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Theory of Ordinary Differential Equations

Prof. G. K. Srinivasan

Grok

Theory of Ordinary Differential Equations

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MA 417: Ordinary Diff. Equations.
Course offered in Autumn 2019

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- Origins, history and Scope of the Theory

- Fundamental Existence / Uniqueness thms.
  Gronwall's lemma
  peano phenomenon

- Linear Systems. Generalities. Wronskians

- Linear Systems with Constant Coefficients
  The Lagrange - Sylvester interpolation method

- Linear Systems with periodic Coefficients
  Floquet Theory

- Oscillation and Comparison theorems.
  Green's functions for two point BVP.

- Non linear Systems:
  Equilibrium points, Stability
  First Integrals and their uses.
  Notion of Complete integrability
  α and ω limit sets of trajectories
  Poincaré recurrence theorem.

This last part is a glimpse into the modern theory of nonlinear dynamics.

Genesis

Balance Laws of physics when expressed in MATHEMATICAL TERMS

↓

Differential Equations

↙ ↘

Systems of ODEs Systems of PDEs

Most famous among the latter are Maxwell's Eq^n^s in Electromagnetism

The most famous among Systems of ODEs: Equations governing motion of Celestial bodies.

Issac Newton: with the Laws of motion enunciated by him, the inverse Sq. law of graviation and Calculus

Newton was able to

(i) Explain Motion of planets

Kepler's Laws followed as a Math. Conseq !

(ii) Precession of Equinoxes.

Astronomy, hitherto an emperical Science transformed into a dynamical Science !

The discovery of Neptune* is Undoubtedly one of the glorious triumphs of Newtonian Mechanics.

(*) For an account of this see Chapt 16, p247 ff of W. M. Smart, Celestial Mechanics John Wiley 1961.

We shall return to the problem of planetary motion (Kepler problem) later. Let us look at a few simple examples

(i) The harmonic oscillator:

$$\ddot{x} + \omega^2 x = f(t) \qquad (0.1)$$

Dots refer to derivatives wrt time t
$f(t)$ is a forcing function

We recast (0.1) as a first order system

put $y_1 = x$, $y_2 = \dot{x}$: $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$\frac{dy}{dt} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix} \qquad (0.2)$$

When $f(t) = 0$ we have a free harmonic oscillator

(ii) The Simple pendulum:

Despite its simplicity the system is one of the basic paradigms in nonlinear ODEs

Demonstrates many important ideas connecting topology with differential equations

Denoting by $x(t)$ the angular deviation of the pendulum from mean position, the governing equation is

$$\frac{d^2x}{dt^2} + \frac{g}{\ell} \sin x = 0 \qquad (0.3)$$

UNLIKE (0.2), this eqn is NON Linear!

It is precisely this non linearity that brings in many important and interesting features.

Before taking up a study of (0.3), let us Continue with MORE examples

(iii) Continuity Equation in hydrodynamics

$v(x,t)$ and $\rho(x,t)$ denote the velocity field and density of a fluid in motion The equation

$$\frac{\partial \rho}{\partial t} + \text{Div}(\rho v) = 0 \qquad (0.4)$$

Which is a PDE is called the Continuity Eqn

Equation (0.4) is simply the mathematical Expression of the Law of Conservation of mass.

Together with other balance laws we get the full system of DEs governing fluid flow

In particular, if ρ is indep. of time, (0.4) simplifies to

$$\text{Div}(\rho v) = 0 \qquad (0.5)$$

We shall later see the immense significance of (0.5) in the theory of ODEs

Examples arising out of Geometry:

DEs arise naturally in geometry
A good account of these things:

John McCleary: Geom. from a diff. viewpoint, Cambridge Univ. press.

(iv) Given a surface $z = f(x,y)$ in $\mathbb{R}^3$ find curves lying on it that (locally) minimize arc lengths.

We know the answer in two cases. The plane and the sphere.

These curves (known as geodesics) are characterized by the fact that their Acceleration has No tangential component.

Naturally this condition would lead us to a system of 2nd order ODEs.

(v) PDEs also arise naturally in Diff. Geom.

(a) $$(1+f_x^2)f_{yy} - 2f_x f_y f_{xy} + (1+f_y^2)f_{xx} = 0 \qquad (0.6)$$

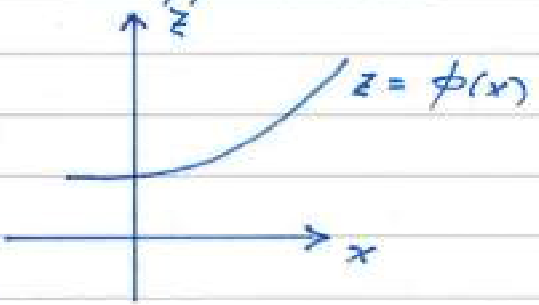
Spells out the fact that the surface $z = f(x,y)$ has Zero Mean Curvature

(See John Opreá's book: Math. Soap films AMS, 2000)

Exercise: Suppose the surface happens to be a surface of revolution:

Take a curve $z = \phi(x)$

in the x-z plane and rotate it about z-axis



An example is the paraboloid $z = x^2 + y^2$

If the surface $z = f(x,y)$ is a surface of revolution, what is the form of the equation (0.6)?

Do you get an ODE in this case?

Find ALL surfaces of revolution that satisfy (0.6)

That is find ALL surfaces of revolution with zero Mean Curvature.

The reason why we get an ODE is due to the Circular Symmetry

Usually, if a PDE admits symmetries then, choosing coord adapted to these symmetries result in simpler systems and one can determine solutions that also enjoy the symmetry.

Let us try this out with the Laplace operator $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$

The Laplace operator is rotation invariant:

$$\sum \frac{\partial^2}{\partial x_i^2} = \sum \frac{\partial^2}{\partial y_j^2} \quad \text{if} \quad y = Rx;$$

R is a $(n \times n)$ rotation matrix: $R^T R = I$
(prove this)

Let us find all solutions of $\Delta u = 0$
that are rotation invariant.
That is, $u(x) = u(Rx)$: $R^T R = I$
Or in other words
$u(x) = f(r)$; $r = (x_1^2 + \dots + x_n^2)^{\frac{1}{2}}$

Well, $\frac{\partial u}{\partial x_i} = f'(r) \frac{\partial r}{\partial x_i}$

$= f'(r) \frac{\partial}{\partial x_i} (x_1^2 + \dots + x_n^2)^{\frac{1}{2}}$

$= x_i f'(r)/r$. Continue...

Show that

$$f''(r) + \left(\frac{n-1}{r}\right) f'(r) = 0 \qquad (0.7)$$

Find all solutions of (0.7)

(vi) Polytropic Gas Spheres: The fundamental equation of Stellar astrophysics is the Nonlinear PDE which goes under the name of Lane-Emden-Fowler equation:

$$\Delta u = f(u) \qquad (0.8)$$

It is natural to seek solutions of (0.8) that are rotation invariant

One gets a variant of (0.7) with a nonlinear term on RHS... we shall not get into details.

Ref: S. Chandrasekhar: Intro to Theory of Stellar Structure.

Travelling Waves:

Let $f(\xi)$ be a given function
Say $\exp(-\xi^2)$ for instance
Ex: Sketch the graphs of $f(x-t)$, $f(x-2t)$, $f(x-3t)$, ...
Explain why the function

$$u(x,t) = f(x-ct)$$

is called a travelling wave.

A famous PDE in fluid mechanics is

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x} \qquad (0.10)$$

called Korteweg de Vries Equation

Let us try to find travelling wave solutions of (0.10). Well,

$$u(x,t) = f(x-ct)$$

$$\frac{\partial u}{\partial t} = -cf'(x-ct) \;\therefore\; \frac{\partial u}{\partial x} = f'(x-ct)$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x-ct), \quad \frac{\partial^3 u}{\partial x^3} = f'''(x-ct)$$

So our f must obey the ODE:

$$-cf' = 6ff' + f'''$$

or

$$0 = (cf + 3f^2 + f'')'$$

$$\therefore\; cf + 3f^2 + f'' = \alpha \quad \text{Constant}$$

Mult by f':

$$cff' + 3f^2f' + f''f' = \alpha f'$$

$$\left(f^3 + \frac{cf^2}{2} + \frac{f'^2}{2} - \alpha f\right)' = 0$$

So we get the first order ODE

$$f'^2 = Af^3 + Bf^2 + Cf + E \qquad (0.11)$$

A, B, C, E are certain Constants.

Can you recognize (0.11) ? You'll see this in Complex analysis Courses

(Try looking into Weierstrass' Elliptic functions)

Exercise: The PDE that governs blood flow in Aorta and also in some population models:

$$\frac{\partial u}{\partial t} - c\frac{\partial^2 u}{\partial x^2} = ru(1-u) \qquad (0.12)$$

Fisher - Kolmogorov Eqn

Exercise: Look for travelling wave solutions
Find the ODE for travelling waves.

Similarly: Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (0.13)$$

perform the change of variables

$$T_\lambda(x,t) = (\lambda x, \lambda^2 t) \qquad (0.14)$$

That is

$$X = \lambda x, \quad \tau = \lambda^2 t$$

(0.13) transforms into $\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$

Equation is invariant under transf (0.14)

Note that $\frac{X^2}{T} = \frac{x^2}{t}$. Find a solution of the heat equation of the form $\psi\left(\frac{x^2}{t}\right)$. Find all such solutions.

Well, $u(x,t) = \psi(x^2/t)$

$$\frac{\partial u}{\partial x} = \psi'\left(\frac{x^2}{t}\right)\frac{2x}{t} \quad ; \quad \frac{\partial u}{\partial t} = -\psi'\left(\frac{x^2}{t}\right)\frac{x^2}{t^2}$$

Now proceed further : Get an ODE for ψ

Such solutions are called (generalized) Similarity solutions.

Thus, ODEs arise in

(i) physical problems

(ii) geometrical problems

(iii) From PDEs - travelling waves, similarity solutions etc.

Issues: Solving a DE explicitly is rarely feasible

Even if an explicit formula for solutions is available, it may not yield useful information

E.g. Does the solution have infinitely many zeros ?

Is the solution periodic ?

Let us turn to these issues in the light of simple examples.

Example: Harmonic Oscillator

Consider the equation for the harmonic osc:

$$y'' + y = 0 \qquad (0.9)$$

Assume we know NOTHING about sin t, cos t

prescribe initial conditions:

$$y(0) = 0, \quad y'(0) = 1 \qquad (0.10)$$

Let us mult. the ODE (0.9) by y':

$\frac{d}{dt}(y^2 + y'^2) = 0$ or $y^2 + y'^2$ is a Constant

This Const = 1 when $t = 0$ (thanks to (0.10))

$$\therefore \quad (y(t))^2 + (y'(t))^2 = 1 \qquad (0.11)$$

$$\therefore \quad |y(t)| \leq 1, \quad |y'(t)| \leq 1 \qquad (0.12)$$

Claim: There exists a $T > 0$ such that

$$y(T) = 1 \quad \text{and} \quad y'(T) = 0$$

Well, Suppose such a T does not exist.

$y'(0) > 0 \Rightarrow y(t)$ is monotone increasing

We shall see in the NEXT Chapter that because of (0.12) the solution $y(t)$ exists for all $t \in \mathbb{R}$

Since $y(t) \neq 1$ for any t, $y(t) < 1$ for all $t \in \mathbb{R}$, $t > 0$ in which case

$$y'(t) \neq 0 \text{ for all } t > 0 \text{ (Why?)}$$

Thus $y'(t) > 0$ for all $t > 0$

$\therefore$ $y(t)$ is a strictly increasing function on $(0, \infty)$

$y'' = -y < 0$ So $y'(t)$ is a strictly decreasing function on $(0, \infty)$

So $\lim_{t\to\infty} y(t)$, $\lim_{t\to\infty} y'(t)$ both exist- say α and β.

Exercise: Explain first graphically and then analytically that

$\beta = 0$ and, by virtue of $y'' = -y$ that $\alpha = 0$ as well. Contradiction.

Hint: $\beta > 0 \Rightarrow y(t) \geq \beta t$ for all $t > 0$.

Also, $y'' = -y \Rightarrow \lim_{t\to\infty} y''(t)$ exists $= \gamma$ say

Applying Mean Value thm,

$$y'(n+1) - y'(n) = y''(\xi_n) \quad (0\cdot13)$$

for some $\xi_n \in (n, n+1)$

But LHS of (0·13) goes to zero.

$\therefore y''(\xi_n) \to 0 \quad \therefore \gamma = 0$

$\therefore \alpha = 0$

Now letting $t \to \infty$ in (0·11) we get a Contradiction.

Exercise: Prove that $y(27) = 0$.

The zeros of y cannot have a limit point

Def: $\pi/2 = \inf \{ T > 0 \;/\; y(T) = 1 \}$

(Analytical definition of π)

Derive all the properties of the solution
$y(t) = \sin t$

Having worked out this toy example let us now take up

Case of a simple pendulum

$$y'' + \frac{g}{\ell} \sin y = 0 \qquad (0.14)$$

Unlike the case of harmonic osc we do not know any explicit solutions - except the trivial solutions (!)
Let us again assume that the pendulum starts from mean pos. with velocity c

That is $y(0) = 0, \quad y'(0) = c \qquad (0.15)$
Assume c is positive and small
We expect the pendulum to swing to a max. extent and then return back.

That is, there exists $T > 0$ such that
$y(T) = \sup \{ y(t) / t \in \mathbb{R} \}$
and $y'(T) = 0$

How does one go about establishing these rigorously? Using Mathematical Analysis?

Proceeding exactly as we did with the harmonic oscillator, Mult DE by y'

$$\frac{d}{dt}\left(\frac{y'^2}{2} - \frac{g}{\ell}\cos y\right) = 0 \quad \text{or}$$

$y'^2 - \frac{2g}{\ell}\cos y$ is constant in time.

Setting $t = 0$ we find this constant to be precisely $c^2 - \frac{2g}{\ell}$

Thus

$$y'^2 - c^2 = \frac{2g}{\ell}(\cos y - 1) = -\frac{4g}{\ell}\sin^2\left(\frac{y}{2}\right) \qquad (0.16)$$

Or, $$y'^2 = c^2 - \frac{4g}{\ell}\sin^2(y/2) \qquad (0.16)'$$

Case 1: $c^2 > \frac{4g}{\ell}$ Say $c^2 = \frac{4g}{\ell} + \alpha^2$

$\therefore y'^2 \geq \alpha^2$

So y' cannot be zero and $y'(0) = c > 0$

$\Rightarrow y' > 0$ always.

$\therefore y' \geq \alpha \quad \therefore y(t) \geq \alpha t$

Case 2: $c^2 < 4g/\ell$

We shall show that the amplitude $y(t)$ reaches a maximum value in finite time T

(Exactly as in the case of the harmonic oscillator)

We show, there exists $T > 0$ such that $y'(T) = 0$.

Suppose not. Then $y'(t) > 0$ for all t
So $y(t)$ is monotone increasing

Next, $y'^2 = c^2 - \frac{4g}{\ell} \sin^2(y/2)$

$\Rightarrow \sin^2 y/2 \leq \frac{c^2 \ell}{4g} \neq 1$

In particular $y(t) < \pi$ for all $t \in \mathbb{R}$

$\lim_{t \to \infty} y(t)$ exists $= \theta_0$ say

Mean value thm gives

$y(n+1) - y(n) = y'(\xi_n) \; ; \; n < \xi_n < n+1$

LHS $\to 0$ So $\lim_{t \to \infty} y'(t) = 0$

Ex: How do you know $\lim_{t \to \infty} y'(t)$ exists?
(Look at the ODE)

Now $\lim y'(t)$ exists $\Rightarrow$ By MVT (and ODE) that

$\lim_{t \to \infty} y''(t) = 0$

$\therefore \lim_{t \to \infty} \frac{g}{\ell} \sin y(t) = 0$

$\therefore \frac{g}{\ell} \sin \theta_0 = 0 \quad \therefore \theta_0 = 0$

Contradiction.

Thus, the claim is established: $y'(T) = 0$.
Deduce that $y(T) = \theta_0 \quad (T > 0)$

This shows that without explicitly solving the ODE we must try to obtain properties of solutions using ideas of analysis, topology etc:

Using topological Methods:

One of the important theorems in topology is the Brouwer's fixed pt thm.

Thm: Let $E^n = \{ x \in \mathbb{R}^n / \sum_{i=1}^{n} x_i^2 \leq 1 \}$

E^n is the closed unit ball in $\mathbb{R}^n$

Suppose $f : E^n \to E^n$ is a continuous function then there exists a $p \in E^n$ such that

$$f(p) = p.$$

Ex: Prove this thm when $n = 1$.

More generally if $f : [a,b] \to [a,b]$ is a continuous function then there exists a $p \in [a,b]$ such that $f(p) = p$.

Let us see how this result can be used to establish existence of periodic solution

The example has been taken from the book of M. W. Hirsch, S. Smale and R. L. Devaney:

Differential Equations, dynamical systems and

An introduction to Chaos

Elsevier 2013 (Ex 15 on page 19)

Consider the differential equation

$$\frac{dy}{dt} = f(y,t) \qquad (0.17)$$

$f : [p,q] \times \mathbb{R} \to \mathbb{R}$ is a Continuous function.

For any choice of initial Condition

$$y(t_0) = y_0 \in [p,q] \qquad (0.18)$$

there is a solution $y(t)$

However, to ensure Uniqueness we need a little more.

Assume to keep things simple that the function f has partial derivatives that are also Continuous on $[p,q] \times \mathbb{R}$

The unique solution of (0.17) - (0.18) (taking $t_0 = 0$) is now a function of t as well as the initial cond. y_0.

Thus, the Unique Sol. of (0.17) - (0.18) is denoted by $y(t, y_0)$

Let us now assume that f is periodic in t with period T. That is,

$$f(y, T+t) = f(y,t) : \quad t \in \mathbb{R} \qquad (0.19)$$
$$y \in [p,q]$$

Que: Are there choices $y_0 \in [p,q]$ for which the solution $y(t, y_0)$ is ALSO periodic with period T namely

$$y(t+T, y_0) = y(t, y_0) \quad \forall\, t \in \mathbb{R}$$

Hirsch - Smale - Devaney have given an example (p 19 problem 16)

Where the answer is NO.

Assume that $f(p,t) > 0$ for all $t \in \mathbb{R}$
$f(q,t) < 0$

Claim: There is atleast one choice $y_0 \in [p,q]$ such that the Solution $y(t, y_0)$ is periodic with period T:
$y(t+T, y_0) = y(t, y_0) \quad \forall\, t \in \mathbb{R}$

Well, the general principles that we shall establish in the next Chapter would imply:

(i) $y(t, y_0)$ exists for all $t \in \mathbb{R}$ for each $y_0 \in [p,q]$

(ii) $y(t, y_0)$ is a continuous map of (t, y_0)

Now with $y(0, y_0) \in [p,q]$
as time t evolves,
Can $y(t, y_0)$ exit the interval $[p,q]$?

Exercise: Suppose it does. Say from right.
Say there is a first time $t_1 > 0$ such that $y(t_1, y_0) > p$
Use MVT and the fact that $f(q,t) < 0$ to arrive at a contradiction.

Thus, the Solution $y(t, y_0)$ is trapped inside the interval $[p,q]$

Let us now set up the function
$\phi : [p, q] \to [p, q]$ given by

$$\phi(y_0) = y(T, y_0)$$

ϕ is continuous and so using Brouwer's fixed point theorem, there is a $\bar{y}_0 \in [p, q]$

such that $\phi(\bar{y}_0) = \bar{y}_0$

That is with the initial condition $\bar{y}_0$, the unique sol. $y(t, \bar{y}_0)$ satisfies

$$y(0, \bar{y}_0) = y(T, \bar{y}_0) \qquad (0.20)$$

We shall now show that THIS solution $y(t, \bar{y}_0)$ is periodic with period T

We have to show $y(t, \bar{y}_0) = y(t+T, \bar{y}_0)$ (0.21)
for all $t \in \mathbb{R}$

Call the RHS as $z(t)$

Then both $y(t, \bar{y}_0)$ and $z(t)$ satisfy the Differential Equation.

Setting $t = 0$, both have the same Initial Conditions

So, by the Uniqueness Clause in the fund. Existence-Uniqueness thm in the next chapter we conclude

$$y(t, \bar{y}_0) = z(t) \text{ for all } t \in \mathbb{R}$$

So (0.21) has been established.

We shall use the Brouwer's thm in higher dimensions to obtain certain non trivial results.

Likewise other deep topological results such as Jordan Curve thm can be used to establish results on existence of periodic sol. in planar systems.

Life span of solutions: Let us begin with a simple example:

$$\frac{dy}{dt} = y^2 \; ; \quad y(0) = c \qquad (0.22)$$

One can immediately write down the solution:

$$y(t,c) = \frac{c}{1-ct} \qquad (0.23)$$

Observe that if $c > 0$ then the solution $y(t,c)$ develops a singularity in finite time $t = 1/c$

Note that the ODE (0.22) was defined everywhere. Where did the singularity in (0.23) come from.?

It comes from the Nonlinearity y^2 in (0.22) Usually presence of nonlinear terms would cause the life span of solutions to be finite

In this example we discovered the finiteness of life span after solving the ODE !

Let us now take up the cases

$$(i) \quad \frac{dy}{dt} = y^2 + t^2 \; ; \quad y(0) = 0 \qquad (0.24)'$$

$$(ii) \quad \frac{d^2y}{dt^2} = 6y^2 + t \; ; \quad y(0) = 0, \; y'(0) = 1 \qquad (0.24)''$$

In neither case is it possible to solve the ODE exactly. Equation (0.24)'' is called

Painlevé - I

Let us now establish that for (0.24)' the life span is finite

From the general thm to be established in the next chapter, equation (0.24)' does have a solution in say $[0, c)$

for some $c > 0$.

