, \mathcal{M}, P)$ be a prob. measure sp.
$X : \Omega \to \mathbb{R}$ be a random var.

Let $X \in \mathcal{L}^1$ $[\equiv \mathcal{L}^1(P) \equiv \mathcal{L}^1(\Omega, P)]$

i.e $\displaystyle\int_\Omega |X(\omega)| \, dP < \infty$

we define the expectation or mean $E[X]$ of X
to be $E[X] = \displaystyle\int_\Omega X \, dP$ (also called μ)

If $E[|X|]$ is <u>not</u> finite / does <u>not</u> exist, then
we say $E[X]$ does <u>not</u> exist.

Given $X : \Omega \to \mathbb{R}$, we can define the pushforward
measure on $\mathbb{R}$ by

$$\mu(I_{a,b}) = P(X^{-1}(I_{a,b}))$$

If we take $a = -\infty$, $b = x$, we get $F(x)$.

If X is continuous, $\quad F_X(x) = \int_{-\infty}^{x} f_X(u) \, dm(u)$

We have equipped $(\mathbb{R}, \mathcal{B})$ with the measure
$\mu(u) = f_X(u) \, dm(u)$

$$\int_{\mathbb{R}} \varphi(u) \, \mu(u) = \int_{\mathbb{R}} \varphi(u) \, f_X(u) \, dm(u)$$

for m'ble fn's φ

Note if X is of cont. type

$$E[X] = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$$

eg: Cauchy distr.

$$f_X(x) \sim \frac{1}{x^2+\lambda^2}$$

$$\int_{-\infty}^{\infty} x f_X(x)\, dx = \lim_{a \to \infty} \int_{-a}^{a} x f_X(x)\, dx$$

Clearly, $E[|X|] = \displaystyle\int_{-\infty}^{\infty} \frac{|x|}{x^2+\lambda^2}\, dx = \infty$

Need $E[|X|] = \displaystyle\int_{-\infty}^{\infty} |x| f_X(x)\, dx < \infty$ for $E[X]$ to exist

Thm : Let $X : \Omega \to \mathbb{R}$ be a random var. of cont. type in $L^1(\Omega, P)$. Let $g : \mathbb{R} \to \mathbb{R}$ be a diff. fx^n & assume that X & g satisfy the cond^{n}s of Thm 1.13.2. Let $Y = g(X)$, then

$$E[Y] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

Pf : $$\int_{-\infty}^{\infty} g(x) f_X(x)\, dx = \int_a^b y \underbrace{f_X(g^{-1}(y))\,[g^{-1}(y)]'}_{f_Y(y)}\, dy$$

$$= E[Y]$$

Rem : If $g(x) = x^n$, n even, can modify the argument suitably to get a similar formula (by restricting g to subsets of domain where $g'(x) > 0$ or $g'(x) < 0$)

<u>Moment</u>: If $E[X^n]$ exists, it is called the n^{th} moment of X about the origin.

If $E[|X|^\alpha]$ exists, we call it the α^{th} absolute moment of X about the origin.

<u>Not^n</u>: $m_n = E[X^n]$

$\beta_\alpha = E[|X|^\alpha]$

Note, $E[|X|^p]$ exists means $\int_\Omega |X|^p \, dP < \infty$

In measure theory, m'ble fx^n f s.t $|f|^p \in L^1$ are called L^p-fx^{n}s $\Leftrightarrow$ $f \in L^p(\Omega, P)$ for $p \in [0, \infty)$

Thm: If $E[|X|^p]$ exists for some $p>0$, then $E[|X|^q]$ exists $\forall\ 0<q<p$

Pf: Let $E = \{x : |X(x)| < 1\}$

$$\Omega = E \amalg E'$$

$$\int_\Omega |X|^q \, dP = \int_E \underbrace{|X|^q}_{\leq 1} dP + \int_{E'} \underbrace{|X|^q}_{\geq 1} dP$$

$$\leq \mu(E) + \int_{E'} |X|^p \, dP$$

$$\leq \mu(E) + E[|X|^p] < \infty$$

For a finite measure sp. say (Z, M), $\mathcal{L}^p(M) \subseteq \mathcal{L}^q(M)$ if $q \leq p$

Suppose $E[|X|^p]$ exists.

Let $A_n = \{x : |X(x)| \leq n\}$.

Let $B_n = A_n' = \{x : |X(x)| > n\}$

$$n^p P(B_n) \leq \int_{B_n} |X|^p \, dP = \int_{\Omega} |X|^p \, dP - \int_{A_n} |X|^p \, dP$$

Note, $\displaystyle\int_{A_n} |X|^p \, dP = \int_{\Omega} \mathbb{1}_{A_n} |X|^p \, dP$

Note, $\mathbb{1}_{A_n} |X|^p \leq \mathbb{1}_{A_{n+1}} |X|^p \quad \forall \; n$

If $x \in \Omega$, $X(x) \leq N$ for $N \in \mathbb{Z}_{>0}$

Clearly, $\displaystyle\lim_{n\to\infty} \mathbb{1}_{A_n} |X|^p = |X|^p$

By MCT, $\displaystyle\lim_{n\to\infty} \int_{\Omega} \mathbb{1}_{A_n} |X|^p \, dP = \int_{\Omega} |X|^p \, dP$

$$\Rightarrow \quad \lim_{n\to\infty} n^p P(B_n) = 0$$

The probabilities $P(B_n)$ is called the <u>tail prob.</u> of the ran. var X.

eg: Pareto distr. $(\alpha, \beta > 0)$

$$f(x) = \begin{cases} \dfrac{\beta \alpha^\beta}{x^{\beta+1}} , & x \geq \alpha \\ 0 , & x < \alpha \end{cases}$$

For what n does $E[X^n]$ exists ?

$$E[X^n] = \int_{-\infty}^{\infty} \frac{x^n \beta \alpha^\beta}{x^{\beta+1}} \, dx$$

$$\beta + 1 - n > 1 \Rightarrow \underline{n < \beta}$$

If $\beta > 2$, calculate

1. $E[X] = \mu$
2. $E[(X-\mu)^2] = Var(X)$

$$E[X] = \int_{\alpha}^{\infty} x \frac{\beta \alpha^{\beta}}{x^{\beta+1}} \, dx = \int_{\alpha}^{\infty} \frac{\beta\alpha^{\beta}}{x^{\beta}} \, dx = \beta\alpha^{\beta} \left[\frac{x^{1-\beta}}{1-\beta} \right]_{\alpha}^{\infty}$$

$$= \frac{\beta\alpha}{\beta-1}$$

$$E[(X-\mu)^2] = \int_{\alpha}^{\infty} \frac{(x-\mu)^2 \beta\alpha^{\beta}}{x^{\beta+1}} \, dx$$

Thm : Let $g : \mathbb{R} \to [0, \infty)$ be m'ble.
If $E[X]$ exists, then

$$P(\{x : g(x) \geq \epsilon\}) \leq \frac{E[g(x)]}{\epsilon} \quad \forall \; \epsilon > 0$$

Pf : Let $E_\epsilon = \{x : g(x) \geqslant \epsilon\}$

$$\epsilon \cdot P(E_\epsilon) \leqslant \int_{E_\epsilon} g(x)\, dP \leqslant \int_{\Omega} g(x)\, dP = E[g(x)]$$

Cor : (Markov's Inequality)

$g(x) = |x|^\alpha$, $\epsilon = K^2$, $K > 0$

$$P(\{x : |X| > K\}) \leqslant \frac{E[|X|^2]}{K^\alpha}$$

Cor : (Chebychyev's Inequality)

$\epsilon = k^2\sigma^2$ where $\sigma^2 = Var(X)$

$$P(\{x : |X(x) - \mu| > k\sigma\}) \leqslant \frac{E[(X-\mu)^2]}{k^2\sigma^2} = \frac{1}{k^2}$$

$(\Omega, \mathcal{M}, m)$ measure sp.