Now $y'(0) = 0$, $y''(0) = 2y(0)y'(0) + 2t\Big|_{t=0} = 0$

$y'''(0) = 2 > 0$

Taylor expansion shows that the solution is monotone increasing in a interval $[0, c_0]$ so that $y(t) > 0$ on $(0, c_0]$

$$y'(t) \geq (y(t))^2 + c_0^2 \quad \text{for } t \geq c_0.$$

Integrating over $[c_0, c)$

$$t - c_0 < \int_{c_0}^{t} \frac{du}{u^2 + c_0^2} < \int_{0}^{\infty} \frac{du}{u^2 + c_0^2} \qquad (0.25)$$

So c cannot be arbitrarily large

Because (0.25) must hold for all $c_0 \leq t < c$

Exercise: Deduce that $c < c_0 + \frac{\pi}{2}$

So without solving the ODE we have an estimate on the life span of solutions.

Estimating life span of solutions is generally speaking a difficult problem and an outstanding problem in Celestial Mechanics.

In the next Chapter we shall obtain an important necessary condition for the life span to be finite and illustrate its uses in concrete examples.

Let us state this in a simplified form

Consider a system of ODEs in $\mathbb{R}^n$:

$$\frac{dy}{dt} = f(y) \qquad (0.27)$$

$f : \Omega \to \mathbb{R}^n$ is a smooth function defined on a connected open domain in $\mathbb{R}^n$

Let us prescribe some initial condition

$$y(0) = c \in \mathbb{R}^n$$

If the solution $y(t, c)$ has finite life span namely, if the solution exists on $[0, T)$ and on no larger interval $[0, T + a)$ where $a > 0$; (T finite)

Then, the solution curve $y(t)$ must exit every compact subset of Ω.

That is to say,

Given any compact subset $K \subset \Omega$ there exists a value $t_K \in [0, T)$ such that

$$y(t_K) \notin K$$

Let us turn this around:

Suppose there exists a compact subset $L \subset \Omega$ such that

$$y(t) \in L \quad \text{for all} \quad t \in [0, T)$$

then $T = +\infty$. The life span of solution is infinite.

Let us illustrate this with the pendulum equations.

The system of ODEs is

$$\frac{dy_1}{dt} = y_2 \; ; \quad \frac{dy_2}{dt} = -\frac{g}{\ell}\sin y_1 \qquad (0.28)$$

Thm: For every choice of initial conditions

$$y_1(0) = c_1, \quad y_2(0) = c_2 \qquad (0.29)$$

the solution of (0.28)-(0.29) lives on $[0, \infty)$

That is to say, the life span is infinite.

proof: Assume that for some c_1, c_2 the life span is finite say $[0, T)$

Mult. first equation in (0.28) by $\frac{g}{\ell}\sin y_1$,

Second by y_2 and add:

$$\frac{d}{dt}\left(\frac{y_2^2}{2} - \frac{g}{\ell}\cos y_1 \right) = 0$$

$$\therefore \quad \frac{y_2^2}{2} - \frac{g}{\ell}\cos y_1 = \frac{c_2^2}{2} - \frac{g}{\ell}\cos c_1 = a_1$$

a_1 is some constant

$$\therefore \quad \frac{y_2^2}{2} < \frac{g}{\ell}\cos y_1 + a_1 < a_2 \; ; \; a_2 \text{ is a const.}$$

$$\therefore \quad y_2^2 < 2a_2 \qquad (0.30)$$

Now

$$|y_2| \leq \sqrt{2a_2}$$

$\frac{dy_1}{dt} = y_2$ gives at once

$$y_1(t) = y_1(0) + \int_0^t y_2(t)\,dt \; ; \; 0 \leq t < T$$

$$\therefore \quad |y_1(t)| \leq a_1 \quad |c_1| + t\sqrt{2a_2} \; ; \; 0 \leq t < T$$

Hence we get that

$$|y_1(t)| \leq |c_1| + T\sqrt{2a_2} \leq a_3 \qquad (0.31)$$

a_3 is a constant

Equations (0.30) - (0.31) give us that the trajectory (solution curve) $y(t)$ stays for $t < T$ inside the Compact set

$$L = \{ (y_1, y_2) \in \mathbb{R}^2 \;/\; |y_1| \leq a_3,\ |y_2| \leq \sqrt{2a_2} \}$$

But this immediately contradicts the theorem that the solution must exit every Compact subset of $\mathbb{R}^2$ (stated on slide 22)

This idea works in a variety of situations

What we have done is to find a

TRAPPING REGION (!)

a Compact subset L inside which the solution is Constrained to lie.

Example (Euler's equations for the spinning top)

$$A \frac{dx_1}{dt} = (B-C)\, x_2 x_3 \qquad (0.32)$$

$$B \frac{dx_2}{dt} = (C-A)\, x_1 x_3$$

$$C \frac{dx_3}{dt} = (A-B)\, x_1 x_2 \qquad ; 0 < C < B < A \text{ Constants.}$$

Mult. first by x_1, second by x_2, third by x_3 and add:

$$\frac{d}{dt}\left(A x_1^2 + B x_2^2 + C x_3^2 \right) = 0$$

$$\therefore\ A\,(x_1(t))^2 + B(x_2(t))^2 + C(x_3(t))^2 \qquad (0.33)$$

is a Constant E

E can be computed from a knowledge of initial Conditions $(x_1(0), x_2(0), x_3(0))$

$L = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \;/\; Ax_1^2 + Bx_2^2 + Cx_3^2 = E \}$

L is a Compact Subset of $\mathbb{R}^3$

Since $A(x_1(t))^2 + B(x_2(t))^2 + C(x_3(t))^2 = E$

We see that:

The Solution $(x_1(t), x_2(t), x_3(t))$ is Constrained to lie on the ellipsoid L which is a Compact Subset of $\mathbb{R}^3$

Appealing to the fundamental theorem (Slide 22) the Solution must live on $[0, \infty)$

For, if life Span is finite $[0, T) : T < \infty$ then there must exist a time $t_L < T$ for which $(x_1(t_L), x_2(t_L), x_3(t_L)) \notin L$ (Contradiction)

Here we shall close the introductory Chapter on Origins, history and Scope.

In the next Chapter we shall turn to the Statement and proof of the fundamental Existence - Uniqueness theorems and its ramifications. We shall make free use of fundamental results in analysis.

We assume you are familiar with the Contents of the first Seven Chapters of

W. Rudin: Principles of Mathematical Analysis 3rd Edition, McGraw Hill 1976.

I - Fundamental Existence - Uniqueness theorems Gronwall lemma and its applications

In this Chapter we shall establish the fundamental existence and uniqueness theorem for systems of ODEs:

$$\frac{dx}{dt} = F(x,t) \qquad (1.1)$$

x denotes a column vector (of functions of t) in $\mathbb{R}^n$

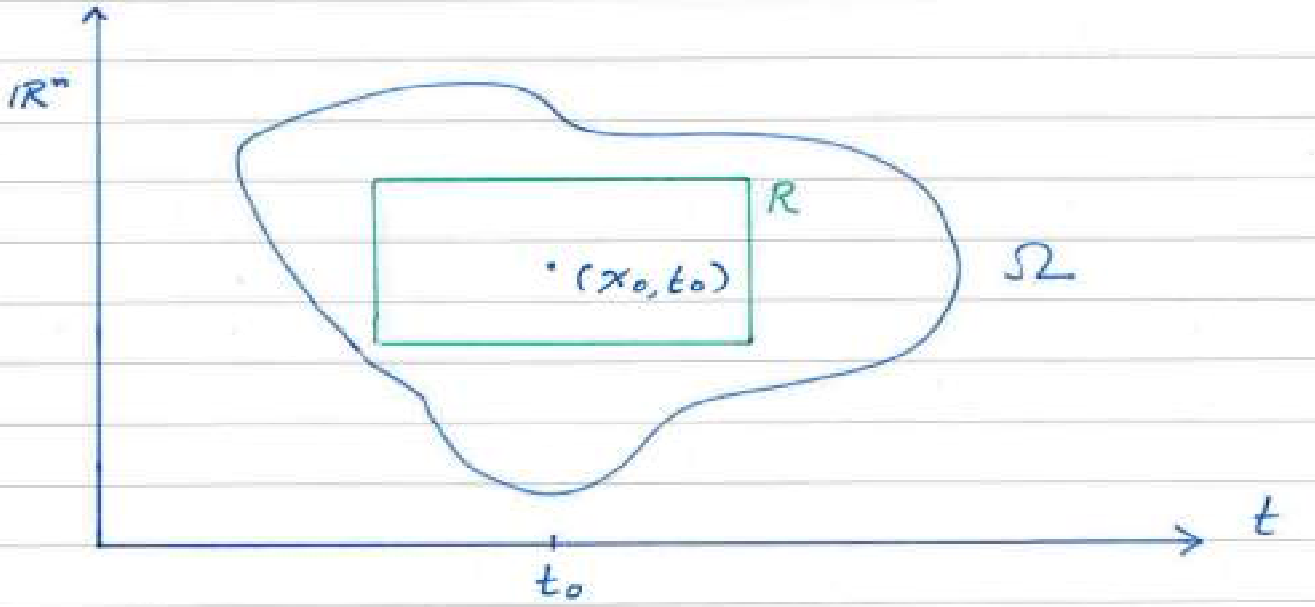
$F: \Omega \to \mathbb{R}^n$ is a continuous function

The domain Ω is an open set in $\mathbb{R}^{n+1}$

We supplement (1.1) with an initial condition

$$x(t_0) = x_0 \qquad (1.2)$$

where $(x_0, t_0) \in \Omega$



Since Ω may have arbitrary shape we work on a Rectangle R contained in Ω as indicated in the figure above

$$R = \{ (x,t) \in \mathbb{R}^{n+1} \;/\; \|x - x_0\| \leq b \;\; |t - t_0| \leq a \}$$

(Closed bounded rectangle contained in Ω)

$$M = \sup_{(x,t) \in R} \|F(x,t)\| \qquad (1.3)$$

Geometrically, we are trying to find a curve $(x(t), t)$ passing through the point (x_0, t_0) such that

$$\frac{d}{dt}(x(t)) = F(x(t), t) \qquad (1.4)$$

holds for all t in a nbd of t_0

This putative nbd of t_0 is $[t_0 - h, t_0 + h]$
we shall presently specify h

The curve $(x(t), t)$ must lie in Ω in order for (1.4) to make sense. Since Ω has a rather arbitrary shape, we shall work in R and seek the curve $(x(t), t)$ in R.

First, $\left\| \frac{dx}{dt} \right\| \leq \|F(x,t)\| \leq M$

$\therefore$ By fundamental thm of Calculus

$$\|x(t) - x(t_0)\| \leq M|t - t_0| \leq Mh \qquad (1.5)$$

If we restrict h such that $\underline{Mh \leq b}$
then (1.5) reveals that-

$$\|x(t) - x_0\| \leq b$$

So that $(x(t), t)$ $(x(t), t)$ would for sure lie in R. Of course $h \leq a$ so we demand

$$h = \text{Min}\left\{ a, \frac{b}{M} \right\} \qquad (1.6)$$

Seek a solution of the ODE (1.1) with Initial Conditions (IC)

On the interval $[t_0 - h, t_0 + h]$

Obviously the solution must be obtained via some techniques of analysis such as

(i) Setting up an iteration scheme*
   - Picard iteration method
(ii) Use of some fixed point theorems
   (for example Schauder's fixed point theorem in functional analysis)
We shall discuss only (i) and defer (ii) to some future course

For the scheme to work we shall need a little more hypothesis on the function F:

We shall assume that F is Lipschitz in the variable x namely,
We assume (Basic Lipschitz Hypothesis)
There exists a Constant $L > 0$ such that
$$\|F(x_1, t) - F(x_2, t)\| \leq L \|x_1 - x_2\| \qquad (1.7)$$
for all pairs $(x_1, t), (x_2, t) \in R$.

Exercise: Show that the Basic Lipschitz hypothesis (1.7) will be satisfied if F is Continuously differentiable on Ω
(Simple application of Lagrange's MVT)

It is Extremely important to note that h is already prescribed (1.6) and is
INDEPENDENT of the Lipschitz Constant L

(*) Other iteration schemes besides Picard scheme are available but we shall only discuss Picard iterates.

We are now ready to state and prove

Thm 1.1 (Picard's thm): Assume $F: \Omega \to \mathbb{R}^n$ is given as above; $(x_0, t_0) \in \Omega$ and R is a compact rectangle with centre (x_0, t_0) and $R \subset \Omega$.

$M = \sup \{ \| F(x,t) \| \;/\; (x,t) \in R \}$

$h = \min \{ a, \frac{b}{M} \}$ (See slides 26, 27)

F satisfies the basic Lipschitz hypothesis (1.7)

Then,

The Initial value problem (IVP)

$$\frac{dx}{dt} = F(x,t), \qquad x(t_0) = x_0 \qquad (1.1)-(1.2)$$

has a unique solution $x(t)$ defined on the interval $[t_0 - h, t_0 + h]$.

Proof: Let us formally integrate (1.1) taking into account IC (1.2):

$$x(t) = x_0 + \int_{t_0}^{t} F(x(s), s)\, ds \qquad (1.8)$$

Let us obtain the solution through a sequence of approximations

The 0th approximant is simply the constant-function x_0

Plugging in this constant on the RHS of (1.8) define

$$x_1(t) = x_0 + \int_{t_0}^{t} F(x_0, s)\, ds \qquad (1.8)_1$$

Plug in this $x_1(s)$ in place of $x(s)$ on RHS of (1.8) to get the 2nd Approximant:

$$x_2(t) = x_0 + \int_{t_0}^{t} F(x_1(s), s)\, ds \qquad (1.8)_2$$

The pattern is clear. From the kth Approximant $x_k(t)$ we obtain the $(k+1)$st Approximant $x_{k+1}(t)$ as:

$$x_{k+1}(t) = x_0 + \int_{t_0}^{t} F(x_k(s), s)\,ds \qquad (1\cdot 8)_{k+1}$$

There are TWO Issues to settle:

(a) The graphs $(x_k(s), s)$ lie in R for $s \in [t_0-h, t_0+h]$: $k \in \mathbb{N}$

This is to ~~establish~~ ensure that the expressions $F(x_k(s), s)$ all make sense when $s \in [t_0-h, t_0+h]$

(b) The Sequence of functions $(x_k(t))_{k=1}^{\infty}$ defined on $[t_0-h, t_0+h]$ Converges.

In fact we shall establish Uniform Conv!

In particular, the limit function $(x(t))$ would be Continuous on $[t_0-h, t_0+h]$

(Rudin: Principles. Math. Analysis Chapter 7)

proof by induction: Using $(1\cdot 8)_1$,

$$\|x_1(t) - x_0\| \le \left\| \int_{t_0}^{t} F(x_0, s)\,ds \right\| \le |t-t_0| M$$

$\therefore \|x_1(t) - x_0\| \le Mh \le b$ for $|t-t_0| \le h$

So $(x_1(t), t) \in R$ for $|t-t_0| \le h$.

Inductively assume $(x_k(t), t) \in R$ for $|t-t_0| \le h$

Using $(1\cdot 8)_{k+1}$ we get

$$\|x_{k+1}(t) - x_0\| \le \left\| \int_{t_0}^{t} F(x_k(s), s)\,ds \right\| \le M|t-t_0| \le Mh \le b$$

Showing $(x_{k+1}(t), t) \in R$ for $|t-t_0| \le h$

Again, $\| x_1(t) - x_0(t) \| \leq M |t - t_0|$ (1.9)

Next

$$x_2(t) - x_1(t) = \int_{t_0}^{t} (F(x_1(s), s) - F(x_0(s), s)) ds$$

Now taking norm, and using the basic Lipschitz hypothesis we get at once

$$\| x_2(t) - x_1(t) \| \leq L \int_{t_0}^{t} \| x_1(s) - x_0(s) \| ds$$

(we assume $t > t_0$. The case $t < t_0$ is similar)

Using (1.9) we get

$$\| x_2(t) - x_1(t) \| \leq LM \int_{t_0}^{t} (t - t_0) dt$$

$$= LM \frac{(t-t_0)^2}{2!} \qquad (1.10)$$

Clearly an induction argument could be used. Exercise. But here we shall go one more step

$$x_3(t) - x_2(t) = \int_{t_0}^{t} (F(x_2(s), s) - F(x_1(s), s)) ds.$$

Taking norms ...

$$\| x_3(t) - x_2(t) \| \leq \int_{t_0}^{t} ML \frac{(s-t_0)^2}{2!} ds$$

$$= ML \frac{(t-t_0)^3}{3!} \qquad (1.10)'$$

prove using ~~3rd~~ Induction:

$$\| x_{k+1}(t) - x_k(t) \| \leq ML \frac{(t-t_0)^{k+1}}{(k+1)!} \qquad (1.10)''$$

By Weierstrass' M-test,

$\sum_{k \geq 0} \| x_{k+1}(t) - x_k(t) \|$ converges uniformly for $|t - t_0| \leq h$

as well as absolutely.

Thus $\sum_{k \geq 0} (x_{k+1}(t) - x_k(t))$ Conv. Uniformly

on $[t_0 - h, t_0 + h]$

Thus, the Picard Iterates $(x_k(t))_{k \geq 0}$ Converge unif. on $[t_0 - h, t_0 + h]$

Let $x(t)$ denote the limit function.
Since each iterate is Continuous, so is the limit function $x(t)$

Now we let $k \to \infty$ in the defining equation $(1\cdot8)_{k+1}$, namely

$$x_{k+1}(t) = x_0 + \int_{t_0}^{t} F(x_k(s), s)\, ds$$

The limit function $x(t)$ satisfies the eqn

$$x(t) = x_0 + \int_{t_0}^{t} F(x(s), s)\, ds \qquad (1\cdot11)$$

Now since F and $x(s)$ are Continuous, Fund. Thm of Calc gives RHS of (1·11) is Continuously differentiable!

That is, the LHS $x(t)$ is Continuously differentiable on $[t_0 - h, t_0 + h]$

We can now differentiate (1·11) and we get

$$\frac{dx}{dt} = F(x(t), t)$$

Thus the function $x(t)$ satisfies the ODE (1·1) on the interval $[t_0 - h, t_0 + h]$.

Setting $t = t_0$ in (1·11) we get

$$x(t_0) = x_0$$

The Initial Conditions are satisfied.

Existence of the solution of IVP is thereby Completed.

To prove uniqueness we shall use the following useful result* which has a wide range of applications

Ex: Prove uniqueness directly using the hint provided in the footnote)

Thm 1.2 (Gronwall lemma): See Hirsch-Smale Diff Eqns, dynamical sys....

Suppose $u, v : [a, b] \to \mathbb{R}$ are continuous non negative functions such that for $t \geq t_0$

$$u(t) \leq C + k \int_{t_0}^{t} u(s)\, v(s)\, ds \qquad (1\cdot12)$$

Where C, k are non neg constants then

$$u(t) \leq C \exp k \int_{t_0}^{t} v(s)\, ds \qquad (1\cdot12)'$$

proof: See Hirsch-Smale: Diff Eqns, dynamical systems and linear algebra, Academic Press (1974) p. 169.

Let $U(t) = C + k \int_{t_0}^{t} u(s)\, v(s)\, ds \geq u(t) \geq 0.$

Assume $C > 0$ to begin with.

$$U'(t) = k\, u(t) v(t) \leq k\, U(t) v(t)$$

Divide by $U(t)$ and integrate (Note $U > 0$)

$$\log \left(\frac{U(t)}{U(t_0)} \right) \leq k \int_{t_0}^{t} v(s)\, ds$$

Exponentiating we get $(1\cdot12)'$.

Next, take up the Case $C = 0$.

(*) One can directly establish Uniqueness. Suppose $y(t), z(t)$ are solutions then

$$y(t) - z(t) = \int_{t_0}^{t} (F(y(s), s) - F(z(s), s)\, ds \ldots$$

We certainly have $u(t) \leq \frac{1}{n} + k \int_{t_0}^{t} u(s)v(s)\,ds$ Apply the

by the positive C case we get

$$u(t) \leq \frac{1}{n} \exp k \int_{t_0}^{t} v(s)\,ds.$$ Letting $n \to \infty$

we get the $C = 0$ case as well. As the first APPLICATION we come to:

Proof of Uniqueness Clause in Picard's Thm:

Suppose $y(t), z(t)$ are two solutions of the IVP (1.1) - (1.2) then both satisfy the equation

$$x(t) = x_0 + \int_{t_0}^{t} F(x(s), s)\,ds$$

$$\therefore y(t) - z(t) = \int_{t_0}^{t} F(y(s), s) - F(z(s), s)\,ds$$

$$\therefore |y(t) - z(t)| \leq L \int_{t_0}^{t} |y(s) - z(s)|\,ds$$

The $C = 0$ case of Gronwall's lemma now gives

$$0 \leq |y(t) - z(t)| \leq 0 \cdot \exp L \int_{t_0}^{t} ds \equiv 0$$

$$\therefore |y(t) - z(t)| = 0 \quad \text{for} \quad t_0 \leq t \leq t_0 + h$$

To handle the case $t_0 - h \leq t \leq t_0$ simply make the change of variables

$$s = 2t_0 - t$$

When t ranges over $[t_0 - h, t_0]$ check that s ranges over $[t_0, t_0 + h]$

The transf. equation also satisfies the basic Lipschitz hypothesis. . . .

Continuous dependence on parameters and Initial Conditions.

Before taking up the general theorem, let us discuss a simple example which is interesting.

Forced harmonic oscillator: Let us consider the ODE

$$x'' + \omega^2 x = \sin at \qquad (1.13)$$

Let us prescribe IC: $x(0) = 0,\ x'(0) = 0$ (1.13)'

The unique solution of the IVP (1.13)-(1.13)' is a function of 3 variables (t, a, ω)

The frequencies ω, a are to be regarded as parameters.

The solution can be immediately written down when $a \neq \pm\omega$ namely

$$\frac{\sin at}{\omega^2 - a^2} - \frac{a}{\omega^2 - a^2}\frac{\sin \omega t}{\omega} = x(t, a, \omega) \qquad (1.14)$$

How to obtain the solution when $a = \omega$?