$f : \Omega \to \mathbb{C}$ m'ble

Def. $$\|f\|_p = \left[\int_\Omega |f|^p \, dm \right]^{1/p}, \quad 0 < p < \infty$$ if it exists.

On the space of m'ble fn^n's s.t $\|f\|_p < \infty$, we introduce the eq. rel^n $f \sim g$ iff $f(x) = g(x)$ almost everywhere i.e $\exists$ a set of measure 0 say $E \, (= E(f, g))$ s.t $f(x) = g(x)$ $\forall \, x \in E'$

The space $L^p(\Omega, m)$ is the set of equivalence classes of m'ble fn^n's & $\| \ \|_p$ is a norm on $L^p(\Omega, m)$

(need to show the triangle inequality)

Thm: (Holder's inequality)

$$\int_\Omega fg \, dm \leq \left[\int_\Omega |f|^p \, dm\right]^{1/p} \left[\int_\Omega |g|^q \, dm\right]^{1/q}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ $(p>1)$ $\forall$ m'ble $f, g : \Omega \to \mathbb{C}$

If $f \in \mathcal{L}^p(m)$, $g \in \mathcal{L}^q(m)$, then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Special case: (Cauchy-Schwartz inequality)

$p = q = 2$, $$\int_\Omega fg \, dm \leq \left[\int_\Omega |f|^2 \, dm\right]^{1/2} \left[\int_\Omega |g|^2 \, dm\right]^{1/2}$$

Rem: $\mathcal{L}^2(\Omega, m)$ is an inner product space.

$\mathcal{L}^p(\Omega, m)$ is a complete normed linear space.

(Banach space)

Complete inner product spaces are called Hilbert spaces.

Let $f: \Omega \to \mathbb{C}$ be m'ble. We say that $a \in \mathbb{R}$ is an <u>essential bound</u> for f if $m(|f|^{-1}(a,\infty)) = 0$

Let U^{ess} be the set of essential bounds of f.

Def. $\|f\|_\infty = \inf U_f^{ess}$

On the space of m'ble fn's, $f: \Omega \to \mathbb{C}$ we set $f \sim g$ iff $f(x) = g(x)$ almost everywhere.

This makes the space of essentially bounded fn's (i.e those for the essential supremum $\|f\|_\infty$ is finite) into a (complete) normed linear space.

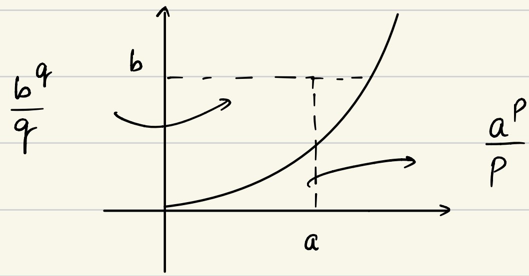
If $p > 1$ & q s.t $\frac{1}{p} + \frac{1}{q} = 1$, p & q are called conjugate exponents.

When $p = 1$ we set $q = \infty$

Pf: Assume that $p \in (1,\infty)$ & $\|f\|_p \cdot \|g\|_q < \infty$
(the other cases are easy)

Young's inequality: $\quad ab \le \frac{a^p}{p} + \frac{b^q}{q} \quad$ if $\frac{1}{p} + \frac{1}{q} = 1$,

$a, b \in [0,\infty)$



$y = x^{p-1}$

$$f(x) = x^{p-1} \Rightarrow f^{-1}(x) = x^{\frac{1}{p-1}} = x^{q-1}$$

Choose $\quad a = \frac{|f(x)|}{\|f\|_p}, \quad b = \frac{|g(x)|}{\|g\|_q}$

By Young's ineq.,
$$\frac{|f(x)||g(x)|}{\|f\|_p \|g\|_q} \le \frac{|f(x)|^p}{p\|f\|_p^p} + \frac{|g(x)|^q}{q\|g\|_q^q}$$

Integrating both sides,

$$\int_{\Omega} |f(x)g(x)|\, dm \leq \|f\|_p \|g\|_q \left(\frac{1}{p} + \frac{1}{q}\right) = \|f\|_p \|g\|_q$$

Ex: Show that equality holds only if $\exists\, a, b \in \mathbb{R}$ s.t $a|f|^p + b|g|^q = 0$ almost everywhere.

Cor: (Minkowski's inequality)

If $f, g \in \mathcal{L}^p(\Omega, m)$, $p \in [1, \infty)$, $f+g \in \mathcal{L}^p(\Omega, m)$ & $\|f+g\|_p \leq \|f\|_p + \|g\|_p$

Pf: If $p > 1$, x^p is a convex fn.

$$|f(x) + g(x)|^p = 2^p \left| \frac{f(x)}{2} + \frac{g(x)}{2} \right|^p$$

$$\leq 2^p \left[\frac{|f(x)|^p}{2} + \frac{|g(x)|^p}{2} \right]$$

$$\leq 2^{p-1} \left[|f(x)|^p + |g(x)|^p \right]$$

$\Rightarrow f+g \in \mathcal{L}^p$

$$\|f+g\|_p^p = \int_\Omega |f+g|^p \, dm$$

$$\Rightarrow \int_\Omega |f+g|^{p-1} |f+g| \, dm \leq \int_\Omega |f+g|^{p-1} |f| \, dm + \int_\Omega |f+g|^{p-1} |g| \, dm$$

$$\leq \|f\|_p \|f+g\|_p^{p-1} + \|g\|_p \|f+g\|_p^{p-1}$$

$$\Rightarrow \quad \|f+g\|_p \leq \|f\|_p + \|g\|_p$$

Hence, $L^p(\Omega, m)$ is a normed linear space.

<u>Cor</u>: (Lyapunov's inequality)

If $1 \leq s \leq t$ & $f \in L^t(\Omega, P)$, then $\|f\|_s \leq \|f\|_t$

<u>Pf</u>: $(\Omega, \mathcal{M}, P)$ - prob. measure sp.

$$\int_\Omega |f|^s \, dP = \int_\Omega |f|^s \cdot 1 \, dP$$

$f = |f|^s$, $g = 1$, $p = t/s$, $q = t/t-s$

By Holder's inequality, $\int_\Omega |f|^s dP \leq \|f\|_t^s \cdot 1$

$$\Rightarrow \left[\int_\Omega |f|^s dP \right]^{1/s} \leq \|f\|_t$$

$$\Rightarrow \|f\|_s \leq \|f\|_t$$

i.e. $E[|X|^s]^{1/s} \leq E[|X|^t]^{1/t}$

$(\Omega, \mathcal{M}, P)$: prob. measure sp.

If $f : \Omega \to \mathbb{R}^n$. Then $f = (f_1, \dots, f_n)$

$f_i : \Omega \to \mathbb{R}$, $1 \leq i \leq n$

If $X : \Omega \to \mathbb{R}^n$ is a random var. (i.e m'ble fn^n)

iff $X_1, \dots, X_n : \Omega \to \mathbb{R}$ are ran. var.

Let $X : \Omega \to \mathbb{R}^n$ be a ran. var.

$$F(x) = F(x_1, x_2, \dots, x_n)$$
$$= P(X^{-1}((-\infty, x_1] \times (-\infty, x_2] \times \dots \times (-\infty, x_n]))$$

is called the distribution function of X.

We see easily that F satisfies the following ppts :

D1) $F(x) \geqslant 0 \quad \forall\, x = (x_1, \dots, x_n) \in \mathbb{R}^n$

D2) F is strictly increasing & right-cont.
in each coordinate

D3) $\lim_{x_1,\dots,x_n \to \infty} F(x) = 1$, $\lim_{\substack{x_i \to -\infty \\ \exists\, 1 \le i \le n}} F(x) = 0$

D4) (n-increasing ppt.)

$n=2$, $x_2 \geq x_1$, $y_2 \geq y_1$

$$F(x_2, y_2) - F(x_2, y_1) + F(x_1, y_1) - F(x_1, y_2) \geq 0$$

Generalised in n-variables

If $f : \Omega \to \mathbb{R}$ satisfying D1 to D4, then it is the distribution fx^n of some ran. var X.

eg: Let

$$F(x, y) = \begin{cases} 1, & x+y \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

Take $(x_1, y_1) = (1/3, 1/3)$ & $(x_2, y_2) = (1, 1)$

$$F(1,1) - F(1, 1/3) + F(1/3, 1/3) - F(1/3, 1) = 1 - 1 + 0 - 1$$
$$= -1 < 0$$

f does <u>not</u> satisfy D4.

Note, if X is a discrete ran. var. i.e each of the X_i's are discrete ran. var. $(1 \le i \le n)$, then one can define the pmf of X by

$$p_{i_1, \dots, i_n} = P(X^{-1}(x_{i_1}, \dots, x_{i_n})) \quad \text{where} \quad x_{i_k} \in X_{i_k}(\Omega)$$

$$\sum_{i_1, \dots, i_n} p_{i_1, \dots, i_n} = 1$$

Joint PDF:

Let $X = (X_1, \dots, X_n)$ be a n-dim. ran. var.
We will say that X is of cont. type if

$$F(x_1, \dots, x_n) = \int_{-\infty}^{x_n} \int_{-\infty}^{x_{n-1}} \cdots \int_{-\infty}^{x_1} f(u_1, \dots, u_n) \, du_1 \dots du_n$$

for some fn^n $f: \Omega \to [0, \infty)$
where F is distr fn^n of X.

The fn^n f is called the joint-PDF of X.

Any non-negative fn^n g s.t $\int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} g(u_1, \dots, u_n)\, du_1 \dots du_n = 1$

is the joint PDF of a n-dim ran. var.

Marginal PDF:

Let $X : \Omega \to \mathbb{R}^2$ be a ran. var. of cont. type with PDF $f(x_1, x_2)$. $(X = (X_1, X_2))$

We can define the marginal PDF of X_1 & X_2 as

$$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_2$$

$$f_2(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1$$

By Fubini's thm, f_1 & f_2 are PDFs.

eg: $X = (X_1, X_2)$ is ran. var with joint PDF

$$f(x_1, x_2) = \begin{cases} 2 \ , & \text{if } 0 < x_1 < x_2 < 1 \\ 0 \ , & \text{otherwise} \end{cases}$$

$$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_2 = \int_{x_1}^{1} 2\, dx_2 = 2(1 - x_1)$$

$(x_1 > 0)$

$$f_1(x_1) = \begin{cases} 2(1 - x_1) \ , & 0 < x_1 < 1 \\ 0 \ , & \text{otherwise} \end{cases}$$

$$f_2(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1 = \begin{cases} \displaystyle\int_{0}^{x_2} 2\, dx_1 = 2x_2 \ , & 0 < x_2 < 1 \\ 0 \ , & \text{otherwise} \end{cases}$$

Marginal CDF :

Let $X = (X_1, X_2)$ be a ran. var. with F as distr. fx^n

The marginal distr. of the X_i, $i = 1, 2$ is defined to be $F_1(x_1) = \lim_{x_2 \to \infty} F(x_1, x_2)$ & $F_2(x_2) = \lim_{x_1 \to \infty} F(x_1, x_2)$

If F is of cont. type.

$$F_1(x_1) = \int_{-\infty}^{x_1} f_1(t)\, dt \quad \& \quad F_2(x_2) = \int_{-\infty}^{x_2} f_2(t)\, dt$$

Conditional distr. fx^n:

The conditional distr. fx^n of a ran. var. X_1 given $X_2(x) = x_2$ is defined as

$$F_{X_1|X_2}(x \mid x_2) = \lim_{\epsilon \to 0^+} \frac{P(X_1^{-1}((-\infty, x]) \cap X_2^{-1}((x_2-\epsilon, x_2+\epsilon)))}{P(X_2^{-1}((x_2-\epsilon, x_2+\epsilon)))}$$

if it exists.

If X is of cont. type with PDF f, f is cont. at (x_1, x_2), f_2 is a cont. fx^n with $f_2(x_2) > 0$

$$f_{X_1|X_2}(x|x_2) = \lim_{\epsilon \to 0^+} \frac{P(X^{-1}((-\infty, x]) \cap X_2^{-1}((x_2-\epsilon, x_2+\epsilon)))}{P(X_2^{-1}((x_2-\epsilon, x_2+\epsilon)))}$$

$$= \lim_{\epsilon \to 0^+} \frac{\int_{-\infty}^{x} \int_{x_2-\epsilon}^{x_2+\epsilon} f(u_1, u_2)\, du_1\, du_2}{\int_{x_2-\epsilon}^{x_2+\epsilon} f(u_2)\, du_2}$$

$$= \lim_{\epsilon \to 0^+} \frac{\int_{-\infty}^{x} \int_{x_2-\epsilon}^{x_2+\epsilon} f(u_1, u_2)\, du_1\, du_2 / 2\epsilon}{\int_{x_2-\epsilon}^{x_2+\epsilon} f(u_2)\, du_2 / 2\epsilon}$$

$$= \frac{\int_{-\infty}^{x_1} f(u, x_2)\, du}{f_2(x_2)} = \int_{-\infty}^{x_1} \frac{f(u, x_2)}{f_2(x_2)}\, du$$

Hence the PDF $f_{X_1|X_2}(x_1, x_2) = \dfrac{f(x_1, x_2)}{f_2(x_2)}$

n-dim random var $X = (X_1, \dots, X_n)$

So far $X_i : \Omega \to \mathbb{R}$ for fixed Ω.

However, we can take $X_i : \Omega_i \to \mathbb{R}$

$X_1(\omega_1) \in (-\infty, x_1]$ & $X_2(\omega_2) \in (-\infty, x_2]$

We can take $X^{-1}((-\infty, x_1] \times (-\infty, x_2])$

$X_1^{-1}((-\infty, x_1]) \times X_2^{-1}((-\infty, x_2])$

If there is a measure on $\Omega_1 \times \Omega_2$, say P, then we can define

$$F(x_1, x_2) = P(X_1^{-1}((-\infty, x_1]) \times X_2^{-1}((-\infty, x_2]))$$

Q. Given 2 measure sp. $(\Omega_1, \mathcal{M}_1, m_1)$ & $(\Omega_2, \mathcal{M}_2, m_2)$, can one construct a measure on $\Omega_1 \times \Omega_2$?

Pf. Given $A_1 \in \mathcal{M}_1$, $A_2 \in \mathcal{M}_2$, the set $A_1 \times A_2$ $(\subseteq \Omega_1 \times \Omega_2)$ is called a m'ble rectangle.

Let $\mathcal{F}$ be the algebra on $\Omega_1 \times \Omega_2$ generated by all m'ble rectangles.