If we believe that the solution $x(t, a, \omega)$ depend continuously on the parameters ω, a then,

we can allow $\omega \to a$ in (1.14) and we get the result:

$$\lim_{\omega \to a} \frac{1}{\omega^2 - a^2}(\omega \sin at - a \sin \omega t)$$

$$= \frac{1}{2a} \lim_{\omega \to a} \frac{\omega \sin at - a \sin \omega t}{\omega - a} \quad \text{using L'Hospital Rule}$$

$$= \frac{1}{2a}(\sin at - at \cos at) \qquad (1.15)$$

We are spared the tedious job of using Method of Undetermined Coefficients.

It also explains why the secular terms $at\cos at$ arise. The amplitude grows as time evolves ...

With this as a motivation, let us turn to the general theory

$$\frac{dx}{dt} = F(x, t, c_1, \ldots, c_k) \qquad (1.16)$$

with IC: $x(t_0) = x_0$. (1.16)'

Now F is a function of $n+1+k$ variables $c_1, \ldots, c_k$ being parameters (in the above example $c_1 = \omega$, $c_2 = a$)

So we assume $F: \Omega \times \Pi \to \mathbb{R}^n$ is a continuous function: Ω is a connected domain in $\mathbb{R}^{n+1}$ containing the point (x_0, t_0)

Π is a parameter domain in $\mathbb{R}^k$.

$(c_1, \ldots, c_k)$ varies over Π.

For simplicity we shall assume that Π is a compact box in $\mathbb{R}^k$ (product of k closed bounded intervals $[\alpha_j, \beta_j]$; $j = 1, 2, \ldots, k$)

$$M = \sup_{(x,t,c) \in R \times \Pi} \| F(x, t, c) \|$$

R is the rectangle $\{(x,t) / \|x - x_0\| \le b, \; |t - t_0| \le a\}$ (1.17)

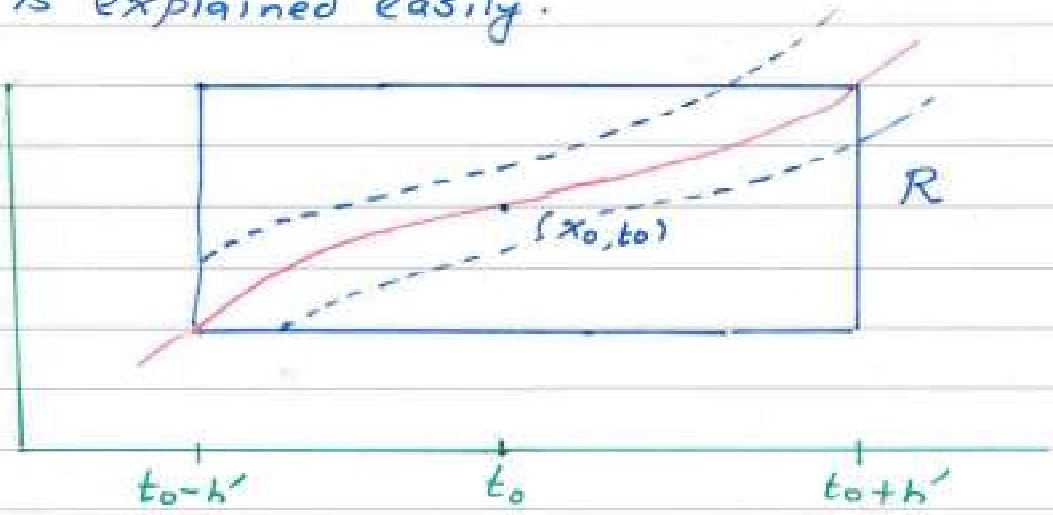
and this time we take

$$h = \text{Min}\left\{ a, \frac{b}{2M} \right\} \qquad (1.18)$$

note !

For reference let us call $h' = \text{Min}\left\{ a, \frac{b}{M} \right\}$ (1.18)'

The reason for the diminished choice (1.18) is explained easily.



The red curve denotes the solution of the "unperturbed problem" passing through (x_0, t_0) with parameter values $c_1 = 0, \ldots, c_k = 0$
(For simplicity the box Π is assumed to be centred at origin)

The dotted curves denote the solution of the same IVP but with values $c_1, \ldots, c_k$ different from origin (but close to origin)
Note that the dotted curves EXIT our reference rectangle R even before t exits $[t_0 - h', t_0 + h']$
In other words if $c \neq 0$
the solution $x(t, x_0, c)$ is not defined on the whole interval $[t_0 - h', t_0 + h']$

By cutting down the interval further to
$[t_0 - h, t_0 + h]$ $[t_0 - h, t_0 + h]$
with $h = \min\left\{a, \frac{b}{2M}\right\}$
Note

We shall see that for all $c \in \Pi$, ALL

The solutions $x(t, x_0, c)$ live on the Common interval $[t_0 - h, t_0 + h]$
So it makes sense to discuss Continuity of

$x(t, x_0, c)$ as a function of t and $c_1, \ldots, c_k$

Thm 1.3 (Dependence on parameters)
Assume $F: \Omega \times \Pi \to \mathbb{R}^n$ is continuous
(Ω, Π as before)
F satisfies a Lipschitz Condition
$\| F(x_1, t, c) - F(x_2, t, c) \| \leq L \| x_1 - x_2 \|$
for all $(x_1, t, c), (x_2, t, c) \in R \times \Pi$

R is a Compact rectangle
$\{ (x,t) / \| x - x_0 \| \leq b : |t - t_0| \leq a \} \subset \Omega$.
$M = \sup_{R \times \Pi} \| F(x, t, c) \|$

$h = \min\left(a, \frac{b}{2M} \right)$

Then, the IVP $\frac{dx}{dt} = F(x, t, c)$ (1.19)
(Initial Cond Change)
$x(t_0) = \tilde{x}_0 \; ; \; \| \tilde{x}_0 - x_0 \| \leq b/2$
has a unique solution $x(t, \tilde{x}_0, c)$
which depends on $(t, c) \in [t_0 - h, t_0 + h] \times \Pi$
The solution is Continuous with respect to
(t, c) on $[t_0 - h, t_0 + h] \times \Pi$

proof: We have to simply re examine the proof of Picard's thm.
Picard Iterates are as before

$$\tilde{x}_0(t) \equiv \tilde{x}_0$$
$$\tilde{x}_{k+1}(t, c) = \tilde{x}_0 + \int_{t_0}^{t} F(\tilde{x}_k(s), s, c)\, ds$$

Each $\tilde{x}_k$ is Cont. wrt (t, c)

We need to check that ALL the iterates are defined on the common interval $[t_0-h, t_0+h]$ for all $c \in \Pi$

Well, it is certainly true for the 0th iterate

$$\|\tilde{x}_{k+1}(t) - x_0\| \le \|\tilde{x}_0 - x_0\| + \int_{t_0}^{t} \|F(\tilde{x}_k(s), s, c)\|\, ds$$

$$\le \frac{b}{2} + |t-t_0| M \le b.$$

Thus, $(\tilde{x}_k(t), t) \in R$ for all $k = 0, 1, 2, \ldots$ for all $c \in \Pi$ and $|t-t_0| \le h$

Even the initial cond $\tilde{x}_0$ can be arbitrary as long as $\|\tilde{x}_0 - x_0\| \le b/2$.

Picard Iterates are all Cont.
The Lipschitz Cond again gives Unif. Conv.
So the $\tilde{x}_k(t)$ Conv. Unif to some $x(t, \tilde{x}_0, c)$

This limit function is Cont wrt (t, c)
Infact it is Continuous wrt ALL the $n+1+k$ arguments $(t, \tilde{x}_0, c)$
on $[t_0-h, t_0+h] \times \{\|\tilde{x}_0 - x_0\| \le \frac{b}{2}\} \times \Pi$
and provides a Sol. of the IVP (1.19)

The above argument is elegant but does not provide quantitative information
Gronwall lemma on the other hand provides more precise information

Suppose $y(t)$, $z(t)$ are solutions of the ODE $\frac{dx}{dt} = F(x, t, c)$

with IC $x(t_0) = y_0$, z_0 respectively.

Then

$$y(t) = y_0 + \int_{t_0}^{t} F(y(s), s, c)\, ds$$

$$z(t) = z_0 + \int_{t_0}^{t} F(z(s), s, c)\, ds$$

$$\therefore\ y(t) - z(t) = (y_0 - z_0) + \int_{t_0}^{t} (F(y(s), s, c) - F(z(s), s, c))\, ds.$$

Taking norms and using Lipschitz Conditions,

$$\| y(t) - z(t) \| \leq \| y_0 - z_0 \| + \int_{t_0}^{t} L \| y(s) - z(s) \|\, ds$$

$$\therefore\ \| y(t) - z(t) \| \leq \| y_0 - z_0 \| \exp L(t - t_0)$$

Thus,

a change in the IC can cause atmost-exponential divergence in the solutions.

That exponential divergence can happen is easily illustrated by a simple (almost trivial) example:

$$\frac{dx}{dt} = x \quad : \quad x(t_0) = a$$

Sol: $x(t, a) = a e^{t}$

So if we look at two Initial Conditions a_1, a_2 the Corresp. solutions diverge as

$$| x(t, a_1) - x(t, a_2) | = | a_1 - a_2 |\, e^{t}$$

As t grows, the distance between $x(t, a_1)$ and $x(t, a_2)$ separate exponentially.

Significance and Scope of Lipschitz Condition.

Consider the IVP: $x' = x^{2/3} : x(0) = 0$

The Constant function $x(t) \equiv 0$ is evidently a solution.

However, it is readily checked that

$z(t) = t^3/27$ is also a solution of the Initial value problem.

One can easily cook up infinitely many solutions of the Initial value problem:

For each $\alpha \geq 0$, define

$$z_\alpha(t) = \begin{cases} 0 & \text{if } t \leq \alpha \\ (t-\alpha)^3/27 & \text{if } t \geq \alpha \end{cases}$$

Check that $z_\alpha(t)$ is a solution of the IVP.

We have a one parameter family of solutions!

Exercise: The function $F(y) = y^{2/3}$ does not satisfy the Lipschitz Condition on any neighborhood of the origin.

Que: Is the Lipschitz condition which is sufficient to ensure uniqueness of solutions of the IVP really necessary?

Ans: No.

The Lipschitz Condition can be weakened significantly but cannot be dropped altogether as the above example indicates.

That is to say, Continuity of $F(x,t)$ ALONE does not give uniqueness of solutions although Continuity suffices for existence of solutions as we shall presently demonstrate.

Two important (weakened) hypothesis that ensure existence - uniqueness of solutions of IVP - in both cases F is Continuous plus a bit more

(i) Osgood's Condition

(ii) Wintner's Condition

We shall discuss these in the list of exercises to this Chapter

The book G. Birkhoff and G. C. Rota: Ordinary differential equations, Ginn and Co. Boston 1962 Contains several illuminating examples on uniqueness under milder assumptions on F.

Differentiability of Solutions wrt parameters and Initial Conditions - The Variational equation

Let Ω be open connected set in $\mathbb{R}^n$

Π a Compact box in $\mathbb{R}^k$ (parameter domain)

$(x_0, t_0) \in \Omega$ and

$$R = \{ (x,t) / \|x - x_0\| \leq b : |t - t_0| \leq a \}$$

$$M = \sup_{R \times \Pi} \| F(x,t,c) \|$$

$F : \Omega \times \Pi \to \mathbb{R}^n$ is Continuous and satisfies on $R \times \Pi$ a Lipschitz Condition as described earlier

$$\| F(x_1, t, c) - F(x_2, t, c) \| \leq L \| x_1 - x_2 \|$$

(x_1, t, c) and (x_2, t, c) lie in $R \times \Pi$

Let $B = \{ x \in \mathbb{R}^n / \|x - x_0\| \leq b/2 \}$

$$h = \text{Min}(a, b/2M)$$

We know that for each choice $p \in B$ the IVP

$$\frac{dx}{dt} = F(x,t,c) \qquad (1.20)$$

$$x(t_0) = p \qquad (1.21)$$

has a unique solution $x(t, p, c)$ which is Continuous on $[t_0 - h, t_0 + h] \times B \times \Pi$

It is natural to enquire whether the solution $x(t, p, c)$ would be differentiable wrt the parameters $c_1, \ldots, c_k$ and the IC $p = (p_1, \ldots, p_n)$

Of course we have to assume that the function $F(x,t,c)$ is of class C^1 namely, it has continuous partial derivatives wrt all the arguments.

It turns out that the solution $x(t,p,c)$ is again of class C^1 with respect to all the arguments.

proof is a bit lengthy and will be taken up later as an appendix if time permits.

The important thing is that given a fixed value p_0 and c_0 the derivatives of the solution $x(t,p,c)$ at $p=p_0, c=c_0$ satisfy a linear ODE called the Equation of Variation

This observation is extremely imp and would play an important role in the proof of Liouville's theorem later in the course.

Notation: $D_x F(x,t,c)$ denotes the $(n \times n)$ matrix $\left[\frac{\partial F_i}{\partial x_j}\right]_{1 \le i,j \le n}$ evaluated at (x,t,c).

The idea is to differentiate the DE (!) wrt c_j or p_k. Well

$$\frac{d}{dt}(x(t,p,c)) = F(x(t,p,c), t, c) \quad (1.22)$$

$$x(t_0,p,c) = p. \quad (1.23)$$

Let us differentiate (1.22) with respect to c_ℓ

$$\frac{\partial}{\partial c_\ell}\frac{d}{dt}(x(t,p,c)) = \frac{\partial}{\partial c_\ell} F(x(t,p,c), t, c)$$

Proceeding formally

$$\frac{d}{dt}\left(\frac{\partial x}{\partial c_\ell}\right) = \left(D_x F\right)\left(\frac{\partial x}{\partial c_\ell}\right) + \frac{\partial F}{\partial c_\ell} \qquad (1.24)$$

$D_x F$ evaluated along the solution $x(t, p_0, c_0)$

We are seeking $\frac{\partial x}{\partial c_\ell}$ at (t, p_0, c_0)

So (in principle) $D_x F(x(t, p_0, c_0), t, c_0)$ is a Known function of t, call it $A(t)$

Likewise $\frac{\partial F}{\partial c_\ell}(x(t, p_0, c_0), t, c_0)$ is again a known function of t, call it $B_\ell(t)$

So equation (1.24) assumes the form

$$\frac{d}{dt}\xi_\ell = A(t)\xi_\ell + B_\ell(t) \qquad (1.25)$$

where $$\xi_\ell = \left.\frac{\partial x}{\partial c_\ell}\right|_{(t, p_0, c_0)} \qquad (1.26)$$

Observe that (1.25) is a inhomogeneous Linear DE (!)

Equation (1.25) is called the Variational Eqn.

What about Initial Conditions for (1.25)?

Well,

$$\xi_\ell\Big|_{t=t_0} = \frac{\partial}{\partial c_\ell}\left(x(t, p_0, \overset{c}{\cancel{c_0}})\right)_{t=t_0}$$

$$= \frac{\partial}{\partial c_\ell}\left(x(t_0, p_0, c)\right)$$

$$= \frac{\partial}{\partial c_\ell} p_0 = 0$$

Now let us take up $\frac{\partial}{\partial p_k} x(t, p, c)$

This time we diff the p_k DE wrt p_k:

$$\frac{\partial}{\partial p_k}\left(\dot{x}(t,p,c)\right) = \left(D_x F\right)\left(\frac{\partial x}{\partial p_k}\right) \quad \Big| \quad \text{dot refers to } \frac{d}{dt}$$

Again proceeding formally

$$\frac{d}{dt}\left(\frac{\partial x}{\partial p_k}\right) = \left(D_x F\right)\left(\frac{\partial x}{\partial p_k}\right) \quad \Big| \quad D_x F \text{ evaluated along } x(t, p_0, c_0)$$

$D_x F$ is a known function of t say $A(t)$

Thus $W_k = \left.\frac{\partial x}{\partial p_k}\right|_{(t, p_0, c_0)}$ satisfies the Linear ODE:

$$\frac{dW_k}{dt} = A(t) W_k \qquad (1.27)$$

What about Initial Conditions for (1.27)?

Well, $W_k\big|_{t=t_0} = \left.\frac{\partial}{\partial p_k}\left(x(t,p,c)\right)\right|_{t=t_0}$

$$= \frac{\partial}{\partial p_k}\left(x(t_0, p, c)\right)$$

$$= \frac{\partial p}{\partial p_k} = \hat{e}_k \qquad (1.28)$$

$\hat{e}_k$ is the kth standard unit vector in $\mathbb{R}^n$

(1.27) is the Variational Equation for $\frac{\partial x}{\partial p_k}$ $(k = 1, 2, \ldots, n)$

Let us illustrate this through Simple examples:

$x'' + x + \epsilon x^3 = 0$: $\epsilon > 0$

Equation governing the oscillations of a Hard Spring

Let us prescribe IC: $x(0) = 0$, $x'(0) = 1$

The unique solution will be a function of two variables t and ϵ

Let us denote the solution by $x(t, \epsilon)$

$x(t, 0)$ is obviously $\sin t$

It is natural to seek a Taylor series expansion for $x(t, \epsilon)$:

$$x(t, \epsilon) = x(t, 0) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \dots \quad (1.29)$$
$$= \sin t + \epsilon x_1(t) + \epsilon^2 x_2(t)$$

What is $x_1(t)$?

Clearly $$x_1(t) = \frac{\partial}{\partial \epsilon} x(t, \epsilon) \Big|_{\epsilon = 0} \quad (1.30)$$

$x_1(t)$ must satisfy the variational Equation.

Recall how this variational eqn was obtained:

Diff. the DE (1) wrt ϵ

$$\frac{d^2}{dt^2}\left(\frac{\partial x}{\partial \epsilon}\right) + \left(\frac{\partial x}{\partial \epsilon}\right) + (x(t, \epsilon))^3 + 3\epsilon (x(t, \epsilon))^2 \frac{\partial x}{\partial \epsilon} = 0$$

Setting $\epsilon = 0$ we get: $\left.\frac{\partial x}{\partial \epsilon}\right|_{\epsilon = 0} = x_1(t)$

$$\frac{d^2}{dt^2} x_1(t) + x_1(t) + \sin^3 t = 0 \quad (1.31)$$

(1.31) is a linear ODE.

IC for (1.31) obtained from the IC for the original Eqn by diff wrt ϵ

But the IC for the original eqn indep of ϵ

So $x_1(t)$ is obtained as

$$x_1'' + x_1 = -\sin^3 t \quad (1.32)$$
$$x_1(0) = 0, \quad x_1'(0) = 0 \quad (1.32)'$$

It is a routine matter to solve (1.32)-(1.32)'

(though a bit tedious)

Next, $$x_2(t) = \frac{1}{2} \frac{\partial^2}{\partial \epsilon^2} x(t, \epsilon) \Big|_{\epsilon = 0} \quad (1.33)$$

Diff the DE $x'' + x + \epsilon x^3 = 0$
twice wrt ϵ and set $\epsilon = 0$

$$x_2'' + x_2 + \frac{1}{2}(2.3 \sin^2 t \cdot x_1(t)) = 0$$

with zero initial conditions.
Again, finding $x_2(t)$ is routine though tedious.

Note that already in $x_1(t)$ the secular term $t \cos t$ would appear
These terms grow with t and so the usefulness of the Taylor series expansion (1.29) is limited — only over compact time intervals for small ϵ.

To cope with this problem elaborate methods have been devised. These problems originated in Celestial Mechanics.
A good place to start learning about these things is the book by

P. Glendinning Stability, Instability and Chaos: An intro to the theory of nonlinear DEs, Cambridge University Press 1999.

Exercise. Consider the pendulum equation

$$x'' + \frac{g}{\ell} \sin x = 0 \qquad (1.34)$$

with IC: $x(0) = 0$; $x'(0) = c$ (1.34)'
Compute $\frac{\partial x}{\partial c}$ at $c = 2\sqrt{\frac{g}{\ell}}$

First, Can you solve the IVP (1.34) - (1.34)' when $c = 2\sqrt{g/\ell}$?
This is the only value of c for which the pendulum equation can be solved explicitly.

Existence in the absence of Lipschitz Condition

We now come to an important result - proof of existence of solutions of IVP

$$\frac{dx}{dt} = F(x,t); \quad x(t_0) = x_0 \qquad (1.35)$$

When $F: \Omega \to \mathbb{R}^n$ is merely assumed to be Continuous

As before we work on a compact rectangle R with center at (x_0, t_0):

$$R = \{(x,t) / \|x - x_0\| \leq b ; |t - t_0| \leq a\} \subset \Omega$$

we need two basic results from analysis

(i) The Weierstrass' approximation theorem stating that a Continuous map $\phi: R \to \mathbb{R}^n$ can be Uniformly approximated by a sequence of polynomial maps $P_m: R \to \mathbb{R}^n$; $m = 1, 2, 3, \ldots$ namely

$$\sup_{(x,t) \in R} |P_m(x,t) - \phi(x,t)| \to 0 \text{ as } m \to \infty$$

(ii) Ascoli Arzela theorem which asserts that if (ϕ_m) is a sequence of Continuous functions on a Compact metric space X taking values in $\mathbb{R}^n$ such that (ϕ_m) is equi Continuous and Uniformly bounded, then we can extract a subsequence Converging uniformly on X.

Both these results are proved in Chapter 7 of Rudin's principles of Math. Analysis (1976)

Thm 1.4: Ω, F as above

$$M = \sup_{(x,t) \in R} \|F(x,t)\| \; ; \quad h = \min\left(a, \frac{b}{2M}\right)$$

Then the IVP (1.35) has a solution on the interval $[t_0 - h, t_0 + h]$

First, Select a sequence $(P_m(x,t))$ of polynomials converging uniformly to $F(x,t)$ on the Compact-rectangle R.