Def. a pre-measure m on $\mathcal{F}$ by

$$m(A_1 \times A_2) = m_1(A_1) \cdot m_2(A_2)$$

If E is a finite disjoint union of rectangles,
i.e $E = \coprod_{i=1}^{n} R_i$

Def. $$m(E) = \sum_{i=1}^{n} m(R_i)$$

<u>Ex</u>: Check that m is well-defined on $\mathcal{F}$
i.e if $E = \coprod_{i=1}^{n} R_i = \coprod_{j=1}^{m} R_j'$, then

$$\sum_{i=1}^{n} m(R_i) = \sum_{j=1}^{m} m(R_j')$$

Thus, given m_1, m_2 on Ω_1 & Ω_2 we have defined a pre-measure on $\mathcal{F}$. We want a measure on the σ-algebra $\mathcal{M}$ generated by $\mathcal{F}$.

Caratheodory Extension Thm says that we need to check that m is cont. at $\emptyset$.

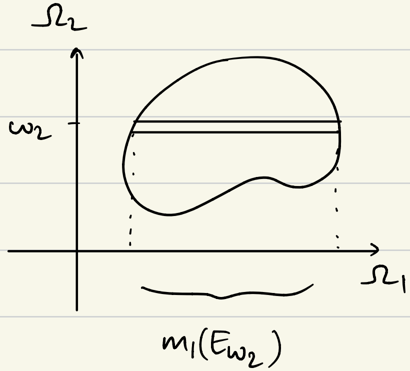
<u>Lem</u>: The pre-measure m on $\mathcal{F}$ is cont. at $\emptyset$.

<u>Pf</u>: Let $E \in \mathcal{F}$. We define

$$E_{\omega_2} = \{ \omega_1 \in \Omega_1 : (\omega_1, \omega_2) \in E \} \quad \text{for } \omega_2 \in \Omega_2$$

Note that as a fx^n of ω_2, $m_1(E_{\omega_2})$ is a m'ble fx^n.
In fact, it is a simple fx^n.

$$m(E) = \int_{\Omega_2} m_1(E_{\omega_2}) \, dm_2$$



Suppose E_n is a non-increasing seq. in $\mathcal{F}$ s.t $\bigcap_{n=1}^{\infty} E_n = \emptyset$

($E_n \to \emptyset$ as $\lim_{n\to\infty} E_n = \emptyset$)

Need to show $\lim_{n\to\infty} m(E_n) = 0$

$$m(E_n) = \int_{\Omega_2} m_1(E_{n,\omega_2})\ dm_2$$

$\because$ $E_n \to \emptyset$, $E_{n,\omega_2} \to \emptyset$

Hence, $m_1(E_{n,\omega_2}) \to 0$ for every $\omega_2 \in \Omega_2$

Assume $m_1(E_1) < \infty$, $m_1(E_{n,\omega_2}) < \infty$

$\Rightarrow m_1(E_{n,\omega_2}) < m_1(E_{1,\omega_2}) \quad \forall n$

By DCT, $m(E_n) \to 0$

Assume if necessary, that $m_1(\Omega_1) < \infty$ & $m_2(\Omega_2) < \infty$ to simplify.

This shows m extends to a measure on $\mathcal{M}$.
This extension will be unique if m_1, m_2 are finite measures.

Let $(\Omega_i, \mathcal{M}_i, m_i)$, $i=1,2$ be measure sp. & let $(\Omega, \mathcal{M}, m)$ be the prod. measure sp.

Let $f: \Omega_1 \times \Omega_2 = \Omega \to \mathbb{C}$ be m'ble

Def. $g_{\omega_1}: \Omega_2 \to \mathbb{C}$ & $h_{\omega_2}: \Omega_1 \to \mathbb{C}$

$$g_{\omega_1}(\omega_2) = h_{\omega_2}(\omega_1) = f(\omega_1, \omega_2)$$

Then, g_{ω_1} (resp. h_{ω_2}) is m'ble fxn of ω_2 (resp. ω_1).
If $f \in \mathcal{L}^1(\Omega, m)$, then g_{ω_1} (resp. h_{ω_2}) is integrable for almost all ω_1 (resp. ω_2)

Further,
$$G(\omega_1) = \int_{\Omega_2} g_{\omega_1} \, dm_2 \quad \&$$
$$H(\omega_2) = \int_{\Omega_1} h_{\omega_2} \, dm_1$$

are m'ble fxn of ω_1 & ω_2 resp., finite a.e & in $\mathcal{L}^1(m_1)$ & $\mathcal{L}^2(m_2)$ resp.

Finally,
$$\int_{\Omega} f \, dm = \int_{\Omega_1} G \, dm_1 = \int_{\Omega_2} H \, dm_2$$

Conversely, if $f: \Omega \to [0,\infty)$ is a m'ble fn & either G or H is in $L^1(m_1)$ or $L^1(m_2)$, then the above equality holds.

This is called <u>Fubini's thm.</u>

<u>Independence</u>

Let $X_1, \dots, X_n$ be ran. var. We will say that they are mutually or completely indep. if

$$f_X(x_1, \dots, x_n) = f_{X_1}(x_1) \dots f_{X_n}(x_n)$$
$$f(x_1, \dots, x_n) = f_1(x_1) \dots f_n(x_n)$$

If $n=2$, we usually say X_1, X_2 are pairwise indep. (or just indep.)

In the language of measure theory, X_1 & X_2 are indep. if the pushfwd. measure on $\mathbb{R}^2$ induced by $X = (X_1, X_2)$ is the product measure $m_1 \times m_2$ where m_i is the pushfwd. measure of X_i on $\mathbb{R}$

Ppn : If X_1, X_2 are of continuous type, they are indep. iff $f(x_1, x_2) = f_1(x_1) f_2(x_2)$

Cor : If X_1, X_2 are indep. ran. var., then

$$f_{X_2|X_1}(x_2 | x_1) = f_2(x_2)$$

Thm : If $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are (Borel) m'ble & if X_1 & X_2 are indep. then $f_1(X_1)$ & $f_2(X_2)$ are indep.

Pf : $F_{f_1(X_1), f_2(X_2)}(x_1, x_2) = P(X_1^{-1}(f_1^{-1}((-\infty, x_1])) \cap X_2^{-1}(f_2^{-1}((-\infty, x_2])))$

$$= F_{f_1(X_1)}(x_1) F_{f_2(X_2)}(x_2)$$

Buffon's Needle Problem

Let R be the ran. var. that represents the distance of the center of the needle to the nearest line.

Suppose θ be the ran. var. that represents the angle the needle makes with the vertical.

$$R \sim U(0, d/2)$$
$$\theta \sim U(0, \pi)$$

R & θ are indep.,

$$f_{R,\theta}(r, \theta) = f_R(r)\, f_\theta(\theta)$$
$$= \begin{cases} \dfrac{1}{d/2} \cdot \dfrac{1}{\pi}, & \text{if } 0 \leq r \leq d/2 \\ & \phantom{\text{if }} 0 \leq \theta \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

Needle intersects the line if $r \leq l/2 \sin(\theta)$.

$$P = \int_0^{\pi} \int_0^{l/2 \sin(\theta)} f_{R,\theta}(r, \theta)\, dr\, d\theta$$

Let $\{X_n\}$ be a seq. of ran. vars. They are said to be indep. if $\forall m \geq 2$, the ran. vars. $X_1, \ldots, X_n$ are mutually indep.

A seq. of ran. vars. $\{X_n\}$ is said to be <u>identically distributed</u> if they have the same distribution fnⁿ.

eg: 1. $X_1 \sim N(0,1)$ then $-X \sim N(0,1)$

2. $\Omega = [6]$, $P(\omega) = 1/6$.
Let $X(\omega) = \omega$. Let $Y = 7 - X$
Clearly, $Y \sim U([6])$

<u>Independent & Identically distributed</u> (iid):
A seq. of indep. & identically distr. ran. vars. with common law $\mathcal{L}(X)$ is defined to be a seq. $\{X_n\}$ of ran. vars. which is indep. & s.t for some ran. var X.
$F_{X_n} = F_X \quad \forall n \in \mathbb{Z}_{>0}$

The ran. vars. $X : \Omega \to \mathbb{R}^n$, $Y : \Omega \to \mathbb{R}^m$ s.t. $X = (X_1, \dots, X_n)$, $Y = (Y_1, \dots, Y_m)$ are said to be indep. if we have

$$F(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m) = F_1(x_1, \dots, x_n)\, F_2(y_1, \dots, y_m)$$

where F is the joint distr. of X & Y
F_1 is the joint distr. of X
F_2 is the joint distr. of Y

Thm: If $g : \mathbb{R}^n \to \mathbb{R}$ & $h : \mathbb{R}^m \to \mathbb{R}$ are m'ble fn's & if $X = (X_1, \dots, X_n)$ & $Y = (Y_1, \dots, Y_m)$ are indep., then $g(X)$ & $f(Y)$ are indep. ran. vars.

Let $X : \Omega \to \mathbb{R}^n$ be a ran. var. & $g : \mathbb{R}^n \to \mathbb{R}^m$ be m'ble
Then, $Y = g(X) = g \circ X : \Omega \to \mathbb{R}^m$ is an m-dim ran. var.

Under certain cond^{n}s, we can find f_Y in terms of f_X:

- $m = n$
- $g \in C^1$ (all partial derivatives exist & are cont.)

Recall, $J(g) = \det\left(\frac{\partial g_i}{\partial x_j}\right)_{1 \le i,j \le n}$

Thm: Let $X = (X_1, \dots, X_n) : \Omega \to \mathbb{R}^n$ be a ran. var. of cont. type & let $g : \mathbb{R}^n \to \mathbb{R}^n$ be m'ble.
Let $Y = g(X) = (Y_1, \dots, Y_n)$ with $Y_i = g \circ X_i$

Assume $g \in C^1(\mathbb{R})$ s.t $g : X(\Omega) \to (g \circ X)(\Omega) = Y(\Omega) \subseteq \mathbb{R}^n$ is bijective & that $J[g^{-1}(y)](y) \neq 0 \ \forall \ y \in Y(\Omega)$

Then Y is a ran. var. of cont. type &

$$f_Y(y) = f_X(g^{-1}(y)) \, |J(g^{-1}(y))|$$

Pf: Let $B = \{y_1, y_2, \dots, y_n\}$

Then, $F_Y(y) = P(Y \leq B) = P(g(X) \leq B) = P(X \leq g^{-1}(B))$

$$= \int_{g^{-1}(B)} f_X(x)\, dx = \int_B f_X(g^{-1}(y))\, |J(g^{-1})(y)|\, dy$$

$$= \int_{-\infty}^{y_1} \int_{-\infty}^{y_2} \cdots \int_{-\infty}^{y_n} f_X(g_1^{-1}(u_1), \dots, g_n^{-1}(u_n)) \left| \det\left(\frac{\partial g_i^{-1}}{\partial u_j}\right)(u) \right| du_1 \dots du_n$$

eg: Let $X_1, X_2 \sim \mathcal{U}(0,1)$ be indep.

Let $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$.

Find the PDF of Y & the two marginal densities

$$f_{X_i}(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{X_1, X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$$

$$g(x_1, x_2) = (x_1 + x_2, x_1 - x_2) = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$g^{-1}(y_1, y_2) = A^{-1}(y_1, y_2) = \left(\frac{y_1+y_2}{2}, \frac{y_1-y_2}{2}\right)$$

$$[J(g^{-1})](y) = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -1/2$$

$$|[J(g^{-1})](y)| = 1/2$$

$$f_Y(y_1, y_2) = \begin{cases} \frac{1}{2} f_{Y_1}\left(\frac{y_1+y_2}{2}\right) f_{Y_2}\left(\frac{y_1-y_2}{2}\right), & 0 < \frac{y_1+y_2}{2} < 1, \; 0 < \frac{y_1-y_2}{2} < 1 \\ = 1/2 & \\ 0, & \text{otherwise} \end{cases}$$

$$f_{Y_1}(y_1) = \begin{cases} \displaystyle\int_{-y_1}^{y_1} \frac{1}{2}\, dy_2 = y_1, & 0 < y_1 < 1 \\ \displaystyle\int_{y_1-2}^{2-y_1} \frac{1}{2}\, dy_2 = 2 - y_1, & 1 < y_1 < 2 \end{cases}$$

$$f_{Y_2}(y_2) = \begin{cases} y_2 + 1, & -1 < y_2 \leq 0 \\ 1 - y_2, & 0 < y_2 < 1 \end{cases}$$

$(\Omega, \mathcal{M}, P)$: prob. measure sp.

$X : \Omega \to \mathbb{R}$ be ran. var.

If $B \subseteq \mathbb{R}$ is a Borel set,

$$m_X(B) = P(X^{-1}(B))$$

Then, $(\mathbb{R}, \mathcal{B}, m_X)$ is a prob. measure sp.

$$m_X(B) = \int_{\mathbb{R}} \mathbb{1}_B \, dm_X = \int_{\Omega} \mathbb{1}_{X^{-1}(B)} \, dP = P(X^{-1}(B))$$

$$\int_{\mathbb{R}} \mathbb{1}_B \, dm_X = \int_{\Omega} (\mathbb{1}_B \circ X) \, dP$$

$$\int_{\mathbb{R}} \mathbb{1}_B(x) \, dm_X = \int_{\Omega} \mathbb{1}_B(X(\omega)) \, dP$$ for every Borel set $B \in \mathcal{B}$

If $s : \mathbb{R} \to [0,\infty]$ is a simple fn^n, say $s = \sum_{i=1}^{k} c_i \mathbb{1}_{B_i}$,

then, $$\int_{\mathbb{R}} s(x)\, dm_X = \int_{\Omega} s(X(\omega))\, dP$$

Let $f : \mathbb{R} \to [0,\infty]$ be any m'ble fn^n.

Clearly, the above eq^n holds for f by MCT.

Let $f : \mathbb{R} \to \mathbb{R}$ be any m'ble fn^n.