We may assume $\sup_R \| P_m - F \| \leq M \quad \forall m \in \mathbb{N}$

Clearly each $P_m(x,t)$ satisfies the basic Lipschitz Condition so that the IVP

$$\frac{dy}{dt} = P_m(y,t) ; \quad y(t_0) = x_0 \qquad (1.36)$$

has a unique solution $x_m(t)$ defined on a certain interval $[t_0 - h, t_0 + h]$

We show that $h = \text{Min}\left(a, \frac{b}{2M}\right)$ serves as a common h for all the $(x_m(t))$

Well,

$$\sup_{(x,t) \in R} \| P_m(x,t) \| \leq \sup_{(x,t) \in R} \| F(x,t) \| + \sup_{(x,t) \in R} \| P_m(x,t) - F(x,t) \|$$

$$\therefore \sup_{(x,t) \in R} \| P_m(x,t) \| \leq 2M$$

This shows that $x_m(t)$ exists on $[t_0 - h, t_0 + h]$ for each $m \in \mathbb{N}$.

Claim: The Sequence $(x_m(t))$ is unif. bounded.

Well, $x_m(t) = x_0 + \int_{t_0}^{t} P_m(x_m(s), s)\, ds \qquad (1.37)$

$$\therefore \| x_m(t) \| \leq \| x_0 \| + |t - t_0| \sup_R \| P_m(x,t) \|$$

$$\leq \| x_0 \| + |t - t_0| \cdot 2M \quad . \quad \text{Done}$$

Claim: The Sequence $(x_m(t))$ is equicontinuous

From (1.37) we get-

$$x_m(t_1) - x_m(t_2) = \int_{t_1}^{t_2} P_m(x_m(s), s)\, ds$$

$$\therefore \| x_m(t_1) - x_m(t_2) \| \leq |t_1 - t_2| \sup_R |P_m(x,t)|$$

Done.

By Ascoli Arzela-Theorem we can extract a subsequence $(x_{n_k}(t))$ converging uniformly on $[t_0-h, t_0+h]$.

$$x_{n_k}(t) = x_0 + \int_{t_0}^{t} P_{n_k}(x_{n_k}(s), s)\,ds$$

$$= x_0 + \int_{t_0}^{t} \left(P_{n_k}(x_{n_k}(s), s) - F(x_{n_k}(s), s)\right) ds + \int_{t_0}^{t} F(x_{n_k}(s), s)\,ds \qquad (1.38)$$

Now the Middle piece

$$\int_{t_0}^{t} \left(P_{n_k}(x_{n_k}(s), s) - F(x_{n_k}(s), s)\right) ds$$

in norm is dominated by

$$|t - t_0| \sup_R \| P_{n_k}(x,t) - F(x,t) \| \to 0 \quad \text{as } k \to \infty$$

The last piece $\int_{t_0}^{t} F(x_{n_k}(s), s)\,ds \to \int_{t_0}^{t} F(x(s), s)\,ds$

Since F is unif. Cont on the Compact R and $x_{n_k} \to x$ uniformly (Exercise: Show that the seq of integrals converges to the stated value)

So letting $n_k \to \infty$ in (1.38) we infer that

$$x(t) = x_0 + \int_{t_0}^{t} F(x(s), s)\,ds. \qquad (1.39)$$

RHS is of class C^1 by fund thm of calc

$\Rightarrow x(t) \in C^1$

So we are allowed to differentiate (1.39):

$\frac{dx}{dt} = F(x(t), t)$. Evidently $x(t_0) = x_0$

So we have a solution of the IVP. Proof of the thm is complete.

Remarks. We have proved existence of solutions to the IVP merely under the assumption of Continuity
Uniqueness was established under the additional hypothesis of Lipschitz Continuity. Note that use of Gronwall lemma required Lipschitz Cond. We have also indicated that under much weaker conditions one has Uniqueness.

Suppose we have a system of ODEs

$$\frac{dx}{dt} = F(x,t) \quad : \quad F \text{ continuous}$$

for which Uniqueness of IVP is guaranteed. Can we then infer Continuity of the Solution $x(t, x_0)$ wrt all the $(n+1)$ arguments. One can prove that it is indeed so. See for example I. G. Petrowski: Lectures on Ordinary Diff. Equations

DEs (ordinary or partial) come equipped with Side Conditions : IC, boundary Conditions etc; These Side Conditions form a Space called the data Space Γ.
Suppose we have existence-Uniqueness namely for each choice $u \in \Gamma$ the problem has a Unique Solution $T(u)$ belonging to some Specified Space S. S is called the Solution Space.
We then have a map $T : \Gamma \to S$
Usually (in the case of PDEs), Γ and S are Banach Spaces and if T is linear if the DE is linear the Closed graph thm then would give us Continuity of T.
The problem is then said to be well posed in the Sense of Hadamard.

Continuation of Solutions. Global Solutions. Life Span of Solutions.

Let us begin with the observation that the interval $[t_0-h, t_0+h]$ given by Picard's thm (on which the solution is guaranteed to exist) is far from optimal.

The following example illustrates this fact-

Example: $\frac{dx}{dt} = x^2+t^2; \quad x(0)=0$

$R = \{(x,t) \in \mathbb{R}^2 / \ |x| \le b; \ |t| \le a\}$

Theorem says that with

$$h = \text{Min}\left(a, \frac{b}{M}\right); \quad M = \sup_R (x^2+t^2) \qquad (1.40)$$

the solution is guaranteed to exist on $(-h, h)$

$M = a^2+b^2$ so $h = \text{Min}\left(a, \frac{b}{a^2+b^2}\right)$ (1.41)

Note that $\frac{b}{a^2+b^2} \le \frac{1}{2a}$ for $a, b > 0$.

To get the largest possible h out of (1.40)-(1.41) One has to take $b=a$ and then

$$h = \text{Min}\left(a, \frac{1}{2a}\right) \qquad (1.42)$$

Of all possible choices of a, RHS of (1.42) is largest when $a = 1/\sqrt{2}$ in which case $h = \frac{1}{\sqrt{2}}$

The Best Possible value of h obtainable from Picard's thm is $h = 1/\sqrt{2}$

All we can get from Picard's theorem is that the Solution of the IVP exists on $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

However this is far from Optimal

We shall in fact show that the solution lives at least on the interval $(-1, 1)$ (!)

Well, let $[0,T)$ be the maximal interval of existence to the right of origin.
Namely, the solution does not exist on $[0, T+\alpha)$ for any $\alpha > 0$ and $T < \infty$.
That $T < \infty$ was established in the last chapter

Now, $\frac{dx}{dt} \leq x^2 + T^2$ for $0 \leq t < T$

integrate over $[0, t)$

$$\int_0^{x(t)} \frac{du}{u^2+T^2} \leq t$$

$$\frac{1}{T} \tan^{-1}\left(\frac{x(t)}{T}\right) \leq t \qquad (1.43)$$

Recall the theorem stated in the previous chapter that when $t \to T-$, the solution $x(t) \to \infty$
So letting $t \to T$ in (1.43) we conclude

$$\frac{\pi}{2T} \leq T$$

$$\boxed{\text{or} \quad T \geq \frac{\sqrt{\pi}}{\sqrt{2}}} \qquad (1.44)$$

$\frac{\sqrt{\pi}}{\sqrt{2}}$ is > 1 and the claim is established

We now discuss the existence of global solutions or maximal intervals of existence

Suppose that the solutions of the IVP for

$$\frac{dx}{dt} = f(x,t)$$

satisfy the uniqueness clause - for example f continuous on Ω and satisfies a Lipschitz estimate $\| f(x,t) - f(y,t) \| \leq L \| x - y \|$

Given any solution $x(t)$ defined on say $I_0 = [t_0 - h, t_0 + h]$ one can extend it to a maximal solution.

Suppose J is an interval such that $J \cap \mathring{I}_0 \neq \emptyset$

$\mathring{I}_0$ = interior of $I_0 = (t_0 - h, t_0 + h)$

J is also an open interval.

Assume $y : J \to \mathbb{R}^n$ is differentiable and satisfies $\frac{dy}{dt} = f(y, t)$ for $t \in J$

Now, if $y(t_1) = x(t_1)$ for at least one $t_1 \in J \cap \mathring{I}_0$ then uniqueness $\Rightarrow y(t) = x(t)$ for all $t \in J \cap \mathring{I}_0$

So by gluing lemma the function

$$z : J \cup I_0 \to \mathbb{R}^n$$

$$z(t) = \begin{cases} x(t) & \text{if } t \in I_0 \\ y(t) & \text{if } t \in J \end{cases}$$

is differentiable and satisfies the IVP

$$\frac{dx}{dt} = f(x, t); \quad x(t_0) = x_0.$$

z is defined on the larger interval $J \cup I_0$.

We say z is an extension of $x : I_0 \to \mathbb{R}^n$

Take the set of all extensions $w : I_w \to \mathbb{R}^n$ namely, each $w : I_w \to \mathbb{R}^n$

Satisfies the ODE $\frac{dx}{dt} = f(x, t)$ on I_w

$t_0 \in I_w$ and $w(t_0) = x(t_0) = x_0$.

These solutions can all be glued together

For, if $v : I_v \to \mathbb{R}^n$ are two such extensions

$w : I_w \to \mathbb{R}^n$

$v(t) = w(t)$ on $I_v \cap I_w$

We have a solution of the IVP on the union of all the intervals I_w (containing t_0)

This is then the maximal or global solution of the IVP.

Case of non Uniqueness:

Define $f(y) = \begin{cases} 2(\sqrt{y} + y^{3/2}) : & y \geq 0 \\ 0 & y < 0 \end{cases}$

Lipschitz Cond. fails in any nbd of origin.

For each $\alpha > 0$, define

$$y_\alpha(t) = \begin{cases} 0 & \text{if } t \leq \alpha \\ \tan^2(t-\alpha) & \text{if } t > \alpha \; ; \; t < \alpha + \frac{\pi}{2} \end{cases}$$

So the zero solution on $(-\infty, 0)$ has several extensions $(y_\alpha)_{\alpha > 0}$ Each y_α has its own (maximal) domain of existence $(-\infty, \alpha + \pi/2)$

Each y_α is a maximal Continvation of the zero solution on $(-\infty, 0)$

In this kind of situation one must employ Zorn's lemma to obtain a maximal extension of a given solution $x : I \to \mathbb{R}^n$ of the IVP

$$\frac{dx}{dt} = f(x,t) : \quad x(t_0) = x_0, \quad t_0 \in I.$$

Now we prove the important thm that we have been talking about and using occasionally.

1.5. Thm: Let $x(t)$ be a solution of the IVP

$$\frac{dx}{dt} = f(x,t) : \quad x(t_0) = x_0$$

Where $f : \Omega \to \mathbb{R}^n$ Continvous $(x_0, t_0) \in \Omega$

the rectangle R as before

Let (α, β) be the maximal interval Containig t_0 to Such that there is a solution of the above IVP defined on (α, β). Then

(i) Either $\alpha = -\infty$ and $\beta = +\infty$

(ii) $\alpha > -\infty$ or $\beta < +\infty$

In Case (ii) the following happens

Assume $\beta < \infty$ (case $\alpha > -\infty$ is similar)
For Each Compact Subset $K \subset \Omega$,

there exists $t_0 \leq t_K < \beta$ (or $t_0 \geq t_K > \alpha$ in case $\alpha > -\infty$) such that $(x(t_K), t_K) \notin K$
That is to say, the trajectory
$\{(x(t), t) \mid \alpha < t < \beta\}$ has limit points on $\partial\Omega$ or goes off to infinity (or both) as $t \to \beta-$

proof: We need a little fact from real analysis (See W. Rudin, Principles of Math Analysis, Chapter 4 - exercises)

Let X be a Complete Metric Space and
$\phi : (a,b) \to X$ be a Continuous map
(a,b) is a bounded interval in $\mathbb{R}$
Then ϕ extends Continuously to $[a,b]$ if and only if ϕ is Uniformly Continuous.

Let us first discuss the proof of this fact.
If ϕ extends Continuously to $\hat{\phi} : [a,b] \to X$
then $\hat{\phi}$ is Unif. Cont ($\because$ $[a,b]$ is Compact)
and so is $\phi = \hat{\phi}|_{(a,b)}$.

Conversely if $\phi : (a,b) \to X$ is Unif. Cont.
Need to define $\hat{\phi}(b)$:
Well, let $t_n \to b-$ so (t_n) is Cauchy
Uniform Cont. of ϕ now implies $(\phi(t_n))$ is a Cauchy Sequence in X
Since X is Complete, $(\phi(t_n))$ Converges to $p \in X$
Declare $\hat{\phi}(b) = p$.

We must show that p is independent

of the choice of the sequence $t_n \to b$-

Suppose (s_n) is another sequence in (a,b) such that $s_n \to b$-. We know that $(\phi(s_n))$ also converges to say $q \in X$

We have to prove $p = q$ which would establish that $\hat{\phi}$ is well-defined.

Interlace the two sequences

$$t_1, s_1, t_2, s_2, t_3, s_3, \ldots \qquad (1.45)$$

which converges to b-

The image sequence

$$\phi(t_1), \phi(s_1), \phi(t_2), \phi(s_2), \ldots \qquad (1.46)$$

must converge in X and so its two subsequences

$\phi(t_1), \phi(t_2), \phi(t_3), \ldots$

$\phi(s_1), \phi(s_2), \phi(s_3), \ldots$

both must converge to the SAME limit- which proves $p = q$

We leave the continuity of $\hat{\phi}$ as an exercise

We are now ready to prove the theorem

Assume the contrary: There exists $K \subset \Omega$, K compact such that there is NO $t_k \in (\alpha, \beta)$ for which $(x(t_k), t_k) \notin K$

That is

$(x(t), t) \in K$ for all $t \in (\alpha, \beta)$

Let $\mu = \sup_{(x,t) \in K} \| f(x,t) \|$

$$\frac{dx}{dt} = f(x,t) \Rightarrow \left\| \frac{dx}{dt} \right\| \leq \mu$$

for $t \in (\alpha, \beta)$

By Mean Value theorem, $x : (\alpha, \beta) \to \mathbb{R}^n$ is Uniformly Continuous so extends

Continuously to $x : [\alpha, \beta] \to \mathbb{R}^n$.

Evidently x maps $[\alpha, \beta]$ into $K \subset \Omega$

$x(\beta) = p$ say.

Let $y(t)$ be ~~the~~ Solution of the IVP

$$\frac{dy}{dt} = f(y,t) : y(\beta) = p.$$

y is defined on say $[\beta - h, \beta + h]$

Now define

$$z(t) = \begin{cases} x(t) & \text{if } \alpha < t \leq \beta \\ y(t) & \text{if } \beta \leq t < \beta + h \end{cases}$$

Then $z : (\alpha, \beta + h) \to \mathbb{R}^n$ is Continuous.

It is certainly differentiable on

$(\alpha, \beta) \cup (\beta, \beta + h)$

and satisfies the ODE on (α, β) and $(\beta, \beta + h)$

Let us see what is happening at β

A simple application of Mean Value thm

Confirms that

left hand derivative and right hand derivative of $z(t)$ at $t = \beta$ are equal

Infact left hand derivative

equals $\lim_{x \to \beta -} x'(t)$

$$= \lim_{x \to \beta -} f(x(t), t) = f(p, \beta)$$

Likewise right hand derivative $= f(p, \beta)$

So the ODE is satisfied throughout

$(\alpha, \beta + h)$

First Integrals

We shall discuss this only for autonomous systems namely systems of the form

$$\frac{dx}{dt} = f(x) \qquad (1.47)$$

f is a smooth map $\Omega_0 \to \mathbb{R}^n$; Ω_0 open in $\mathbb{R}^n$
The point is that the indep variable t DOES NOT occur explicitly in the ODE

So $\Omega = \Omega_0 \times \mathbb{R}$ open set in $\mathbb{R}^{n+1}$
$F(x,t) = f(x)$ (no t-dependence)

A first integral for (1.47) is a smooth function $\phi : \Omega_0 \to \mathbb{R}$ such that

$$\nabla\phi \cdot f(x) = 0 \qquad (1.48)$$

and $\nabla\phi \neq 0$ and Ω_0 connected. This condition excludes Trivial Cases such as the constant function ϕ.

Example: Consider the pendulum equations

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1$$

Check that $\frac{x_2^2}{2} - \frac{g}{\ell}\cos x_1 = \phi(x_1, x_2)$

is a First integral. In this case the ϕ has a simple interpretation – it is the total energy of the system.

In general for a two dim. autonomous system

$$\dot{x}_1 = P(x_1, x_2)$$
$$\dot{x}_2 = Q(x_1, x_2)$$

A first integral can be obtained by integrating the ODE

$$Q(x_1, x_2)\,dx_1 - P(x_1, x_2)\,dx_2 = 0. \qquad (1.49)$$

Try this out with the Lotka-Volterra Eqns

$$\frac{dx}{dt} = ax - bxy; \quad \frac{dy}{dt} = -cy + exy \qquad (1.50)$$

$x>0, y>0.$ (1.501)

a, b, c, e are positive Constants.

(1.49) is called the Predator-Prey System. governs the evolution of population of species in a two species ecology - a predator & a prey.

For example Sharks and Sardines.

See the discussion in the book of P. Glendinning cited earlier.

Equation (1.49) in this case reads:

$$-y(c-ex)\,dx - x(a-by)\,dy = 0 \qquad (1.51)$$

Check that

$$ex + by - c\log x - a\log y = \phi(x,y) \qquad (1.52)$$

is a first integral

Proper Maps: Let X, Y be metric Spaces

A map $f: X \to Y$ is said to be proper if $f^{-1}(K)$ is Compact in X Whenever K is Compact in Y

Show that a Continuous proper map is a closed map.

Check that the map ϕ given by (1.52) is a proper map.

The interest in proper maps Comes from the fact that

If a system of ODEs $\frac{dx}{dt} = f(x)$

admits a first integral which is a proper map
Then the life span of solutions is $\mathbb{R}$

Well, let $x(t)$ be a solution with maximal domain (α, β)
Let us compute $\frac{d}{dt}\phi(x(t))$

$$= \nabla\phi(x(t)) \cdot x'(t)$$
$$= \nabla\phi(x(t)) \cdot f(x(t))$$
$$= 0 \quad \text{(by def of first integral)}$$

So $\phi(x(t))$ is constant in time

$$\therefore \phi(x(t)) = \phi(x(t_0)) \quad \forall\ t \in (\alpha, \beta)$$
$$t_0 \in (\alpha, \beta)$$

So the curve $x(t)$ lies Entirely on the level set

$$\phi^{-1}(c); \quad c = \phi(x(t_0)) \quad (1.53)$$

ϕ Being a proper map, the level set $\phi^{-1}(c)$ is compact
Now $\beta < \infty \Rightarrow (x(t), t)$ lies in the Compact subset $K = \phi^{-1}(c) \times [t_0, \beta]$

This is a Contradiction Since $(x(t), t)$ must exit K for some $t_k \geq t_0$ and $t_k < \beta$.
The proof that $\alpha = -\infty$ is similar

Exercise: Show that the solutions of the equation for a hard spring $x'' + x + \epsilon x^3 = 0$ exist on $\mathbb{R}$

First write $x_1 = x, \quad x_2 = x'$:

$$\dot{x}_1 = x_2; \quad \dot{x}_2 = -x_1 - \epsilon x_1^3$$

Use (1.49)

Peano Phenomenon and Kneaser's thm:

This is the Concluding part of this Chapter.

To begin with we take a Scalar ODE

$$\frac{dy}{dt} = F(y,t) \qquad (1.54)$$

F is a Continuous function defined on an open Set in $\mathbb{R}^2$ and as usual we work with a rectangle

$$\{ (y,t) \in \mathbb{R}^2 \;/\; |y-y_0| \le b \;;\; |t-t_0| \le a \}$$

Suppose that $V : (t_0-h, t_0+h) \to \mathbb{R}$

$W : (t_0-h, t_0+h) \to \mathbb{R}$

are Two Solutions of (1.54) with the Same Initial Condition $y(t_0) = y_0$ namely

$$V(t_0) = W(t_0) = y_0 .$$

Then,

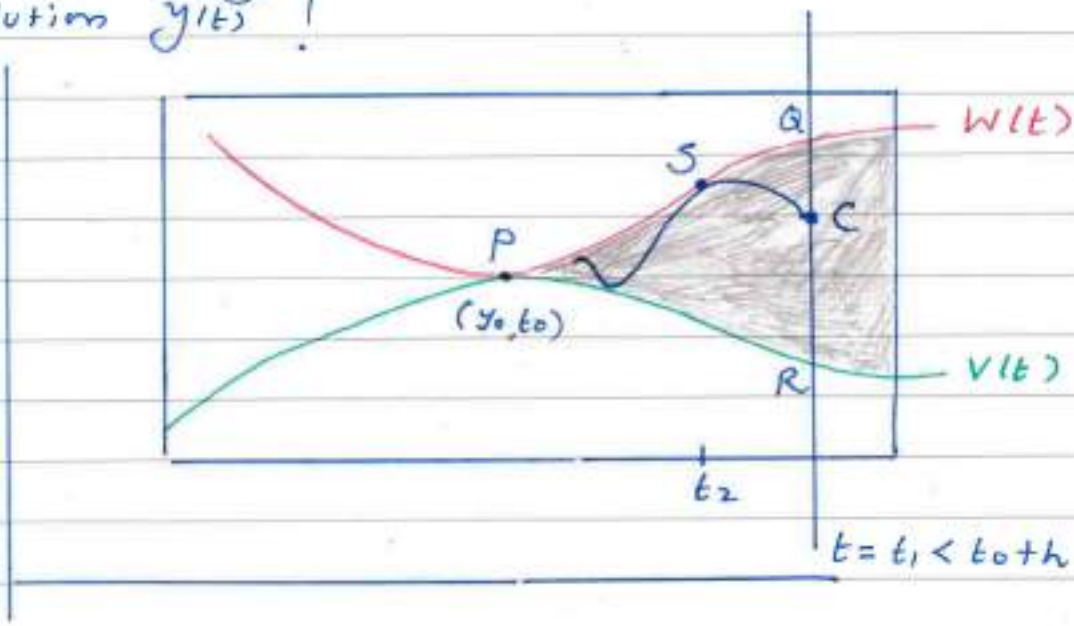
Define $y(t) = \begin{cases} V(t) & \text{if } t_0-h \le t \le t_0 \\ W(t) & \text{if } t_0 \le t \le t_0+h \end{cases}$

$y(t)$ is Continuous on (t_0-h, t_0+h)

Ex: Check that $y(t)$ is diff on (t_0-h, t_0+h)

To discuss differentiability at t_0 use MVT and the fact that $V(t)$, $W(t)$ are Solutions of (1.54) on (t_0-h, t_0) and (t_0, t_0+h) respectively

Note that along with $V(t)$, $W(t)$ we now have a third Solution $y(t)$!