Write $f = f^+ - f^-$. The above eq^n holds for f^+ & f^- & hence for f. Hence

$$\int_{\mathbb{R}} f(x)\, dm_X = \int_{\Omega} f(X(\omega))\, dP$$

Take $f(x) = x$, $$\int_{\mathbb{R}} x\, dm_X = \int_{\Omega} X(\omega)\, dP = E[X]$$

If m_X is given by a pdf say $\varphi(x)$, then
(i.e $dm_X = \varphi(x)\, dm$)

$$E[X] = \int_{\mathbb{R}} x\varphi(x)\, dm,$$ where m : Lebesgue measure

Sim., for higher moments

$$E[X^n] = \int_{\mathbb{R}} x^n \, dm_X = \int_{\Omega} X^n(\omega) \, dP$$

(take $f(x) = x^n$)

Replace $(\Omega, \mathcal{M}, P)$ by $(\mathbb{R}, \mathcal{B}, m_X)$ &
$(\mathbb{R}, \mathcal{B}, m_X)$ by $(\mathbb{R}, \mathcal{B}, m_Y)$ where $Y = g(X)$
for some m'ble g.

$$\int_{\mathbb{R}} g(y) \, dm_Y = \int_{\mathbb{R}} g(X(x)) \, dm_X$$

(Most general change of var. formula)

Suppose X & Y are indep. ran. var.

$$E[X]E[Y] = \int_{\mathbb{R}} x \, dm_X \cdot \int_{\mathbb{R}} y \, dm_Y = \int_{\mathbb{R}\times\mathbb{R}} xy \, dm_X \, dm_Y$$

$$= \int_{\mathbb{R}\times\mathbb{R}} xy \, dm_{X,Y} \quad , \quad dm_{X,Y} : \text{prod. measure}$$

$$= E[XY]$$

$$\Rightarrow \quad E[X]E[Y] = E[XY]$$

$$\mathrm{Var}(X+Y) = E[(X+Y-E[X+Y])^2]$$

$$= E[((X-E[X]) + (Y-E[Y]))^2]$$

$$= E[(X-E[X])^2] + E[(Y-E[Y])^2] + 2\underbrace{E[(X-E[X])(Y-E[Y])]}_{\mathrm{Cov}(X,Y)}$$

$$= \mathrm{Var}(X) + \mathrm{Var}(Y)$$

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y] - E[Y]E[X] + E[X]E[Y]$$

$$= 0 \quad \text{(by indep.)}$$

We def. $\text{Cov}(X,Y)$ (if it exists) to be
$E[(X-E[X])(Y-E[Y])]$

If $E[X^2]$ & $E[Y^2]$ exist, Cauchy-Schwarz says
$\text{Cov}(X,Y)$ exists.

$$\text{Cov}(X,Y) = E[XY] - E[X]E[Y]$$

Def. correlation coeff ρ as

$$\rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

If $\rho = 0$, we say X & Y are uncorelated.

Let $(\Omega, \mathcal{M}, P)$ be a prob. measure sp. A seq. of m'ble fn's $f_n : \Omega \to \mathbb{R}$ is said to <u>converge in probability</u> to a m'ble fn $f : \Omega \to \mathbb{R}$ if

$$\lim_{n \to \infty} P(\{\omega : |f_n(\omega) - f(\omega)| > \epsilon\}) = 0$$

for any $\epsilon > 0$

We write $f_n \xrightarrow{P} f$

<u>eg</u>: Let $A_{n,l} = \begin{cases} \left[\frac{l}{n}, \frac{l+1}{n}\right], & \text{if } l = 0 \\ \left(\frac{l}{n}, \frac{l+1}{n}\right], & \text{if } l > 0 \end{cases}$, $\begin{array}{l} n \in \mathbb{Z}_{>0} \\ 0 \leq l \leq n-1 \end{array}$

Then, $\mathbb{1}_{A_{n,l}} \xrightarrow{P} 0$ but $\mathbb{1}_{A_{n,l}} \overset{a.e}{\not\to} 0$ (almost everywhere)

(seq.: $A_{10}, A_{20}, A_{21}, A_{30}, A_{31}, A_{32}, \ldots$)

Lem: Let X_n be a seq. of ran. var. s.t $E[|X_n|] \to 0$ as $n \to \infty$, then $X_n \xrightarrow{P} 0$

Pf: Let $\delta > 0$ & let $A_n = \{\omega \in \Omega : |X(\omega)| \geq \delta\}$.
Suppose $X_n \xrightarrow{P} 0$. Then $\exists\, \epsilon > 0$ & a subseq. A_{n_k} s.t
$P(A_{n_k}) > \epsilon \quad \forall k$

$$E[|X_{n_k}|] \geq P(A_{n_k}) \cdot \delta \geq \delta\epsilon > 0 \quad \forall k \to \text{Contd}^n$$

Pp^n: (Weak Law of Large Numbers - weak form)
Let X_n be a seq. of i.i.d ran. var. with
$E[X] = \mu$ & $Var(X_n) = \sigma^2$ (i.e $X_n \in L^2(\Omega)$)
Let $S_n = X_1 + \dots + X_n$. Then $\left| \frac{S_n}{n} - \mu \right| \xrightarrow{P} 0$

Pf: Notice $E[S_n/n] = \mu$
Further $Var(S_n) = Var(X_1 + \dots + X_n) = n\sigma^2$
$Var(S_n/n) = Var(S_n)/n^2 = \sigma^2/n$

Choose $K = \delta/\sigma$ in Chebyshev's ineq.

$$P\left(\left\{x : \left|\frac{S_n}{n} - \mu\right| \geq \delta\right\}\right) \leq \frac{\mathrm{Var}(S_n/n)}{\delta^2} = \frac{\sigma^2}{n\delta^2} \longrightarrow 0$$

as $n \to \infty$

<u>Rem</u>: In proof of the weak form, we only need

$\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mu_i \quad (\mu_i = E[X_i])$

Suppose $\frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2 \to 0$. Then $S_n/n \xrightarrow{P} \mu$

<u>Rem</u>: Take $f(x) = \frac{c}{(a^2+x^2)^{1+\epsilon}}$

If X is ran. var. with pdf $= f$, then $E[X]$ exists but $E[|X|^2]$ need <u>not</u> exist.

Yet, we can apply WLLN due to the following stronger form.

Thm: (Weak Law of Large Numbers - strong form)
Assume X_n is a seq. of i.i.d ran. var. with $E[|X_n|] < \infty$
$E[X_n] = \mu$. Then $S_n/n \xrightarrow{P} \mu$

Pf: We want to show

$$E\left[\left|\frac{S_n}{n} - \mu\right|\right] \to 0 \quad \text{as} \quad n \to \infty$$

(then the lemma says the thm is true)

X: ran. var.,
$$X_c(\omega) = \begin{cases} X(\omega), & \text{if } |X(\omega)| \leq c \\ 0, & \text{otherwise} \end{cases}$$

X_c is a ran. var as $X_c = Id_c \circ X$

$$Id_c(x) = \begin{cases} x, & \text{if } |x| < c \\ 0, & \text{otherwise} \end{cases}$$

This shows $X_{n,c}$ are ran. var.

Recall that if $\{X_n\}$ is a seq. of indep. ran. var.,
then so is $g(X_n)$ for every m'ble fn^n $g: \mathbb{R} \to \mathbb{R}$.
Take $g = Id_c$, then $\{X_{n,c}\}$ is a seq. of indep. ran. var.

Let F_n : distr. fn^n of X_n.

$\because$ the X_n are identically distr., $F_n = F$ for some CDF $\forall\, n \in \mathbb{Z}_{>0}$

<u>Note</u> : $E[|X_{n,c}|] = \int_\Omega |X_n|\, dP = \int_{\mathbb{R}} |x|\, dm_{n,c}$

$m_{n,c}$: pushfwd measure of $X_{n,c}$

It is determined by $F_{n,c}$ where $F_{n,c}$ is the distr. corres. to $X_{n,c}$.