Let us now take a $t_1 < t_0 + h$, $t_1 \geq t_0$

$P = (y_0, t_0)$; Q is the point along the graph $(w(t), t)$ with $t = t_1$

R is the point on the graph $(v(t), t)$ with $t = t_1$

Claim: For each point c lying on the vertical line segment joining R and Q. There is a solution of the ODE (1.54) call it $y_c(t)$ satisfying the IC $y_c(t_0) = y_0$.

Thus

$$y_c(t_1) = c \quad ; \quad y_{c_1}(t_1) \neq y_{c_2}(t_1) \text{ if } c_1 \neq c_2$$

and we see that $\{ y_c(t) / \; R \leq c \leq Q \}$ provides us with an uncountable family of solutions of (1.54) with IC $y_c(t_0) = y_0$

When uniqueness fails, it does so rather spectacularly! This is called Peano phenomenon

Proof of the Claim: Apply existence of solutions to the IVP at the point (c, t_1) in the rectangle depicted. The point itself is named c in the figure.

Let (α, β) be the maximal interval of existence of the solution of the ODE with

$$y(t_1) = c$$

$\alpha < t_1 < \beta$.

The graph $(y(t), t)$ for $\alpha < t < t_1$ lies on the left of the line $t = t_1$ and initially lies inside the shaded region. as $t \to \alpha+$ this curve must exit the shaded region either by passing through P in which case $y(t_0) = y_0$

and we have a solution of the IVP

$$\frac{dy}{dt} = F(y,t): \quad y(t_0) = y_0$$

with additional cond. $y(t_1) = c$

Else,

The curve $(y(t), t)$ must cut the red curve - graph $(w(t), t)$ at some $t_2 \in (t_0, t_1)$ at the point $(w(t_2), t_2) = S$.

Now declare $z(t) = \begin{cases} w(t) & \text{if } t_0 \leq t \leq t_2 \\ y(t) & \text{if } t_2 \leq t \leq t_1 \end{cases}$

Then $z(t)$ is continuous on $(t_0 - h, t_1)$

Satisfies the ODE (1.54)

$$z(t_0) = w(t_0) = y_0$$
$$z(t_1) = y(t_1) = c$$

The third case is the curve $(y(t), t)$ must cut the green curve - graph $(v(t), t)$ at some $t_2 \in (t_0, t_1)$

namely $S = (v(t_2), t_2)$ (the point of intersection S lies on $(v(t), t)$ as well as $(y(t), t)$)

Declare $z(t) = \begin{cases} v(t) & \text{if } t_0 \leq t \leq t_2 \\ y(t) & \text{if } t_2 \leq t \leq t_1 \end{cases}$

As before $z(t)$ is a solution of the IVP $\frac{dz}{dt} = F(z,t): \quad z(t_0) = v(t_0) = y_0$.

and $z(t_1) = c = y(t_1)$

We have constructed our solution $y_c(t)$.

Question now is of course whether this spectacular phenomenon (of there being uncountably many solutions of IVP in the absence of uniqueness) persists in higher

dimensions as well.

The answer is YES and is called Kneaser's -Thm. (1923)
A remarkable example was given two years later by M. Lavrentieff of an ODE (1.54) with $F : \mathbb{R}^2 \to \mathbb{R}$ Continuous
AND
through each point $(y_0, t_0) \in \mathbb{R}^2$ there pass infinitely many solutions
An example is given in the book:
P. Hartman: Ordinary Diff. Equations, 2nd Edition, SIAM 2002
A thorough treatment of Peano phenomenon is available in
L. C. Piccinni, G. Stampaccia and G. Vidossich, ODEs in $\mathbb{R}^n$, problems and methods, Springer Verlag NY 1984 (p 159 ff)

Another important ref is
W. Walter: ODEs, Springer Verlag, NY 1998.

A complete proof of Kneaser's -Thm is available in the notes of my MA 417 Course (Autumn 2019) uploaded on my website
https://www.math.iitb.ac.in/~gopal/

Problems:
We shall begin with a problem on Gronwall lemma. This is from the book
J. Hale: Ordinary Differential Equations
John Wiley and Sons, 1969 (p 35)

I) Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies

$$|f(y,t)| \leq \phi(t)\,|y|$$

for all $(y,t) \in \mathbb{R}^2$.

ϕ is a non neg function in $L^1(\mathbb{R})$ so that for instance $\int_a^\infty \phi(s)\,ds < \infty$

prove

(i) If $y(t)$ is any solution of

$$\frac{dy}{dt} = f(y,t)$$

then $\sup_{t \geq a} |y(t)|$ is finite.

In particular solution exists on $[a,\infty)$

(ii) Show that $\lim_{t\to\infty} y(t)$ exists.

(iii) Demonstrate this via $f(y,t) = \frac{y}{1+t^2}$

(iv) If $\int_a^\infty |\psi(t)|\,dt < \infty$ then

Solutions of $\dot{y} = -y + \psi(t)\,y$ grow like ce^{-t}. Hint: put $y = xe^{-t}$.

The first is easy:

$$y(t) = y(a) + \int_a^t f(y(s),s)\,ds$$

$$|y(t)| \leq |y(a)| + \int_a^t \phi(s)\,|y(s)|\,ds$$

Now Gronwall lemma gives us at once

$$|y(t)| \leq |y(a)| \exp \int_a^t \phi(s)\,ds$$

$$\leq |y(a)| \exp M : \quad M = \int_a^\infty \phi(s)\,ds. ; \quad (1.55)$$

(ii) now follows from (i)

Suppose $[\alpha, \beta)$ is the Max interval of existence to the right of a then

$\beta < \infty \Rightarrow (y(t), t)$ must exit the Compact-set

$$\{(y,t) \in \mathbb{R}^2 \;/\; |y| \le |y(a)| e^{M} : a \le t \le \beta\}$$

But this will immediately contradict (1.55)

I leave (iii) - (iv) to the viewers.

(II) Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ is Continuous and satisfies

$$|f(x,t) - f(y,t)| \le \phi(t)|x-y|$$

where ϕ is unbounded in any nbd of t_0 but $\int_{t_0-h}^{t_0+h} \phi(s)\,ds < \infty$

Assume ϕ cont. on (t_0, ∞) and $(-\infty, t_0)$.

Show that the IVP $\frac{dy}{dt} = f(y,t) : y(t_0) = y_0$

has a Unique Solution.

Discuss the case $f(y,t) = \sqrt{t}\sin(t^{-1}y) : t \neq 0$
$= 0$ if $t = 0$.

The specific example is left for the viewer.

Suppose $y_1(t)$, $y_2(t)$ are both Solutions of the IVP in a nbd (t_0-h, t_0+h)

$$y_1(t) - y_2(t) = \int_{t_0'}^{t_0+h} (f(y_1(t),t) - f(y_2(t),t))\,dt + (y_1(t_0') - y_2(t_0')) : t_0' > t_0$$

Take absolute value and estimate

$$|y_1(t) - y_2(t)| \le |y_1(t_0') - y_2(t_0')| + \int_{t_0'}^{t_0+h} \phi(s)\,|y_1(s) - y_2(s)|\,ds$$

Apply Gronwall lemma

$$|y_1(t) - y_2(t)| \le |y_1(t_0') - y_2(t_0')| \exp \int_{t_0'}^{t_0+h} \phi(s)\,ds.$$

Now let $t_0' \to t_0+$ and we get (by Cont. of y_1, y_2) that

$|y_1(t) - y_2(t)| = 0$ for $t_0 < t < t_0 + h$.

(III) $\dot{x} = 3t x^{1/3}$: $x(1) = x_0 \in (0,1)$

find the largest $\delta > 0$ such that the IVP has a unique solution on $|t-1| < \delta$

What happens when $x_0 > 1$

(Hirsch-Smale: DE, dynamical systems and linear algebra. Academic press. See p177)

(IV) Let $f: \mathbb{R} \to \mathbb{R}$ be continuously differentiable

$f(a_1) = 0$: $f(a_2) = 0$; $a_1 < a_2$

and $f(x) > 0$ for $a_2 < x < \infty$

Now suppose $x(0) \in (a_1, a_2)$

then the solution of the IVP $\frac{dx}{dt} = f(x)$ with $x(0) \in (a_1, a_2)$ has life span $\mathbb{R}$

whereas if $x(0) > a_2$ the life span is $T = \int_{x(0)}^{\infty} \frac{du}{f(u)}$.

(See L.S. Pontryagin: ODEs, Addisson Wesley).

Since f is of class C^1, we have uniqueness of solutions of IVP.

Note that $x_1(t) \equiv a_1$

$x_2(t) \equiv a_2$

are both solutions of the ODE.

So for the solution $x(t)$ with $x(0) \in (a_1, a_2)$, $x(t)$ cannot exit (a_1, a_2)

Because if it exits at time $t = t_1$ from either one of the two end points

Uniqueness will be violated.

So $x(t)$ is trapped in (a_1, a_2).

So if the life span is T and $T < \infty$
$(x(t), t)$ lies in the compact set
$[a_1, a_2] \times [0, T]$ for all $0 \le t < T$
Contradiction since $(x(t), t)$ must exit Every
Compact Subset as $t \to T-$

On the other hand if $x(0) > a_2$
$\dot{x}(0) = f(x(0)) > 0$
So $x(t)$ increases at $t=0$ and it is clear
that $\dot{x}(t) > 0$ for all $t > 0$.
Thus $x(t) \ge x(0) > a_2$ for all $t > 0$.
From the ODE

$$\frac{dx}{dt} = f(x)$$

we get that $\int_{x(0)}^{x(t)} \frac{du}{f(u)} = t$

Let $[0, T)$ be the Max interval of existence
as $t \to T-$, $x(t) \to x(T)$

(i) $\int_{x(0)}^{x(T)} \frac{du}{f(u)} = T$ if $T < \infty$

Whereas (ii) $\int_{x(0)}^{x(\infty)} \frac{du}{f(u)} = +\infty$ if $T = +\infty$
$x(\infty) = \lim_{t \to \infty} x(t)$

(Exists since $x(t)$ is monotone inc.)
(possibly $x(\infty) = \infty$)

Note that in (i), $T < \infty \Rightarrow x(T) = \lim_{t \to T-} x(t) = \infty$
and life span is

$$\int_{x(0)}^{\infty} \frac{du}{f(u)}$$ as asserted.

Case (ii): $T = +\infty$. $x(\infty) = \lim_{t \to \infty} x(t)$

We show that $\lim_{t \to +\infty} x(t)$ is also $+\infty$

Suppose not.. $x(t) \to p$. as $t \to \infty$
p is a finite real number $> x(0) > a_2$.
But $\frac{dx}{dt} = f(x(t)) \to f(p)$ as $t \to \infty$

So by MVT, $\frac{dx}{dt} \to 0$ as $t \to \infty$

Well, $x(n+1) - x(n) = x'(\xi_n)$: $n < \xi_n < n+1$
LHS $\to 0$ ...
$\therefore f(p) = 0$ which is a Contradiction since $f > 0$ on (a_2, ∞)

So (ii) now reads

$$\int_a^{\infty} \frac{du}{f(u)} = +\infty$$

In both cases the assertion is proved.
We close this chapter here.
More problems can be found in the notes uploaded on my website

https://www.math.iitb.ac.in/~gopal/
Watch out for errors.
An Errata Sheet will also be uploaded and there would be periodic revisions.
We shall now move on to the Study of linear systems of ODEs.

II - Linear Systems of ODEs (Generalities)

In this chapter we shall study systems of ODEs of the form

$$\frac{dy}{dt} = A(t)y + b(t) \qquad (2.1)$$

where $A(t)$ is an $n \times n$ matrix whose entries are continuous functions defined on a fixed interval $J \subseteq \mathbb{R}$ and $b(t)$ is a column vector

$b : J \to \mathbb{R}^n$ continuous

Obviously (2.1) satisfies Lipschitz Condition on every subdomain $\mathbb{R}^n \times [t_0 - h, t_0 + h]$ (see the previous chapter)

So for $t_0 \in J$ and $y_0 \in \mathbb{R}^n$ the ODE (2.1) with Initial Conditions

$$y(t_0) = y_0 \qquad (2.2)$$

has a unique solution on a nbd of t_0.

Thm 2.1: Solutions of (2.1) - (2.2) exists on the entire interval J

proof Suppose $[t_0, \beta)$ is the maximal interval of existence to the right of t_0

We must show β is the right-end point of J. Suppose not.

Then $[t_0, \beta] \subsetneq J$

$$y(t) = y_0 + \int_{t_0}^{t} A(s)y(s)\,ds + \int_{t_0}^{t} b(s)\,ds$$

Let $m_1 = \sup_{[t_0,\beta]} \|b(s)\|$; $m_2 = \sup_{[t_0,\beta]} \|A(s)\|$

$$\therefore \|y(t)\| \leq \|y_0\| + (\beta - t_0)m_2 + m_1 \int_{t_0}^{t} \|y(s)\|\,ds$$

$$= C + m_1 \int_{t_0}^{t} \|y(s)\|\,ds \quad \text{for all} \quad t \in [t_0, \beta)$$

$$C = \|y_0\| + (\beta - t_0)m_2$$

Gronwall lemma now gives

$$\|y(t)\| \leq C \exp m_1(t - t_0) \leq C \exp m_1(\alpha - t_0)$$

$$\therefore \|y(t)\| \leq A : \quad A = C \exp m_1(\alpha - t_0).$$

So the trajectory $(y(t), t)$ is confined to stay within the compact subset

$$K = \{ (y,t) \in \mathbb{R}^n \times J \;/\; \|y\| \leq A : \; t_0 \leq t \leq \beta \}$$

as t varies over $[t_0, \beta)$

But we know there exists $t_k \in [t_0, \beta)$ s.t $(y(t_k), t_k) \notin K$. Contradiction

The theorem is proved

The associated homogeneous System:

The System $$\frac{dy}{dt} = A(t)y \qquad (2.3)$$

is called the homogeneous linear system associated to (2.1)

The following result is quite obvious

Thm 2.2: Let V be the set of all solutions of the ODE (2.3).

V is a vector space over $\mathbb{R}$

That is to say if $v_1(t)$, $v_2(t)$ are solutions of (2.3) and $c_1, c_2 \in \mathbb{R}$ then $c_1 v_1(t) + c_2 v_2(t)$ is also a solution of (2.3)

Wronskian: Suppose $y_1(t), \ldots, y_n(t)$ are n-solutions of the homogeneous DE (2.3) then

$$W(t) = \det [y_1(t), \ldots, y_n(t)] \qquad (2.4)$$

is called the Wronskian of $y_1(t), \ldots, y_n(t)$

Note that each $y_j(t)$ is a $(n \times 1)$ matrix whose entries are Cont. diff. functions on J

We now prove an important result

Thm 2.3 (The Abel-Liouville formula):
The Wronskian $W(t)$ satisfies the first order ODE

$$W'(t) = (\text{Trace } A(t))\, W(t) \qquad (2.5)$$

so that

$$W(t) = W(t_0) \exp \int_{t_0}^{t} (\text{Trace } A(s))\, ds \qquad (2.5)'$$

In particular $W(t) \equiv 0$ on J
OR $W(t)$ is NEVER zero on J.

Before we begin the proof we need two basic facts from linear algebra

(1) Suppose $v_1, \ldots, v_n \in \mathbb{R}^n$ and A is a $n \times n$ matrix then

$$\det(Av_1, v_2, \ldots, v_n) + \det(v_1, Av_2, v_3, \ldots, v_n) + \ldots + \det(v_1, v_2, \ldots, v_{n-1}, Av_n) \qquad (2.6)$$
$$= (\text{Trace } A) \det(v_1, \ldots, v_n)$$

To prove this observe that both sides of (2.6) are multilinear and alternating as a function

$$\mathbb{R}^n \times \ldots \times \mathbb{R}^n \to \mathbb{R}$$
(n factors)

So LHS of (2.6) = C (RHS of (2.6)) (2.6)'
for some constant C
Setting $v_1 = \hat{e}_1$, $v_2 = \hat{e}_2$, ..., $v_n = \hat{e}_n$

LHS of (2.6) = $a_{11} + \ldots + a_{nn}$
RHS of (2.6) = Trace A
we immediately conclude that $C = 1$ in (2.6)'
and equation (2.6) follows.

The other fact we need is that if an $(n \times n)$ matrix of functions has

Columns $C_1(t), \ldots, C_n(t)$

Then

$$\frac{d}{dt} \det(C_1(t), \ldots, C_n(t)) \qquad (2\cdot 7).$$

$$= \det(C_1'(t), C_2(t), \ldots, C_n(t)) +$$

$$\det(C_1(t), C_2'(t), C_3(t), \ldots, C_n(t)) + \ldots +$$

$$+ \det(C_1(t), C_2(t), \ldots, C_{n-1}(t), C_n'(t)).$$

Proof of the Abel Liouville formula: Using (2·7) we get at once

$$W'(t) = \det[y_1'(t), y_2(t), \ldots, y_n(t)]$$

$$+ \det[y_1(t), y_2'(t), y_3(t), \ldots, y_n(t)] + \ldots +$$

$$\ldots + \det[y_1(t), y_2(t), \ldots, y_{n-1}(t), y_n'(t)]$$

$$= \det[A(t)y_1, y_2, \ldots, y_n] + \det[y_1, Ay_2, y_3, \ldots, y_n]$$

$$+ \ldots + \det[y_1, y_2, \ldots, y_{n-1}, Ay_n]$$

$$= (\text{Trace } A(t)) \det[y_1(t), \ldots, y_n(t)]$$

$$= (\text{Trace } A(t)) W(t) \qquad (2\cdot 8)$$

Note that we have used (2·6)

Integrating (2·8) we get at once

$$W(t) = W(t_0) \exp \int_{t_0}^{t} (\text{Trace } A(s))\, ds. \qquad (2\cdot 9).$$

Case of nth order scalar ODE:

$$y^{(n)} + a_1(t) y^{(n-1)} + \ldots + a_n(t) y = 0 \qquad (2\cdot 10)$$

As usual we convert (2·10) into a System of ODEs:

Put $y_1 = y$

$y_2 = y'$, $y_3 = y''$, $\ldots$ $y_n = y^{(n-1)}$

and form the column vector

$$Y(t) = [y_1(t), y_2(t), \ldots, y_n(t)]^T$$
$$= [y(t), y'(t), \ldots, y^{(n-1)}(t)]^T$$

$$\frac{dY}{dt} = A(t)Y \; ; \quad A(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_n & -a_{n-1} & \cdots & \cdots & \cdots & -a_1 \end{bmatrix}$$

The above displayed matrix is often called the Companion matrix associated with (2.10)

$$\text{Trace } A(t) = -a_1(t)$$

and

$$W(t) = \det W(t_0) \exp -\int_{t_0}^{t} a_1(s)ds \qquad (2.11)$$

Note that if $\phi_1, \ldots, \phi_n$ are n-solution of (2.10) then the Wronskian of these is

$$\det \begin{bmatrix} \phi_1(t) & \phi_2(t) \cdots & \phi_n(t) \\ \phi_1'(t) & \phi_2'(t) \cdots & \phi_n'(t) \\ \cdots & \cdots \cdots & \cdots \\ \phi_1^{(n-1)}(t) & \phi_2^{(n-1)}(t) \cdots & \phi_n^{(n-1)}(t) \end{bmatrix} \qquad (2.11)'$$

familiar from UG courses on DEs.

Note that (2.11) is the Abel-Liouville formula for the scalar nth order ODE (2.10)

We shall now prove one of the most important results in linear ODEs namely, that the vector Space V of ALL Solutions of the homogeneous DE

$$\frac{dY}{dt} = A(t)Y$$

is an n-dimensional Vector Space.
The proof relies heavily on the principles laid down in the previous Chapter.

Thm 2.4: Consider a homogeneous linear system

$$\frac{dY}{dt} = A(t)Y \qquad (2.12)$$

$A(t)$ is an $(n \times n)$ matrix with entries as continuous functions on an interval $J \subseteq \mathbb{R}$.

(i) The space of solutions is n-dimensional

(ii) A list of solutions $\tilde{Y}_1, \ldots, \tilde{Y}_n$ of (2.12) is linearly indep (and hence forms a basis) iff

$$W(t) = \det[\tilde{Y}_1(t), \ldots, \tilde{Y}_n(t)] \neq 0 \text{ for } t \in J.$$

Proof: First we show that the space of solutions V has dimension $\geq n$

Let $Y_j(t)$ be the unique solution of IVP

$$\frac{dY}{dt} = A(t)Y \; ; \quad Y(t_0) = \hat{e}_j \qquad (2.13)$$

$\hat{e}_j$ = is the standard unit vector in $\mathbb{R}^n$

The solution Y_j exists throughout J.

$Y_1(t_0), \ldots, Y_n(t_0)$ are linearly indep.

And hence the functions $Y_1(t), \ldots, Y_n(t)$ are lin. indep

$W(t_0) = 1$ so by Abel-Liouville formula $W(t) \neq 0$ For ANY $t \in J$.

Now we show

$\text{Dim } V \leq n$

Let $Y(t)$ be any Sol. of (2.12). Evaluate at $t_0 \in J$

$$Y(t_0) = [c_1, \ldots, c_n]^T$$

Look at

$$Z(t) = Y(t) - \sum_{j=1}^{n} c_j Y_j(t) \qquad (2.14)$$

$Y_j(t)$ as above. Then, $Z(t)$ satisfies the ODE and $Z(t_0) = 0$

By Uniqueness of Sol. of IVP

We get that $Z(t) \equiv 0$ which means

$$Y(t) = \sum_{j=1}^{n} c_j Y_j(t) \quad \text{for all } t \in J$$

$\therefore Y(t) \in \mathbb{R}\text{-Span}\{Y_1(t), \ldots, Y_n(t)\}$

So $\dim V = n$ and $Y_1(t), \ldots Y_n(t)$ forms a basis for V

Next, let $\tilde{Y}_1, \tilde{Y}_2, \ldots \tilde{Y}_n$ be n solutions of the linear system of ODEs $\frac{dY}{dt} = A(t)Y$

If $\tilde{Y}_1, \ldots, \tilde{Y}_n$ are linearly dep then their Wronskian is trivially zero (columns lin. dep)

If $\tilde{Y}_1, \ldots, \tilde{Y}_n$ are linearly indep we show $W(t) \neq 0$ at an every point $t \in J$

Well, suppose not. $W(t_0) = 0$ for some $t_0 \in J$

At t_0: $\tilde{Y}_1(t_0), \ldots, \tilde{Y}_n(t_0)$ lin dep.

$\therefore c_1 \tilde{Y}_1(t_0) + \ldots + c_n \tilde{Y}_n(t_0) = 0$

for some $c_1, \ldots, c_n \in \mathbb{R}$ NOT ALL zero.

Now

$Z(t) = c_1 \tilde{Y}_1(t) + \ldots + c_n \tilde{Y}_n(t)$ satisfies the ODE and $Z(t_0) = 0$

By uniqueness, $Z(t) = 0$ for all $t \in J$

$\therefore c_1 \tilde{Y}_1(t) + \ldots + c_n \tilde{Y}_n(t) = 0$ for all $t \in J$

Contradicting lin. indep of $\tilde{Y}_1, \ldots, \tilde{Y}_n$

The theorem is completely established

Solving the Inhomogeneous Eqn.

Fundamental Matrix and the Method of Variation of parameters.

Some of these are covered in UG courses for scalar nth order Equations.

2.5: Thm (Structure of Sp. of Solutions):

As before let V be the vector space of all solutions of $\frac{dY}{dt} = A(t)Y$ (2.15)

and $P(t)$ be any one solution of the inhomogeneous equation $\frac{dY}{dt} = A(t)Y + b(t)$ (2.15)'

Then $S = \{ P(t) + Y(t) \mid Y(t) \in V \}$ (2.16)
is the set of all solutions of (2.15)'

It is clear through direct subst. that every member of S satisfies (2.15)'
Conversely let $Z(t)$ be any Sol. of (2.15)'
So that

$$\frac{d}{dt}(Z(t) - P(t)) = Z'(t) - P'(t)$$
$$= A(t)Z(t) + b(t) - (A(t)P(t) + b(t))$$
$$= A(t)(Z(t) - P(t))$$

Thus $Z(t) - P(t) \in V$
$\therefore Z(t) \in S$.

So the problem of solving (2.15)' involves TWO steps
(i) Solve (2.15) Completely. Find a basis of V
(ii) Find any one Sol. of (2.15)'

Ex: $y'' + y = 1$ (2.16)
Assoc. homogeneous eqn $y'' + y = 0$ whose solutions are all $c_1 \cos t + c_2 \sin t$
Thus V is Completely known.
By inspection $y'' + y = 1$ has the Special Solution 1 (Constant function 1)
So $S = \{ 1 + c_1 \cos t + c_2 \sin t \mid c_1, c_2 \in \mathbb{R} \}$
is the Complete Set of Solutions of (2.16)

Of course here we were lucky to be able to guess the particular solution of $y'' + y = 1$
However for the example

$$y'' + y = \tan x : \quad (-\pi/2, \pi/2)$$

the situation is not so easy to handle

The Method of Variation of parameters provides us a systematic way to handle such situations
The method was devised by J.L. Lagrange in the context of problems in Celestial Mechanics
(See H. Pollard: Celestial Mech. Carus Math. Mono. 1976)
See p49 ff the method goes by the name
Variation of elements.

Let us recapitulate this in the context of 2nd order linear ODEs

$$y'' + P(x)y' + Q(x)y = R(x) \qquad (2.17)$$

Assume that $v_1(x)$, $v_2(x)$ are two linearly indep solutions of

$$y'' + P(x)y' + Q(x)y = 0 \qquad (2.17)'$$

Idea:

(i) Seek a particular solution of (2.17) in the form $y_p(x) = u_1(x)v_1(x) + u_2(x)v_2(x)$ (2.18)
(Basic Ansatz)

Subst (2.18) into (2.17)

For making calculations easier one postulates

(ii) $$u_1'v_1 + u_2'v_2 = 0 \qquad (2.19)$$

One gets then

$$u_1'v_1' + u_2'v_2' = R(x) \qquad (2.19)'$$

Solving (2.19) - (2.19)' for u_1', u_2': integrating gives us u_1, u_2 and thereon y_p.

Two fundamental Questions:

(i) What is the Motivation for (2.18) ?
(ii) What is the interpretation of (2.19) ?

Both questions have very elegant answers
The Ansatz $y_p = u_1 v_1 + u_2 v_2$ is Motivated by Celestial Mechanics and (ii) Eqn (2.19) has a simple geometrical interpretation as a Envelope Cond.

These are explained in my Youtube Video
Basic Ordinary Differential Equations: Gopala Krishna Srinivasan, Lecture 5 (CEP 2009)
See from 35:00 onwards.

We now pass on to systems:
Suppose $Y_1(t), \ldots, Y_n(t)$ are n-linearly indep. solutions of $\frac{dY}{dt} = A(t)Y$

The Matrix $\Pi(t) = [Y_1(t), \ldots, Y_n(t)]$
is called the Fundamental Matrix
We shall assume $\Pi(t_0) = I_n$
Let us now seek a particular solution of

$$\frac{dY}{dt} = A(t)Y + b(t) \qquad (2.20)$$

in the form $z(t) = \sum c_j(t) Y_j(t)$ (2.21)

The coefficients $c_1(t), \ldots, c_n(t)$ in (2.21) have to be determined.

$$\frac{dz}{dt} = \sum c_j'(t) Y_j(t) + \sum c_j(t) Y_j'(t)$$

$$= \sum c_j'(t) Y_j(t) + A(t) \sum c_j(t) Y_j(t)$$

$$= \sum c_j'(t) Y_j(t) + A(t) z(t)$$

Since z satisfies (2.20) we get

$$\sum c_j'(t) Y_j(t) = b(t) \qquad (2.22)$$

The last equation can be written as

$$\Pi(t)\, C'(t) = b(t)$$

$C'(t) = [c_1', c_2', \ldots, c_n']^T$ Or, Since $\Pi(t)$ is invertible - its Columns are lin. indep.

$$C'(t) = \Pi(t)^{-1} b(t)$$

Integrating,

$$C(t) = \int_{t_0}^{t} \Pi(s)^{-1} b(s)\, ds$$

Whereby we get

$$Z(t) = \int_{t_0}^{t} \Pi(t)\, \Pi(s)^{-1} b(s)\, ds \qquad (2.23)$$

The matrix $\Pi(t)\,\Pi(s)^{-1}$ appearing in (2.23) is often called The transition Matrix

The Complete Solution of the inhomogeneous eqn is thus

$$\sum_j \alpha_j \gamma_j(t) + \int_{t_0}^{t} \Pi(t)\, \Pi(s)^{-1} b(s)\, ds$$

$\alpha_1, \ldots, \alpha_n \in \mathbb{R}$

Exercise:

Let us now fix the IC as Zero.

For each $B(t)$ continuous $\mathbb{R}^n$ valued function on a Compact interval $[a, b]$

The IVP

$$\frac{dy}{dt} = A(t)\, y + B(t); \quad y(t_0) = 0$$

has a unique Solution LB which is evidently a cont. function on $[a, b]$

The map $L: C[a,b] \to C[a,b]$

$B \mapsto LB$

is a linear transformation.

Show that L is a Compact linear Map

It takes bounded sets to pre compact sets

(Use Ascoli Arzela thm)

Exercise: Suppose $A(t)$ is a skew-symmetric matrix with entries being continuous real valued functions on an interval $J \subseteq \mathbb{R}$
Consider the system of ODEs

$$\frac{dy}{dt} = A(t)y$$

Let $\Pi(t)$ be the fundamental matrix for this system. Recall this means $\Pi(t_0) = Id$.
Show that $\Pi(t)$ is an orthogonal matrix for every $t \in J$.

This is important in connection with the Frenet-Serret formula in elementary differential geometry

$$\frac{d}{dt}(\Pi^T\Pi) = \dot{\Pi}^T\Pi + \Pi^T\dot{\Pi}$$
$$= (A\Pi)^T\Pi + \Pi^T(A\Pi)$$
$$= \Pi^T(-A)\Pi + \Pi^T A\Pi$$
$$= 0$$

So $\Pi^T\Pi$ is constant. When $t = t_0$
$\Pi^T\Pi = Id$
$\therefore \Pi^T\Pi = Id$ for all $t \in J$
$\therefore \Pi(t)$ is an orthogonal matrix for each $t \in J$

Periodic Systems: and Floquet theory

We shall only provide a glimpse into this theory. A detail study of these matters would have to wait for a future course.
Let us begin with a simple exercise.
Let $p : \mathbb{R} \to \mathbb{R}$ be a T-periodic cont. function.
Show that solutions of $\dot{y} = p(t)y$ are T-periodic iff $\int_0^T p(s)ds = 0$.

This exercise is really elementary. We shall discuss a profound generalization of this

Let $\phi(t)$ be any non zero solution of (2.24)
So that $\phi(t+T)$ is also a solution of (2.24)
Since the space of solutions is one dimensional

$$\phi(t+T) = C\,\phi(t) \quad \text{for all } t \in \mathbb{R}$$

for some NON-zero Constant C.

Write $C = \exp \lambda$; λ is a Complex Number
So that $\phi(t+T) = (\exp\lambda)\,\phi(t)$
Mult by $\exp(at + aT)$; a to be Chosen presently
Then

$$\phi(t+T)\exp a(t+T) = \exp(\lambda + aT)\,\phi(t)\exp at$$

So if we Choose a Such that

$$\exp(\lambda + aT) = 1 \qquad (2.26)$$

then we get

$$\phi(t+T)\exp a(t+T) = \phi(t)\exp at \qquad (2.27)$$

$a = -\lambda/T$ is the requisite Choice ($a \in \mathbb{C}$)
What we see is that

$\phi(t)$ may not be periodic but
$\phi(t)\exp(at)$ is periodic with period T

The number C above measures the extent to which the function $\phi(t)$ deviates (in a multiplicative sense) from being periodic
We shall see how the Condition (2.25) in the last slide ties up with the Abel-Liouville formula.

We shall now proceed with a Similar Study for linear Systems of ODEs:

Consider now a linear system

$$\frac{dy}{dt} = A(t)\,y \qquad (2.28)$$

where A is a matrix whose entries are continuous periodic functions on $\mathbb{R}$ with a minimal period $T > 0$

Note: Show that the periods of a continuous function $f: \mathbb{R} \to \mathbb{R}$ forms an abelian subgroup of $(\mathbb{R}, +)$ (period. group)

Show that either $f \equiv 0$ OR else the period group is cyclic. In this case this cyclic group is infinite with a unique positive generator T : T will be called the period of f.

The proof of the following is almost trivial.

Thm 2.6: For the system (2.28) with period T

Let $\Phi(t) = [y_1(t), \ldots, y_n(t)]$ (2.29)'

where $y_1, \ldots, y_n$ are linearly indep solutions

There exists a NON singular matrix C such that

$$\Phi(t+T) = \Phi(t)\,C \qquad (2.29)''$$

Further if $y_1, \ldots, y_n$ form a fund. syst. namely if Φ is a fundamental matrix (meaning $\Phi(0) = I_n$) then

$$C = \Phi(T) \qquad (2.30)$$

(2.30) is of course a trivial Conseq. of (2.29)''

proof: Observe that

$y_1(t+T), \ldots, y_n(t+T)$

are also lin. independent (Check) and so

$$y_j(t+T) = \sum_{i=1}^{n} c_{ij}\, y_i(t) \quad \forall\, t \in \mathbb{R} \qquad (2.30)$$

From elementary linear algebra, $C = [c_{ij}]$ is a non singular matrix

Thm 2.7: For the matrix $C = [c_{ij}]$ given by (2.30), we have

$$\det C = \exp \int_0^T \text{Trace}\,(A(s))\,ds \qquad (2\text{-}31)$$

We see that the exercise on page 82 for the scalar case is really a special case of Thm 2.7 Of course we could give an elementary argument here.

Proof: From Abel-Liouville formula we get

$$\det\left(\Phi(t)\right) = \det\left(\Phi(t_0)\right) \exp \int_{t_0}^{t} (\text{Trace } A(s))\,ds$$

Replace t by $t+T$ and t_0 by t

$$\det\left(\Phi(t+T)\right) = \det \Phi(t) \exp \int_t^{t+T} \text{Tr}\,(A(s))\,ds$$

$$= \det\left(\Phi(t)\right) \exp \int_0^T \text{Tr}\,(A(s))\,ds$$

thanks to the periodicity of $A(s)$ with period T.

But $\Phi(t+T) = \Phi(t)\,C$ and $\det \Phi(t) \neq 0$

The stated result follows.

Def (Monodromy Matrix and Floquet exponents) The matrix C appearing in (2.30) is called the Monodromy matrix for the given periodic system.

If ρ is an eigen value of C (obviously $\rho \neq 0$), the numbers

$$\mu = \frac{1}{T} \log \rho$$

are called characteristic exponents

or Floquet exponents. They are not unique since $\log \rho$ is multivalued
($\log \rho$ and $\log \rho + 2\pi i n$ are both logarithms of ρ : $n \in \mathbb{Z}$)

Recall that in the scalar case we wrote
$$c = \exp \lambda$$
To proceed with systems we need

Thm 2.8: Given a non singular matrix C there exists a matrix B such that
$$C = \exp B$$

(Note: $\exp B = I + B + \frac{1}{2!}B^2 + \frac{1}{3!}B^3 + \dots$ (def)
Show that the series on RHS converges absolutely)
One can prove this using Jordan Canonical forms but we shall in the next Chapter discuss a very general method for dealing with such questions.

Thm 2.9: (Floquet)
Let $\Phi(t) = [y_1(t), \dots, y_n(t)]$
where $y_1, \dots, y_n$ form a basis for the solution space of the homogeneous system $\dot{y} = A(t)y$
Then, we have a factorization
$$\Phi(t) = P(t) \exp tB \qquad (2.32)$$
for some $(n \times n)$ matrix B of constants.
(cf. discussion on p 83)
$P(t+T) = P(t)$ for all $t \in \mathbb{R}$ namely, $P(t)$ is a T-periodic matrix.
$\exp(TB) = C$ is the Monodromy matrix of the system.

Suppose we have two sets of solutions

$$\gamma_1, \ldots, \gamma_n ; \qquad z_1, \ldots, z_n$$

each set being a basis for the space of solutions.

Accordingly we have $\underline{\Phi}(t) = [\gamma_1(t), \ldots, \gamma_n(t)]$

$$\underline{\Psi}(t) = [z_1(t), \ldots, z_n(t)]$$

Now

$$\left.\begin{aligned} \underline{\Phi}(t+T) &= \underline{\Phi}(t)\, C \\ \underline{\Psi}(t+T) &= \underline{\Psi}(t)\, C' \end{aligned}\right\} \quad (2.32)'$$

Let us find a relation between C and C':

Well, since $\gamma_1, \ldots, \gamma_n$ and $z_1, \ldots, z_n$ are two bases for the n-dim vect-sp of solutions

$$\gamma_j = \sum_{i=1}^{n} q_{ij}\, z_i$$

Or, $\underline{\Phi}(t) = \underline{\Psi}(t)\, Q$; Q is non singular.

$$\underline{\Phi}(t+T) = \underline{\Psi}(t+T)\, Q.$$

Using $(2.32)'$ we get

$$\underline{\Phi}(t)\, C = \underline{\Psi}(t)\, C' Q$$

$$\therefore \quad \underline{\Psi}(t)\, QC = \underline{\Psi}(t)\, C'Q$$

$$\therefore \quad C = Q^{-1} C' Q$$

The matrices C, C' are similar so the eigen values of C and C' coincide

The eigenvalues are called Characteristic Multipliers of the periodic Sys. and they are well-defined.

Exercise: Prove (2.32) namely

Show that $\underline{\Phi}(t+T)\exp(-(t+T)B) = \underline{\Phi}(t)\exp(-tB)$

and the requisite $P(t)$ is RHS of the above.

Let us now perform a Change of Variables:

Put $y = P(t)z$ in the ODE $\dot{y} = A(t)y$

and we get

$$\dot{z} = Bz \qquad (2.33)$$

In other words we have transformed our linear System into a linear System with Constant Coefficients !

Well, $\dot{y} = \dot{P}z + P\dot{z}$

$\therefore Ay = \dot{P}z + P\dot{z}$

$\therefore APz = \dot{P}z + P\dot{z}$

$\therefore \dot{z} = P^{-1}(APz) - P^{-1}\dot{P}z$

Claim: $P^{-1}AP - P^{-1}\dot{P} = B$.

Start with $\Phi(t) = P(t)e^{tB}$

Diff with respect to t:

$A(t)\Phi(t) = \dot{P}e^{tB} + PBe^{tB}$

$\therefore A(t)Pe^{tB} = (\dot{P} + PB)e^{tB}$

Since e^{tB} non singular (proof ?)

$A(t)P = (\dot{P} + PB)$

$\therefore P^{-1}A(t)P - P^{-1}\dot{P} = B$ as asserted.

Summarizing,

Thm 2.10 : The change of variables $y = P(t)z$ transforms the ODE $\dot{y} = A(t)y$ into a Const Coeff linear System $\dot{z} = Bz$.

Now, let ρ be an eigen value of C and $\xi = [\xi_1, \ldots, \xi_n]^T$ be the corresponding eigen vector :

$C\xi = \rho\xi$

$\Phi(t)C\xi = \rho\Phi(t)\xi$

$\therefore \Phi(t+T)\xi = \rho\Phi(t)\xi$

But $\Phi(t)\xi = y(t)$ is a lin. comb. of $y_1(t), \ldots, y_n(t)$ and so is a solution of the ODE $\dot{y} = A(t)y$

and this sol. $y(t)$ satisfies:

$$y(t+T) = \rho\, y(t) \qquad (2\text{-}34)$$

Equation (2-34) is interesting since it says that after one epoch of time T

The solution $y(t)$ instead of returning back namely, instead of $y(t+T)$ and $y(t)$ agreeing, they do so upto proportionality

(Note ρ is complex) and so $y(t+T)$ and $y(t)$ are collinear with a complex factor ρ One could think of some kind of stretching and "rotation" though this should not be taken too literally!

Hill's equation: $y'' + (a + \phi(t))y = 0 \qquad (2\text{-}35)$

Where $\phi(t+\pi) = \phi(t)$ for all $t \in \mathbb{R}$

This was studied by G. W. Hill in connection with lunar perturbations. It also appears in several other contexts.

A special case is Mathieu's equation*

As usual we transform (2-35) into a first order system setting $y_1 = y$, $y_2 = y'$ so that we get the ODE

$$\dot{\gamma} = \begin{bmatrix} 0 & 1 \\ -a-\phi(t) & 0 \end{bmatrix} \gamma \; : \quad \gamma = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Let $\underline{\Phi}(t)$ be a fundamental matrix ; $\underline{\Phi}(0) = I$

then, Thm 2.7 (p 85) gives at once

$$\det C = \exp \int_0^{\pi} \text{Trace } A(s)\, ds = 1$$

(*) See E. T. Whittaker and G. N. Watson, Course of Modern Analysis, CUP.

C is the monodromy matrix.
Let $y(t)$, $z(t)$ be the solutions of the Hill's equation with the following ICs.

For $y(t)$, Initial Cond: $y(0)=1$, $y'(0)=0$
For $z(t)$, Initial Cond: $z(0)=0$, $z'(0)=1$
Thus

$$\underline{\Phi}(t) = \begin{bmatrix} y(t) & z(t) \\ y'(t) & z'(t) \end{bmatrix} \; ; \; \underline{\Phi}(0) = \underline{I}_2$$

Putting $t=0$ in the defining equation

$$\underline{\Phi}(t+T) = \underline{\Phi}(t)\, C \text{ , gives } C = \underline{\Phi}(\pi)$$

and Trace $C = y(\pi) + z'(\pi)$

Since $y(t)$, $z(t)$ depend Continuously Even analytically (and in particular Smoothly) on the parameter a,
Trace C depends analytically (and hence Smoothly) on a

Quadratic Equation for determining Char. Exponents

$$\rho^2 - (\text{Trace } C)\,\rho + 1 = 0$$

For more details on Hill's Equation we offer Two References (among Many!)

(1) J. Hale, Ordinary differential equations, John Wiley, 1969 (see p117 ff)
(This ref. was cited earlier

(2) R. Grimshaw, Nonlinear ordinary DE, Blackwell, Science Pub, 1990.
(This Could be Consulted in Conjunction with

The book of P. Glendinning cited earlier.

Let us consider a non linear autonomous ODE *

$$y' = f(y) \quad ; \quad y(0) = \alpha \qquad (2.36)$$

Solution is $y(t, \alpha)$

Assume that for $\alpha = 0$ the solution $y(t, 0)$ is periodic with period T.

We wish to examine $\left.\frac{\partial y}{\partial \alpha}\right|_{\alpha=0} = u_0(t)$

Let $\frac{\partial y}{\partial \alpha} = u(t, \alpha)$:

Diff. the ODE (2.36) (for finding variational Eq^n)

$$\frac{d}{dt}\left(\frac{\partial y}{\partial \alpha}\right) = (Df)(y(t,\alpha)) \cdot \frac{\partial y}{\partial \alpha}$$

Setting $\alpha = 0$ we get

$$\frac{d}{dt} u_0(t) = (Df)(y(t,0))\, u_0(t) \qquad (2.37)$$

But $y(t,0)$ is a known periodic solution

Call it $A(t)$

We see at once that (2.37) is precisely

$$u_0' = A(t) u_0$$

A system of linear ODEs with periodic Coefficients.