Let $x < -c$, $X^{-1}_{n,c}((-\infty, x]) = \emptyset$

So, $P(X^{-1}_{n,c}((-\infty, x])) = 0$

$x \geq c$, $X^{-1}_{n,c}((-\infty, x]) = \Omega$

So, $P(X^{-1}_{n,c}((-\infty, x])) = 1$

$-c \leq x < c$,

$X^{-1}_{n,c}((-\infty, x]) = X^{-1}_{n,c}((-\infty, -c]) \amalg X^{-1}_{n,c}((-c, x])$

$$P(X_{n,c}^{-1}((-\infty, x])) = 0 + \underbrace{P(X_{n,c}^{-1}((-c, x]))}$$

$$P(X_n^{-1}((-c, x]))$$
$$= F_n(x) - F_n(-c)$$
$$= F(x) - F(-c)$$

$$F_{n,c}(x) = \begin{cases} 0 & , \quad x < -c \\ F(x) - F(-c) & , \quad -c \leq x < c \\ 1 & , \quad x \geq c \end{cases}$$

This shows that the $X_{n,c}$ s are identically distributed.

This means all the pushfwd. measures $m_{n,c}$ are the same say m_c.

1. $E[|X_{n,c}|] = \int_\Omega |X_{n,c}| \, dP \leq \int_\Omega |X_n| \, dP = E[|X_n|] < \infty$

2. $E[|X_{n,c}|^2] = \int_\Omega |X_{n,c}|^2 \, dP \leq \int_\Omega c^2 \, dP = c^2 < \infty$

So, if X_n is a seq. of i.i.d ran. var., so is $X_{n;c}$ and $E[|X_{n;c}|^2] < \infty$, $E[X_{n;c}^2] = \int_{\mathbb{R}} x^2 \, dm_{n;c}$

$$= \int_{\mathbb{R}} x^2 \, dm_c = \lambda_c^2$$

Also, $E[X_{n;c}] = \int_{\mathbb{R}} x \, dm_{n;c} = \int_{\mathbb{R}} x \, dm_c = \mu_c$

Let $Y_{n;c} = X_n - X_{n;c}$

Since X_n, $X_{n;c}$ are identically distr., so is $Y_{n;c}$

$E[X_{n;c}] = \mu_c$, $E[(X_{n;c} - \mu_c)^2] = \sigma_c^2$

$$\int_{\Omega} X_{n;c} \, dP = \int_{\mathbb{R}} x \, dm_c = \int_{\mathbb{R}} x \, dm_{n;c} = \mu - \mu_c = \nu_c$$

$\mu = \mu_c + \nu_c$ $\quad (|Y_{n;c}| \le |X_{n;c}|)$

$$A_{n,c} = \frac{X_{1,c} + \dots + X_{n,c}}{n}$$

$$B_{n,c} = \frac{Y_{1,c} + \dots + Y_{n,c}}{n}$$

$$E[|B_{n,c}|] \leqslant \frac{\sum_{k=1}^{n} E[|Y_{n,c}|]}{n} = E[|Y_{1,c}|] \leqslant E[|Y_1|]$$

$$\partial n = \int_{\Omega} \left| \frac{S_n}{n} - \mu \right| dP \leqslant \int_{\Omega} |A_{n,c} - \mu_c| \, dP$$

$$+ \int_{\Omega} |B_{n,c} - \nu_c| \, dP$$

$$\leqslant E[|A_{n,c} - \mu_c|] + 2E[|Y_{1,c}|]$$

Using the pp^n for $X_{n,c}$,

$$P(\{w : |A_{n,c} - \mu_c| \geqslant \delta\}) \to 0 \quad \text{for any } \delta > 0.$$

This means, $\exists \, \epsilon_n \to 0$ s.t $P(\{w : |A_{n,c} - \mu_c| \geqslant \delta\}) < \epsilon_n$

Hence, $E[|A_{n,c} - \mu_c|] \leqslant \delta + 2c\epsilon_n$

contribution from the set on which $|A_{n,c} - \mu_c| < \delta$

Let $n \to \infty$, $\quad E[|A_{n,c} - \mu_c|] \leq \delta$ for any $\delta > 0$

Hence, $\quad \lim_{n\to\infty} E[|A_{n,c} - \mu_c|] = 0$

Moreover, $\limsup_{n\to\infty} a_n \leq 2E[|Y_{1,c}|]$

Choose a seq. of +ve nos. c_n s.t $\lim_{n\to\infty} c_n = \infty$

$X_{1,c_n} \to X_1$ pointwise, $\quad |X_{1,c_n}| \leq |X_1|$

& $\int_\Omega |X_1| \, dP = E[|X_1|] < \infty$

By DCT, $\quad E[|X_{1,c_n}|] \to E[|X_1|]$ as $c_n \to \infty$

$\Rightarrow \quad E[|Y_{1,c_n}|] \to 0$

Hence, $\quad \limsup_{n\to\infty} a_n = 0$, i.e.

$$\int_\Omega \left| \frac{S_n}{n} - \mu \right| dP \longrightarrow 0$$

Lemma says $\left| \frac{S_n}{n} - \mu \right| \xrightarrow{P} 0$

Thm : (Strong Law of Large Numbers — weak form)

Let $\{X_n\}_{n \geq 1}$ be a seq. of i.i.d ran. var. with

$E[X_n] = \mu$, $E[X_n^2] = \lambda^2$ & $E[|X_n|^4] < \infty$ $\forall n \in \mathbb{Z}_{>0}$

Then $\left| \frac{S_n}{n} - \mu \right| \xrightarrow{a.e} 0$

(Replace X_n by $X_n - \mu$ and then prove $S_n/n \to 0$)

Lemma 1 : (Borel – Cantelli)

Let $(\Omega, \mathcal{M}, P)$ be a prob. measure sp.

Let $A_n \in \mathcal{M}$ be a seq. of events s.t $\sum_{n=1}^{\infty} P(A_n) < \infty$

Then $\mathbb{1}_{A_n} \xrightarrow{a.e} 0$

Lemma 2 : (A converse to the Borel-Cantelli Lemma)

Suppose the events A_n are mutually indep. & $\sum_{n=1}^{\infty} P(A_n) = \infty$

Then if A is the event that infinitely many of the A_ns occur, $P(A) = 1$

OR

Then $\mathbb{1}_{A_n} \overset{a.e}{\not\to} 0$

Pf: (Lemma 1) Suppose $f_n : \Omega \to [0,\infty)$ is a seq. of m'ble fn's & let $S(\omega) = \sum_{n=1}^{\infty} f_n(\omega)$

Let $B = \{\omega \in \Omega : S(\omega) < \infty\}$.

If $f_n \in L^1(\Omega)$ & $\sum_{n=1}^{\infty} E[f_n] < \infty$, by MCT.

$$E[S] = \int_{\Omega} S \, dP = \int_{\Omega} \left(\sum_{n=1}^{\infty} f_n(\omega)\right) dP$$

$$= \sum_{n=1}^{\infty} \int_{\Omega} f_n(\omega) \, dP = \sum_{n=1}^{\infty} E[f_n] < \infty$$

$\Rightarrow P(B) = 1$

Apply this to $f_n = \mathbb{1}_{A_n}$

$$E[f_n] = \int_{\Omega} \mathbb{1}_{A_n} \, dP = P(A_n)$$

& $\sum_{n=1}^{\infty} P(A_n) < \infty$ by hypothesis.

So, $B = \{\omega : \sum_{n=1}^{\infty} \mathbb{1}_{A_n}(\omega) < \infty\}$ and $P(B) = 1$

Hence, $\forall\ \omega \in B$, $\mathbb{1}_{A_n}(\omega) = 0$ for all but finitely many n. (i.e $P(B^c) = 0$)

OR

$$\mathbb{1}_{A_n} \xrightarrow{a.e} 0$$

Rem: If we have $E[X_n] = \mu$ for some $\mu \neq 0$, then $X' = X - \mu$ would also satisfy SLLN.
So, wlog, we can assume $E[X_n] = 0$.

Pf: (SLLN)

We will prove the equivalent statement that

$$P\left(\left\{\omega \in \Omega : \lim_{n \to \infty} \left|\frac{S_n}{n}\right| \neq 0\right\}\right) = 0$$

Equivalently, we want $P\left(\left\{\omega \in \Omega : \left|\frac{S_n(\omega)}{n}\right| > \epsilon\right\}\right) = 0$
for infinitely many n

$\Rightarrow P(\{\omega \in \Omega : |S_n(\omega)| \geq n\epsilon$ for infinitely many $n\}) = 0$

Let $A_n = \{\omega \in \Omega : |S_n(\omega)| \geqslant n\epsilon\}$

We want $\sum_{n=1}^{\infty} P(A_n) < \infty$. Then Borel-Cantelli says

$\mathbb{1}_{A_n} \xrightarrow{a.e} 0$

$E[S^4] = E[(X_1 + \dots + X_n)^4]$

Consider $E[X_{i_1} X_{i_2} X_{i_3} X_{i_4}]$ where $i_2, i_3, i_4 \neq i_1$

$$E[X_{i_1} X_{i_2} X_{i_3} X_{i_4}] = \underbrace{E[X_{i_1}]}_{0} E[X_{i_2} X_{i_3} X_{i_4}] = 0$$

Only terms surviving are

$$E[X_1^4 + X_2^4 + \dots + X_n^2] + \sum_{j > k} E[X_j^2] E[X_k^2]$$

$$= n\tau + 3n(n-1)\sigma^4 \leqslant cn^2$$

Apply Markov's inequality for $\alpha = 4$.

$$P(A_n) \leq \frac{E[S_n^4]}{n^4 \epsilon^4} \leq \frac{c n^2}{n^4 \epsilon^4} = \frac{c}{n^2 \epsilon^4}$$

$$\Rightarrow \sum_{n=1}^{\infty} P(A_n) < \infty$$

Hierarchy of conv. of seq. of fn^n

1. Uniform conv. : $f_n \xrightarrow{\text{uniform}} f$
2. Pointwise conv. : $f_n \rightarrow f$
3. Almost everywhere conv. : $f_n \xrightarrow{a.e} f$
4. Conv. in probability : $f_n \xrightarrow{P} f$
5. Weak conv. / conv. in distribution : $f_n \xrightarrow{w} f$

Weak convergence / Conv. in distr.

A seq. of prob. measures μ_n on $\mathbb{R}$ is said to conv. weakly to a (prob.) measure μ if

$$\lim_{n\to\infty} \mu_n(I) = \mu(I) \quad \text{for all intervals } I = [a,b]$$

s.t $\mu(\{a\}) = \mu(\{b\}) = 0$.

In this case, we write $\mu_n \xrightarrow{\omega} \mu$

Let $F_n : \mathbb{R} \to [0,1]$ be a seq. of distr. fn's.
We will say that $F_n \xrightarrow{\omega} F$ of $\lim_{n\to\infty} F_n(x) = F(x)$ for all x where F is cont.

Ex: Show that $\mu_n \xrightarrow{\omega} \mu$ iff $F_n \xrightarrow{\omega} F$

Let $\{X_n\}_{n\geq 1}$ be a seq. of ran. var. & let F_n be the corresponding distr. fxn.

We say X_n <u>conv. in law</u> to X with distr. F if $F_n \xrightarrow{w} F$

We write $X_n \xrightarrow{w} X$ in this case.

eg: Let $X_n \sim Ber(1/2)$ be an i.i.d seq. of ran. var.

$\because$ F_ns are identical, $F_n = F \xrightarrow{w} F$ trivially

Check that there is no ran. var. X s.t $X_n \xrightarrow{P} X$

Ppn: If $X_n \xrightarrow{P} X$, then $F_n \to F$

Pf: Given $\epsilon > 0$. Outside a set of measure δ_n with $\lim_{n\to\infty} \delta_n = 0$

$$|X_n(\omega) - X(\omega)| < \epsilon$$

$$\Rightarrow \quad X(\omega) - \epsilon < X_n(\omega) < X(\omega) + \epsilon$$

$$P(X^{-1}((-\infty, x-\epsilon])) - \delta_n < P(X_n^{-1}((-\infty, x]))$$
$$< P(X^{-1}((-\infty, x+\epsilon])) + \delta_n$$

$$\Rightarrow \quad F(x-\epsilon) - \delta_n < F_n(x) < F(x+\epsilon) + \delta_n$$

Let $n \to \infty$

$$F(x-\epsilon) \leq \liminf_{n\to\infty} F_n(x) \leq \limsup_{n\to\infty} F_n(x) \leq F(x+\epsilon)$$

This is true $\forall \epsilon > 0$. Since F is cont. at x,

$$F(x) \leq \liminf_{n\to\infty} F_n(x) \leq \limsup_{n\to\infty} F_n(x) \leq F(x)$$

It follows that $\lim_{n\to\infty} F_n(x) = F(x)$

Thm: (Central Limit Thm)

Let X_n be a seq. of i.i.d. ran. var. with $E[X_n] = \mu$ & $Var(X_n) = \sigma^2 > 0$.

Then,
$$\frac{S_n - n\mu}{\sqrt{n}} \xrightarrow{w} N(\mu, \sigma^2)$$

Wlog take $\mu = 0$, $\sigma = 1$
$$\frac{S_n}{\sqrt{n}} \xrightarrow{w} N(0,1)$$

Rem: Identically distributed is not an essential hypothesis. Enough to have control on $\sum_{i=1}^{n} \sigma_i^2$ (Lindenberg condⁿ)

eg: Life of lightbulb is exponentially distr.
Mean = 10 days.
If light bulb burns out, replace it. Find the prob. that more than 50 bulbs will be replaced in the year.

Let X_n = life of n^{th} light bulb.
Assume that the X_i s are indep. with mean 10
i.e $\lambda = 1/10$.

$S_n = X_1 + \ldots + X_n$ = time when the n^{th} bulb will burn out.

S_{50} has mean $= 50\mu = 500$
Variance $= 50\mu^2 = 5000$

$$P(S_{50} < 365) \approx \Phi\left(\frac{365-500}{\sqrt{5000}}\right) \approx \Phi(-1.91) \approx 0.028$$

m: prob. measure on $\mathbb{R}$

Characteristic fx^n of m

$$\varphi(t) = \int_{\mathbb{R}} e^{-itx} \, dm = \int_{\mathbb{R}} e^{-itx} f(x) \, dx = \hat{f}(t)$$

(Fourier Transform)

(if m is absolutely cont.)

$\varphi(t)$ is cont.

X, Y are two indep. ran. var with distr. F & G,

$X + Y \sim F * G$

$$F * G(x) = \int_{-\infty}^{\infty} F(x-y) G(y) \, dy$$

$$\widehat{F * G}(x) = \hat{F}(x) \hat{G}(x)$$

α, β are prob. measures,

$$\varphi_{\alpha * \beta}(t) = \varphi_\alpha(t) \varphi_\beta(t)$$

$$\frac{S_n}{\sqrt{n}} \longrightarrow \psi_n(t) = \left[\varphi\left(\frac{t}{\sqrt{n}}\right)\right]^n$$

$$\varphi(t) = 1 - \frac{\sigma^2 t^2}{2} + o(t^2)$$

$$\varphi\left(\frac{t}{\sqrt{n}}\right) \approx 1 - \frac{\sigma^2 t^2}{2n} + o(1/n)$$

$$\psi_n(t) \longrightarrow \psi(t) = \exp\left(\frac{-\sigma^2 t^2}{2}\right)$$