We shall close this Chapter here.

Next is the study of linear systems with Constant Coefficients.

(*) f is defined on $\mathbb{R}^n$ and is smooth

We shall assume that solutions are defined on whole of $\mathbb{R}$.

III - Linear Systems with Constant Coefficients.

In this chapter we shall study linear systems in the form

$$\frac{dy}{dt} = Ay + b(t) \qquad (3.1)$$

The coefficient matrix A is now a constant-matrix. It is useful to allow entries of A to be complex numbers.

We shall discuss the so called real Canonical form when A is real but its eigenvalues are not necessarily real.

Some Notions: $M_n(\mathbb{C})$ = set of all $(n \times n)$ matrices with entries as complex numbers.

One can regard $M_n(\mathbb{C}) = \mathbb{C}^{n^2}$ in the obvious way and for $A = [a_{ij}] \in M_n(\mathbb{C})$

$$\|A\| = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}$$

Likewise $M_n(\mathbb{R})$ = set of all $(n \times n)$ matrices with real entries.

$GL_n(\mathbb{C})$ = set of all $(n \times n)$ non singular matrices

Likewise for $GL_n(\mathbb{R})$ ...

Ex: Prove that $GL_n(\mathbb{C})$ is an open set in $M_n(\mathbb{C})$ (Likewise $GL_n(\mathbb{R})$ open in $M_n(\mathbb{R})$)

Further $GL_n(\mathbb{C})$ is dense in $M_n(\mathbb{C})$

$GL_n(\mathbb{R})$ is dense in $M_n(\mathbb{R})$

Thus $d(A,B) = \|A - B\|$, $A, B \in M_n(\mathbb{C})$ defines a metric on $M_n(\mathbb{C})$ and it makes sense to talk about Convergent Sequences and Series of matrices

Of course, an infinite series $A_1 + A_2 + A_3 + \cdots$ is by definition $\lim_{\nu \to \infty} (A_1 + A_2 + \cdots + A_\nu)$ (if it exists)

Notion of absolute conv. is clear.

$\sum_{\nu=1}^{\infty} A_\nu$ Converges absolutely if the series $\sum \|A_\nu\|$ Converges

Prove: Absolute Convergence $\Rightarrow$ Convergence.

Imp Ineq: This will be frequently used

$\|AB\| \leq \|A\| \cdot \|B\|$; $A, B \in M_n(\mathbb{C})$

prove this using Cauchy Schwarz inequality.

In particular

$\|A^k\| \leq \|A\|^k$

Thus

$$\sum_{k=0}^{\infty} \frac{1}{k!} A^k \qquad (3.2)$$

Converges absolutely !

$A^0 = I_n$ of course.

The Sum of the series (3.2) is by definition $\exp A$.

(Important)

Def:
$$\exp A = I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \cdots \qquad (3.3)$$

If $AB = BA$, prove that

$$\exp(A+B) = (\exp A)(\exp B)$$

Thus if A is a given $(n \times n)$ matrix and $s, t \in \mathbb{C}$

$$\exp((s+t)A) = (\exp sA)(\exp tA)$$

Let us now fix A, and Consider

$$\exp tA = I + tA + \frac{1}{2} t^2 A^2 + \frac{1}{3!} t^3 A^3 + \cdots \qquad (3.4)$$

Think of RHS of (3.4) as an infinite series whose terms are functions of t namely, we have a function

$\mathbb{R} \to M_n(\mathbb{C})$ given by $t \mapsto \exp tA$

Each summand is obviously diff. wrt t and

$$\frac{d}{dt}\left(I + tA + \frac{t^2}{2!}A^2 + \dots\right) \qquad (3.5)$$

$$= 0 + \frac{d}{dt}(tA) + \frac{d}{dt}\left(\frac{t^2}{2!}A^2\right) + \dots$$

$$= A + tA^2 + \frac{t^2}{2!}A^3 + \dots = A(\exp tA)$$

Exercise: Justify (3.5). Are we allowed to differentiate term by term? Consult Chapt 7 of W. Rudin's Principles of Math. Analysis

Thus

$$\frac{d}{dt}(\exp tA) = A(\exp tA) \;:\; (\exp tA)\Big|_{t=0} = I_n$$

The Columns of $\exp tA$ are Solutions of-

$$\frac{dy}{dt} = Ay$$

Well, jth col. of $(\exp tA)$ is simply

$C_j(t) = (\exp tA)\hat{e}_j$: $\hat{e}_j$ = jth standard unit vector in $\mathbb{C}^n$

$$C_j'(t) = \frac{d}{dt}(\exp tA)\hat{e}_j = A(\exp tA)\hat{e}_j$$
$$= A\,C_j(t)$$

Thus we get-

Thm 3.1: $\exp tA$ is the Fundamental Matrix for the linear system $\frac{dy}{dt} = At$

The Columns of $\exp tA$ provide us with a basis of solutions of $\frac{dy}{dt} = At$.

The jth Col. $C_j(t)$ of $\exp tA$ Satisfies the IVP: $\frac{dy}{dt} = Ay$; $y(0) = \hat{e}_j$

The Method of variation of parameters now gives us that

$$y_0(t) = \int_0^t \exp((t-s)A)\, b(s)\, ds \qquad (3.6)$$

is a particular solution of the inhomogeneous system $\frac{dy}{dt} = Ay + b(t)$ (3.7)

So it would "appear" that the study of (3.7) is more or less closed but not quite!

Main issue is how to Compute $\exp tA$ for a given matrix A. This will be our next immediate Concern.

Exercise: Suppose v is an eigen vector of A with eigen value λ then

$$\frac{d}{dt}(e^{\lambda t} v) = \lambda e^{\lambda t} v = e^{\lambda t} A v = A(e^{\lambda t} v)$$

Thus $e^{\lambda t} v$ is a solution of $y' = Ay$.

If A is diagonalizable then we get a basis of the set of all solutions of $y' = Ay$

Namely,

if $v_1, \ldots, v_n$ are n lin. indep eigen vectors of A with Corresp eigen values $\lambda_1, \ldots, \lambda_n$

Then $e^{\lambda_1 t} v_1, \ldots, e^{\lambda_n t} v_n$ is a basis of Sol.

Let $\Pi(t) = [e^{\lambda_1 t} v_1, \ldots, e^{\lambda_n t} v_n]$ (3.8)

$\Pi(0) = [v_1, \ldots, v_n]$ is non singular

Explain how to obtain a fundamental Matrix out of (3.8)

Namely at $t=0$, the fund. matrix should reduce to I_n. What we have here is

$$\Pi(0) = [v_1, \ldots, v_n].$$

Some modification is Necessary!
So the problem can be Completed in case A is diagonalizable.

What about the General case? It is customary to appeal to the Jordan Canonical Form from Linear algebra. However finding a Jordan basis is tedious.
Second, the Jordan Structure is extremely Sensitive to perturbations. We shall completely avoid ANY appeal to the Jordan Canonical Form!
Before we take this up let us look at the Scope of the result obtained for the diagonalizable Case

Recall: For an $(n \times n)$ matrix A, TFAE
(i) A is diagonalizable. That is, $\exists$ a basis $v_1, \ldots, v_n$ Consisting of eigen vectors of A

(ii) There exists a non singular matrix P such that $P^{-1}AP = \text{diag}(\lambda_1, \ldots, \lambda_n)$
$\lambda_1, \ldots, \lambda_n \in \mathbb{C}$

(iii) The Minimum polynomial has no repeated roots. Further if we want diagonalizability over $\mathbb{R}$ (i.e $v_1, \ldots, v_n \in \mathbb{R}^n$, if $A \in M_n(\mathbb{R})$ $v_1, \ldots, v_n$ lin indep. Equivalently $P \in GL_n(\mathbb{R})$ for $A \in M_n(\mathbb{R})$) then Min poly. must factorize into linear factors over $\mathbb{R}$

Equivalence of (i) and (iii) is routine
Prove the equivalence of (i) - (iii) Using Lagrange Interpolation formula!

Obviously if $A \in M_n(\mathbb{C})$ has n distinct eigenvalues then A is diagonalizable.

D_0 = set of all $(n \times n)$ matrices over $\mathbb{C}$ with distinct eigenvalues

D = set of all $(n \times n)$ matrices over $\mathbb{C}$ that are diagonalizable. Then $D_0 \subset D$

Thm 3.2: D_0 (and hence D) is diag dense in $M_n(\mathbb{C})$

So, often one can first prove an identity for the case of D_0 and use Thm 3.1 to deduce it for all of $M_n(\mathbb{C})$

Let us illustrate this simple but useful idea.

Thm 3.3: For $A \in M_n(\mathbb{C})$

$$\det(\exp A) = \exp(\text{Trace } A)$$

proof: Case (i) A is diagonalizable

$$P^{-1}AP = \text{diag}(\lambda_1, \ldots, \lambda_n) = \Lambda$$

$$\frac{1}{k!}(P^{-1}AP)^k = \frac{1}{k!}(P^{-1}A^kP) \quad \text{(Trivial)}$$

Sum from $k = 0$ to ∞

$$\therefore \exp \Lambda = P^{-1} \exp A \, P$$

Take det. on both sides:

$$\det(\exp \Lambda) = \det(\exp A)$$

But LHS $= \exp(\lambda_1 + \lambda_2 + \ldots + \lambda_n) = \exp \text{Tr } A$

Case (ii) General A.

$\psi : M_n(\mathbb{C}) \to \mathbb{C}$ and $\phi : M_n(\mathbb{C}) \to \mathbb{C}$ be given by $\psi(A) = \exp(\text{Tr } A)$, $\phi(A) = \det(\exp A)$

ϕ, ψ Both sides are continuous (Proof ?)

ϕ and ψ agree on the dense subset D_0 of $M_n(\mathbb{C})$

$\therefore \phi = \psi$. Proof is Complete.

Of course this could be derived independently.

Exercise: Use Abel-Liouville formula to prove that $\det(\exp A) = \exp(\text{Trace } A)$

Hint: $\exp tA$ is the Fund. Matrix of $\dot{y} = Ay$.

This method of reasoning - using density of D_0 in $M_n(\mathbb{C})$ is often used in ODE theory.

Holomorphic function of a matrix

We have define $e^A = \exp A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$

Suppose $f: \mathbb{C} \to \mathbb{C}$ is an entire function and assume

$$f(z) = c_0 + c_1 z + c_2 z^2 + \dots$$

is the power series representation of f

Define $f(A) = c_0 I + c_1 A + c_2 A^2 + \dots$ (39)

Thm 3.4: The series (39) Converges absolutely for each $A \in M_n(\mathbb{C})$

Immediate: $\| c_\nu A^\nu \| \leq |c_\nu| \|A\|^\nu$

and we know $\sum |c_\nu| \|A\|^\nu$ conv.

Example: $\frac{\sin A}{A} \overset{(\text{def})}{=} f(A)$; $f: \mathbb{C} \to \mathbb{C}$ is the entire function given by

$f(z) = \frac{\sin z}{z}$ if $z \neq 0$; $f(0) = 1$

$\cos A$ is obviously $I - \frac{1}{2!} A^2 + \frac{1}{4!} A^4 - \dots$

Compute $\frac{d^2}{dt^2} \cos tA$ and $\frac{d^2}{dt^2} \left(\frac{\sin tA}{A} \right)$

Examine the relevance to ODEs.

Arises in the theory of small vibrations of a coupled system.

More generally, using ideas of Unif. Conv. and differentiation one can justify term by term differentiation of a Power Series

$$\frac{d}{dt}\left(\sum_{j=0}^{\infty} c_j t^j A^j\right) = \sum_{j=1}^{\infty} c_j j t^{j-1} A^j$$

(Note: A is fixed here)

Exercise: Discuss this in detail.

It is of interest to define $f(A)$ when $f : \Omega \to \mathbb{C}$ is a holomorphic function defined on a connected open set in $\mathbb{C}$. The problem needs to be handled with care.

For example: $\Omega = \mathbb{C} - \{ t \in \mathbb{R} \mid t \leq 0 \}$

$f(z) = \log z$ is holomorphic on Ω

$f(A)$ will make sense if A is non singular.

The definition of $f(A)$ must satisfy the following obvious requirements

(i) When f is a polynomial, $f(A)$ has the obvious meaning

(ii) More generally, if f is an entire function, $f(A)$ is given by (3.9)

(iii) If $f, g : \Omega \to \mathbb{C}$ are holomorphic we need

$$(f+g)(A) = f(A) + g(A)$$
$$(fg)(A) = f(A)g(A)$$

Set $H(\Omega)$ of all holomorphic functions on Ω forms an algebra.

To each member of $H(\Omega)$ we are assigning a member of $M_n(\mathbb{C})$ thereby arriving at a map

$$H(\Omega) \xrightarrow{\Phi_A} M_n(\mathbb{C})$$

This map Φ_A (for fixed A) must be an

algebra homomorphism

In particular, if $f \in H(\Omega)$ and $f(z) \neq 0$ for $z \in \Omega$ then $\frac{1}{f} \in H(\Omega)$

Property (iii) gives $f(A) \cdot \left(\frac{1}{f}\right)(A) = I$

$\therefore$ $f(A)$ is nonsingular : $(f(A))^{-1} = \left(\frac{1}{f}\right)(A)$

So if in particular $f \in H(\Omega)$ and is a rational functions (poles outside Ω) then $f(A)$ has its usual meaning.

Now $H(\Omega)$ and $M_n(\mathbb{C})$ are also metric spaces. The relevant metric on $H(\Omega)$ is one for which

$f_\nu \to f$ iff f_ν conv. unif to f on Compact subsets of Ω.

We shall not explicitly describe this metric See standard textbooks such as

W. Rudin: Real and Complex Analysis

J. B. Conway: Functions of One Complex Variable.

The property described above is the one we shall use. We demand that the map $\Phi_A : H(\Omega) \to M_n(\mathbb{C})$ we are trying to construct be Continuous. (!)

Let $\Omega = \mathbb{C} - \{ t \in \mathbb{R} \,/\, t \leq 0 \}$

$f(z) = \log z$ holomorphic on Ω

$$\sum_{k=0}^{N} \frac{1}{k!} (f(z))^k \to z$$ unif. on Compact subsets of Ω

This implies

$$\sum_{k=0}^{N} \frac{1}{k!} (f(A))^k \to A \text{ in } M_n(\mathbb{C})$$

i.e $\exp f(A) = A$

$\therefore$ $\exp(\text{Trace } f(A)) = \det A$

$\Rightarrow \det A \neq 0$.

We see that if A is singular then $\log A$ cannot be defined in any meaningful manner

In the last chapter, in connection with Floquet-theory, we encountered $\log C$: C is the monodromy matrix (thm 2.8 p 86)

We shall see that if A is non singular $\log A$ is defined (and by the argument above) it is the a matrix that satisfies

$$\exp(\log A) = A.$$

More generally, let $\sigma(A)$ = set of eigen-values of A (the spectrum of A)

If Ω contains $\sigma(A)$ and $f: \Omega \to \mathbb{C}$ is holomorphic we can define $f(A)$ so as to get a continuous algebra homomorphism

$$\phi_A : H(\Omega) \to M_n(\mathbb{C}) : \phi_A(f) = f(A)$$

and $f(A)$ has the usual meaning when f is an entire function.

The algebra homomorphism above is uniquely determined by the properties (i) - (iii) and continuity

Because $\phi_A(f)$ is then uniquely determined for all rational functions f whose poles lie outside Ω.

And now, by a theorem of Runge (see W. Rudin: Real and Complex analysis)

Any $f \in H(\Omega)$ is a limit of a sequence of rational functions with poles outside Ω

The continuity of ϕ_A then implies uniqueness.

Before turning to existence, let us make two simple Observations:

(1) If $f(z) = c_0 + c_1 z + \dots + c_k z^k$; $c_k \in \mathbb{C}$ is a polynomial then

$$f(A) = c_0 I + c_1 A + \dots + c_k A^k$$

and for two polynomials f, g

$$f(A)g(A) = g(A)f(A).$$

(2) Suppose $f(z)$ is a polynomial - assume monic for simplicity and factorize:

$$f(z) = (z - \xi_1) \dots (z - \xi_k); \quad \xi_1, \dots \xi_k \in \mathbb{C}$$

(not necessarily distinct.

$$f(A) = (A - \xi_1 I) \dots (A - \xi_k I)$$

$\det f(A) = 0 \Rightarrow \det(A - \xi_j I) = 0$ for atleast one j

$\Rightarrow \xi_j$ is an eigenvalue of A.

Cor 3.5: If f is a polynomial s.t $f(z) \neq 0$ for $z \in \Omega$ and $\sigma(A) \subset \Omega$ then, $f(A)$ is nonsingular

Define: $\left(\frac{1}{f}\right)(A) = (f(A))^{-1}$ and

$$\left(\frac{g}{f}\right)(A) = g(A)(f(A))^{-1}$$ for polynomials f, g with $g(z) \neq 0$ when $z \in \Omega$

So for each rational function $f : \Omega \to \mathbb{C}$ with poles lying outside Ω, $f(A)$ is defined

The set of all rational functions whose poles lie outside Ω forms an algebra $Q(\Omega) \subset H(\Omega)$ (Metric Sp)

The map $Q(\Omega) \to M_n(\mathbb{C})$ (3.10)

$f \mapsto f(A)$

is an algebra homomorphism.

Thm 3.6: The map (3.10) is continuous

Lemma 3.7: If $f(z)$ is a polynomial

$$f(A) = \frac{1}{2\pi i} \int_C f(\xi)(\xi I - A)^{-1} d\xi \qquad (3.11)$$

where C is the union of circles $C_1, \ldots, C_K$ each C_j traced counter clockwise. Each C_j encloses an eigen value λ_j.

$\lambda_1, \ldots, \lambda_K$ is the list of all distinct eigen values of A.

To see this, it suffices to take

$$f(z) = z^\ell \quad : \quad \ell \in \mathbb{N} \cup \{0\}.$$

$$RHS = (2\pi i)^{-1} \int_C \xi^{\ell-1} \left(I - \frac{A}{\xi}\right)^{-1} d\xi.$$

Now, by Cauchy's thm we can replace C by a large circle $\{|z| = R\}$: $R >> 1$

$$\left(I - \frac{A}{\xi}\right)^{-1} = I + \frac{A}{\xi} + \frac{A^2}{\xi^2} + \cdots$$

Since $|\xi| >> 1$ along the circle series Conv. Unif. Term by term integration now gives the result A^ℓ

Exercise: Supply the details.

Now we insert another important but easy Exercise The resolvant identity:

$$(\xi I - A)^{-1} - (\lambda I - A)^{-1} = -(\xi - \lambda)(\xi I - A)^{-1}(\lambda I - A)^{-1} \qquad (3.12)$$

ξ, λ not in $\sigma(A)$.

prove this. Mult both sides by $(\xi I - A)(\lambda I - A)$ and observe that they Commute.

We are ready to generalize lemma 3.7 to the following

Exercise: Use the Lemma 3.7 to prove the Cayley Hamilton-thm (!)
We have a proof of a result in linear alg. using Complex analysis (!!)

Thm 3.8: If $h = G(\Omega)$ namely, h is a rational functions with poles outside Ω then

$$h(A) = (2\pi i)^{-1} \int_C h(\xi)(\xi I - A)^{-1} d\xi \qquad (3.13)$$

C is exactly as in lemma 3.7 except that the disks D_j with boundary D_j equals C_j
(the disks $D_1, \ldots, D_k$ are pairwise disjoint)

We have to prove that-

$$(RHS)\ g(A) = f(A)$$

But since g is a polynomial, we can use lemma 3.7

$$(RHS)\ g(A) = (RHS) \cdot (2\pi i)^{-1} \int_{C'} g(\xi)(\xi I - A)^{-1} d\xi \qquad (3.14)$$

C' is the union of Circles $C_1', \ldots, C_k'$
C_j' Concentric with C_j
D_j' = disk with bdy C_j' ; $D_1', \ldots, D_k'$ all lie inside Ω and are pairwise disjoint-
C_j lies insides C_j' for $j = 1, 2, \ldots, k$
(Figure in next page)

So, (3.14) reads

$$(2\pi i)^{-2} \int_C h(\lambda)(\lambda I - A)^{-1} d\lambda \int_{C'} g(\xi)(\xi I - A)^{-1} d\xi \qquad (3.15)$$

(λ runs over C) (ξ runs over C')

Use the resolvant identity and (3.15) becomes

$$(2\pi i)^{-2} \int_C \int_{C'} -h(\lambda) g(\xi)(\xi - \lambda)^{-1} \left((\xi I - A)^{-1} - (\lambda I - A)^{-1}\right) d\lambda d\xi. \qquad (3.16)$$

Now,

We integrate $\int_{C_j} \int_{C_j'} h(\lambda) g(\xi) (\xi - \lambda)^{-1} (\xi I - A)^{-1} \, d\lambda \, d\xi$

first wrt λ and then wrt ξ

And $\int_{C_j} \int_{C_j'} h(\lambda) g(\xi) (\xi - \lambda)^{-1} (\lambda I - A)^{-1} \, d\lambda \, d\xi$ is first carried out wrt ξ and then wrt λ

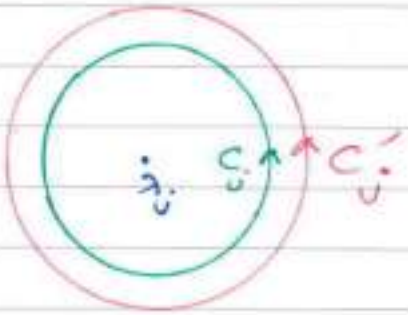
We get in the first case

$\int_{C_j'} \cdots d\xi \int_{C_j} \frac{h(\lambda)}{\xi - \lambda} d\lambda = 0$ since h is holo. on Ω, ξ lies OUTSIDE C

(Recall ξ varies of C_j' and C_j is inside C_j')

In the second case

$\int_{C_j} \cdots d\lambda \int_{C_j'} \cdots$



In the second case we have

$$(2\pi i)^{-2} \int_{C_j} h(\lambda) (\lambda I - A)^{-1} d\lambda \int_{C_j'} g(\xi) (\xi - \lambda)^{-1} d\xi$$

$$= (2\pi i)^{-1} \int_{C_j} h(\lambda) (\lambda I - A)^{-1} (-g(\lambda)) \, d\lambda$$

$$= -(2\pi i)^{-1} \int_{C_j} f(\lambda) (\lambda I - A)^{-1} = -f(A)$$

Note that we also have an extra $-$ sign in (3.16) which gives $f(A)$. Proof is Complete.

Thm (3.6) namely Continuity of the map
$h(z) \mapsto h(A)$ follows at once now
$Q(\Omega) \to M_n(\mathbb{C})$
In fact we have slightly more

Thm 3.6′ : Suppose $(h_m(z))$ is Cauchy in $Q(\Omega)$ then $(h_m(A))$ is Cauchy in $M_n(\mathbb{C})$
proofs of both are similar
Well, thanks to Thm 3.8

$$\| h_m(A) - h_p(A) \| \leq \frac{1}{2\pi} \int_C \sup | h_m - h_p | \cdot \| (\xi I - A)^{-1} \| \, d\xi$$

Sup is taken over $D_1 \cup \cdots \cup D_k$ Compact Subset of Ω. The result follows at once.

We now employ the celebrated Runge approx thm
Given any $h(z) \in H(\Omega)$
there is a sequence $(h_m(z))$ in $Q(\Omega)$
Converging to $h(z)$ unif. on each Compact Subset of Ω.

For each $h(z)$, select one such $(h_m(z))$
$(h_m(z))$ is unif. Cauchy on each Compact subset of Ω.
$\therefore$ $(h_m(A))$ is Cauchy in $M_n(\mathbb{C})$ and hence Converges to say $h(A)$
Assign $h(A)$ to $h(z)$ thereby Obtaining

$$\phi_A : H(\Omega) \to M_n(\mathbb{C})$$
$$h(z) \mapsto h(A)$$

This ϕ_A is then the desired Map.
a Continuous algebra homomorphism
s.t $\phi_A(f)$ has the usual Meaning when

f is a polynomial.

A small detail Remains.
Suppose $(h_m'(\xi))$ and $(h_m''(\xi))$ are two

Sequences in $Q(\Omega)$ both of which Converge Unif on Compact Sets to $h(\xi)$

How can we be certain that the Corresp. Sequences

$(h_m'(A))$ and $(h_m''(A))$ Conv. to the Same limit in $M_n(\mathbb{C})$?

Recall the interlacing argument employed in Chapt I. We shall use it again

Interlace: $h_1', h_1'', h_2', h_2'', \ldots$ (3.17)
(3.17) Conv. Unif. on Compact Subsets so the image Sequence

$h_1'(A), h_1''(A), h_2'(A), h_2''(A), \ldots$ (3.17)'
Conv. in $M_n(\mathbb{C})$

Its two subsequences

$h_1'(A), h_2'(A), h_3'(A), \ldots$
$h_1''(A), h_2''(A), h_3''(A), \ldots$

must then Converge to the SAME limit and the matter is Settled.

The map Φ_A is well defined.

Remark: What we have established here is a Special Case of the holomorphic functional Calculus that is Central result in Functional Analysis It has far reaching Consequences and is EXTREMELY imp in Differential equations.

Ref.: R. D. Richtmyer,
Principles of Advanced Mathematical Physics, vol I, Springer Verlag 1978

Now that we have proved existence and uniqueness of the functional Calculus, we turn to the problem of Computing $f(A)$ for a given $f \in H(\Omega)$ and $(n \times n)$ matrix A with $\sigma(A) \subset \Omega$.

We shall assume that the Min. polynomial of f is known.

Let $m_A(z)$ or $m(z)$ denote the Min. poly of A
In general A will not be diagonalizable
so $m(z)$ would have repeated roots:

$$m(z) = (z-\lambda_1)^{k_1} \cdots (z-\lambda_\ell)^{k_\ell}$$

$\lambda_1, \ldots, \lambda_\ell$ distinct eigen values of A
$k_1, \ldots, k_\ell$ their Multiplicities as roots of the Min poly.

The symbol Σ_A denotes the data $\begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_\ell \\ k_1 & k_2 & \cdots & k_\ell \end{bmatrix}$

first row displays eigen values
2nd row displays Multiplicities - as roots of $m(z)$

Suppose f, g are holomorphic functions on Ω
such that $m(z)$ divides $f-g$
namely $f(z) - g(z) = m(z)\,h(z)$: $h(z)$ holomorphic
then $f(A) = g(A)$

Now $m(z)$ divides $f-g$ iff

$$f^{(j)}(\lambda_i) = g^{(j)}(\lambda_i) \quad \text{for } 0 \le j \le k_i - 1$$
$$\text{and } i = 1, 2, \ldots, \ell$$

We shall construct a polynomial $\Lambda_f(z)$ such that

(i) $\deg \Lambda_f \leq \deg m - 1$ ($m(z)$ = Min Poly)

(ii) $f^{(j)}(\lambda_i) = \Lambda_f^{(j)}(\lambda_i)$: $0 \leq j \leq k_i - 1$
$i = 1, 2, \ldots \ell$

It follows at once that such a Λ_f must be unique.

If $\tilde{\Lambda}_f$ is another then $m(z)$ must divide $\Lambda_f(z) - \tilde{\Lambda}_f(z)$ and the fact that the degrees of Λ_f, $\tilde{\Lambda}_f$ are strictly less than $\deg m$ we get

$$\Lambda_f(z) = \tilde{\Lambda}_f(z).$$

This unique $\Lambda_f(z)$ is called the Lagrange Sylvester interpolating polynomial

We now define $f(A)$:

Def: $f(A) = \Lambda_f(A)$: $\Lambda_f(z)$ = Lagrange-Sylvester interpolating poly.

We must check:

(i) When f is a polynomial, this def agrees with the earlier def

(ii) The map $f \mapsto \Lambda_f(A)$ is an algebra homomorphism $H(\Omega) \to M_n(\mathbb{C})$

The first is easy:

By division algorithm, $f(z) = m(z)q(z) + r(z)$
$\deg r \leq \deg m - 1$

Now $f^{(j)}(\lambda_i) = \left(\frac{d}{dz}\right)^{(j)} (m(z)q(z))\Big|_{z=\lambda_i} + r^{(j)}(\lambda_i)$ (3.18)

Since λ_i is a root of $m(z)q(z)$ of Mult k_i the term $\left(\frac{d}{dz}\right)^{(j)} (m(z)q(z))\Big|_{z=\lambda_i} = 0$

Thus (3.18) holds for $0 \leq j \leq k_i - 1$; $i = 1, 2, \ldots, \ell$

$f(A) = r(A) = \Lambda_f(A)$ as desired

We shall now
(i) Explicitly describe the Construction of $\Lambda_f(z)$
(ii) Prove that the map

$$H(\Omega) \to M_n(\mathbb{C}) \qquad (3.19)$$
$$f \longmapsto \Lambda_f(A)$$

is a ring homomorphism
(iii) is Continuous
Finally,
Look at some Concrete examples

Define $m_i(z) = \dfrac{m(z)}{(z-\lambda_i)^{k_i}}$; $i = 1, 2, \ldots \ell$

Notice that $m_i(z)$ depend only on the matrix A and this is fixed throughout the discussion $\sigma(A) \subset \Omega$.

It is not difficult to Guess the form of $\Lambda_f(z)$:
It is

$$\Lambda_f(z) = \sum_{i=1}^{\ell} \left(a_0^i + a_1^i (z-\lambda_i) + \cdots + a_{k_i-1}^i (z-\lambda_i)^{k_i-1} \right) m_i(z)$$

(3.20)

Let us put $z = \lambda_i$. Note that $m_j(\lambda_i) = 0$ if $j \neq i$

So $\Lambda_f(\lambda_i) = a_0^i m_i(\lambda_i)$: $m_i(\lambda_i) \neq 0$

By def of Λ_f we must have $\Lambda_f(\lambda_i) = f(\lambda_i)$

So that a_0^i is Uniquely determined !
Now let us differentiate (3.20) once and then set $z = \lambda_i$
The defining relation $\Lambda_f'(\lambda_i) = f'(\lambda_i)$
must of course be used.

$$f'(\lambda_i) = a_1^i m_i(\lambda_i) + a_0^i m_i'(\lambda_i)$$

This uniquely gives us a_1^i

Differentiate (3.20) twice and then set $z = \lambda_i$
Use the defining relation $f''(\lambda_i) = \Lambda_f''(\lambda_i)$
we get

$$f''(\lambda_i) = 2a_2^i m_i(\lambda_i) + 2a_1^i m_i'(\lambda_i) + a_0^i m_i''(\lambda_i)$$

This uniquely determines a_2^i etc;
So all the coefficients

$$a_0^i, a_1^i, \ldots, a_{k_i-1}^i \quad ; i = 1, 2, \ldots, l$$

are sucessively computed and the existence as well as an algorithm for $\Lambda_f(z)$ have been delineated.

Since dependence of f appears only through the coefficients $a_0^i, \ldots, a_{k_i-1}^i$ it is clear that the map

$$f \mapsto \Lambda_f \text{ is } \mathbb{C}\text{-linear.}$$

and hence

$$f \mapsto \Lambda_f(A) \text{ is } \mathbb{C}\text{-linear.}$$

We shall now prove

Claim: $\Lambda_{fg}(z) - \Lambda_f(z)\Lambda_g(z)$ is

divisible by $m(z)$ whereby

$$\Lambda_{fg}(A) = \Lambda_f(A)\Lambda_g(A)$$

The map

$$f \mapsto \Lambda_f(A) : H(\Omega) \to M_n(\mathbb{C})$$

is a ring homomorphism.
Recall that if f, g are holo. functions such that

$$f^{(j)}(\lambda_i) = g^{(j)}(\lambda_i) : 0 \leq j \leq k_i - 1 : i = 1, 2, \ldots, l$$

then $(f - g) = m \cdot h$ for some holo. function h
We shall use this.

By def, $f^{(i)}(\lambda_i) = \Lambda_f^{(i)}(\lambda_i)$
for $0 \le i \le k_i - 1$: $i = 1, 2, \ldots, \ell$.

Now $f - \Lambda_f = m h_1$
$g - \Lambda_g = m h_2$ and
$fg - \Lambda_{fg} = m h_3$ for some $h_1, h_2, h_3 \in H(\Omega)$

$\therefore \Lambda_{fg} - \Lambda_f \Lambda_g$

$= \Lambda_{fg} - fg + fg - \Lambda_f \Lambda_g$
$= m h_3 + fg - g\Lambda_f + g\Lambda_f - \Lambda_f \Lambda_g$
$= m \cdot h_3 + g m h_1 + \Lambda_f \cdot m h_2$

$= m \cdot (h_3 + g h_1 + h_2 \Lambda_f) = m h_4$
Where $h_4 \in H(\Omega)$
RHS is a holomorphic function which has a zero at each λ_j with Mult. atleast k_j.
$(j = 1, 2, \ldots, \ell)$
So does LHS.
$\therefore$ Each root of $m(z)$ is also a root of LHS of equal or higher multiplicity
$\therefore$ $m(z)$ clivides $\Lambda_{fg} - \Lambda_f \Lambda_g$
Claim is established.

To prove Continuity of the map $H(\Omega) \to M_n(\mathbb{C})$
$f \mapsto \Lambda_f(\Omega)$
It suffices to show that each of the functions $f \mapsto a_j^i$: $0 \le i \le k_j - 1$
$H(\Omega) \to \mathbb{C}$ $j = 1, 2, \ldots, \ell$
is continuous.

First,

Lemma 3.9: Fix $p \in \Omega$ and $k \in \mathbb{N}$
the map $H(\Omega) \to \mathbb{C}$ given by $f \mapsto f^{(k)}(p)$ is continuous.

Follows at once from Cauchy's formula:
Let D be a closed disc centred at p such that
$$D \subset \Omega$$
Then
$$f^{(k)}(p) = \frac{k!}{2\pi i} \int_C \frac{f(\xi)\,d\xi}{(\xi - p)^{k+1}}$$
Now
$$|f_\nu^{(k)}(p) - f^{(k)}(p)| \leq \frac{k!}{2\pi} \int_0^{2\pi} \frac{|f_\nu(\xi) - f(\xi)|\,|d\xi|}{R^{k+1}} \qquad (3.21)$$
R = radius of the disc and $C = \partial D$ traced counter clockwise
$f_\nu \to f$ in $H(\Omega) \Rightarrow f_\nu \to f$ unif on each compact subset of Ω and in particular on $\partial D = C$
(3.21) now gives us continuity of the map
$$f \mapsto f^{(k)}(p)$$
$$H(\Omega) \to \mathbb{C}.$$
Now returning back to our map $f \mapsto \Lambda_f(A)$
To prove that the coeff maps
$$f \mapsto a_j^i \quad : \quad 0 \leq j \leq k_i - 1 \; ; \quad i = 1, 2, \ldots, l$$
are continuous,
Begin with the simplest case: $j = 0$
we know $a_0^i = (m_i(\lambda_i))^{-1} f(\lambda_i)$ (See p 110)
Continuity follows from Lemma 3.9
Next, take up a_1^i. We know
$$a_1^i = m_i(\lambda_i)^{-1} \left(f'(\lambda_i) - a_0^i m_i'(\lambda_i) \right)$$
from which continuity of a_1^i follows etc.,

The Construction of $\Lambda_f(Z)$ is completed.
The proof of the fact that $f \mapsto \Lambda_f(A)$ is a Cont. ring homomorphism is completed.

Let us summarize the work done so far:

(I) There exists a unique map $H(\Omega) \xrightarrow{\Phi_A} M_n(\mathbb{C})$
(A is a given matrix with $\sigma(A) \subset \Omega$)
such that

(i) $\Phi_A(f) = f(A)$ (usual meaning) when f is a polynomial

(ii) Φ_A is a ring homomorphism

(iii) Φ_A is continuous

In particular

$\Phi_A(f) = \wedge_f(A)$ where $\wedge_f(z)$ is the Lagrange Sylvester interpolating polynomial

$$\wedge_f(z) = \sum_{i=1}^{\ell} \left(a_0^i + a_1^i (z-\lambda_i) + \dots + a_{k_i-1}^i (z-\lambda_i)^{k_i-1} \right) m_i(z) \qquad (3.22)$$

$$a_0^i = f(\lambda_i) / m_i(\lambda_i)$$

$$a_1^i = m_i(\lambda_i)^{-1} \left(f'(\lambda_i) - a_0^i m_i'(\lambda_i) \right) \text{ etc;}$$

In short a_j^i depend linearly on f and it would be more logical to denote it by $a_j^i(f)$

We have avoided the clutter !

Let us club $m_i(z)$ with $(z-\lambda_i)^j$ and write (3.22) as (3.23)

$$\wedge_f(z) = \sum_{i=1}^{\ell} \left(f(\lambda_i) \phi_{i,0} + \dots + f^{(k_i-1)}(\lambda_i) \phi_{i,k_i-1} \right)$$

$\phi_{i,0}, \dots, \phi_{i,k_i-1}$ are some polynomials in z that depend only on A. Degree of all of them $\leq \deg m - 1$

$$\text{Deg } \phi_{i,j} = (\deg m) - k_i + j$$

$0 \leq j \leq k_i - 1 ; \quad i = 1, 2, \dots \ell$

We shall now discuss an efficient way to find these polynomials $\phi_{ij}(z)$ through specific choices of f in (3.23)

Also, with $z = A$ (3.24)

$$f(A) = \Lambda_f(A) = \sum_{i=1}^{l} \left(f(\lambda_i) E_{i,0} + f'(\lambda_i) E_{i,1} + \dots + f^{(k_i-1)}(\lambda_i) E_{i,k_i-1} \right)$$

The matrices E_{ij} (Known as Constituent matrices) are determined by A and are indep of f

We insert here the IMPORTANT

REFERENCE: F. Gantmacher, Theory of Matrices Vol I, AMS Chelsea, 2000

See p101 ff.

We now take up specific examples

The first example is really easy

Ex 1:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad ; \quad A^4 = 0$$

Min polynomial $m(z) = z^4$

Only one eigen value $\lambda_1 = 0$: Mult = 4

So (3.23) reads:

$$\Lambda_f(z) = f(0)\phi_{1,0} + f'(0)\phi_{1,1} + f''(0)\phi_{1,2} + f'''(0)\phi_{1,3}.$$

Taking $f(z) = 1$ we get (since $\Lambda_f = f$ in case f is a polynomial of deg $\leq$ deg m-1)

$$1 = \phi_{1,0} + 0 + 0 + 0 \quad \therefore \quad \phi_{1,0} = 1$$

Take $f(z) = z$

$$z = 0 + \phi_{1,1}(z)$$

Take $f(z) = z^2$ and this gives

$$z^2 = 2\phi_{1,2}$$ and finally with

$f(z) = z^3$ we get $z^3 = 6\phi_{1,3}$

$\therefore \Lambda_f(z) = f(0) + f'(0)z + f''(0)\frac{z^2}{2} + f'''(0)\frac{z^3}{3!}$

So if $f(z) = \exp tz$:

$$\exp tA = I + tA + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!}$$

If $f(z) = \sin z$ we get-

$\sin A = A - \frac{1}{6}A^3$ as expected.

Next let us take up.

Ex 2: $A = \begin{bmatrix} 1 & 1 & -2 & 0 \\ 2 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 2 & 1 \end{bmatrix}$

(Matrix appears in the book of H.D. Ikramov: Problems in linear algebra, Ex 6.4.44 (Mir Publishers) 1983)

Check $(A-I)^2 = 0$

So that $m(z) = (z-1)^2$ (Min poly)

Only one eigen value $\lambda_1 = 1$: multiplicity 2.

So (3.23) reads

$$\Lambda_f(z) = f(1)\phi_{1,0} + f'(1)\phi_{1,1}$$

and

$$f(A) = f(1)E_{1,0} + f'(1)E_{1,1}.$$

Take $f(z) = 1$ and we get

$I = E_{1,0}$

Take $f(z) = z-1$: $f(1) = 0$: $f'(1) = 1$

$A - I = E_{1,1}$

$\therefore f(A) = f(1)I + f'(1)(A-I)$

In particular taking $f(z) = \exp tz$

$$\exp tA = e^t I + te^t(A-I)$$

No Need to deal with Jordan Canonical Form!

Example: Consider the 5×5 matrix

$$A = \begin{bmatrix} 7 & -2 & 1 & 2 & -3 \\ 8 & -2 & 2 & 3 & -4 \\ 6 & -2 & 2 & 2 & -3 \\ -8 & 3 & -1 & -1 & 4 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Min polynomial $(x-1)^3(x-2)^2 = m(x)$

Let us now consider the problem of solving the IVP

$$\frac{dy}{dt} = Ay; \quad y(0) = [1 \quad -1 \quad 0 \quad 2 \quad 1]^T$$

We need to compute e^{tA} and the solution is $e^{tA}[1 \quad -1 \quad 0 \quad 2 \quad 1]^T$

In this case the Lagrange Sylvester interpolation polynomial $\wedge_f(x)$ assumes the form

(3.25)

$$\wedge_f(x) = f(1)\phi_{1,0}(x) + f'(1)\phi_{1,1}(x) + f''(1)\phi_{1,2}(x) + f(2)\phi_{2,0}(x) + f'(2)\phi_{2,1}(x)$$

$$f(A) = f(1)E_{1,0} + f'(1)E_{1,1} + f''(1)E_{1,2} + f(2)E_{20} + f'(2)E_{21}. \quad (3.26)$$

Taking $f(z) = z-1$ we get

$$A - I = E_{1,1} + E_{2,0} + E_{21} \quad (3.27)$$

Take $f(z) = (z-2), (z-2)^2$ in succession gives

$$A - 2I = -E_{1,0} + E_{1,1} + E_{21} \quad (3.27)'$$

$$(A-2I)^2 = E_{1,0} + 2(-1)E_{1,1} + 2E_{1,2} \quad (3.27)''$$

(No need to take $f(x) = 1$ since $1 = (x-1) - (x-2)$ and so can be recovered from (3.27) and (3.27)'.

One must take 5 linearly indep choices of the polynomial $f(z)$
Taking $f(z) = (z-2)^3$ would now give

$$(A-2I)^3 = -E_{1,0} + 3E_{1,1} - 6E_{1,2} \quad (3.27)^{(3)}$$

Finally $f(z) = (z-2)^4$ gives

$$(A-2I)^4 = E_{1,0} - 4E_{1,1} + 12E_{1,2} \quad (3.27)^{(4)}$$

Note that with $f(z) = (z-2)^2$, $(z-2)^3$ and $(z-2)^4$ the equations $(3.27)''$, $(3.27)^{(3)}$ and $(3.27)^{(4)}$ do not involve $E_{2,0}$ and $E_{2,1}$
and these three equations can be solved for $E_{1,0}$, $E_{1,1}$ and $E_{1,2}$ (Check)
$E_{2,0}$ and $E_{2,1}$ can then be found from (3.27)-$(3.27)'$. Complete the Job.

Note: Packages like MATHEMATICA can directly produce the minimal polynomial.
Use that to compute $(A-2I)^3$ etc.

Exercise Let $C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2\pi \\ 0 & 2\pi & 0 \end{bmatrix}$

Compute $\exp C$
Is the function $\exp : M_n(\mathbb{R}) \to GL_n(\mathbb{R})$ injective? Ans: NO.

The SN decomposition: This is an extremely imp result in MANY parts of mathematics specially in diff equations, dynamical systems, Lie groups and Lie algebras.
See for example the book of-

M. W. Hirsch and S. Smale, Differential Equations dynamical systems and linear algebra Academic Press 1974. See p110 ff and also Appendix III (p 331 ff)
Main-Theorem appears on p 333

However we shall prove this using Cauchy's Thm in Complex analysis much in the spirit of this Chapter.

We shall use the formula (Lemma 3.7, p103, formula (3.111))

$$f(z) = (2\pi i)^{-1} \int_C f(\xi)(\xi I - A)^{-1} d\xi$$

C as described on p103.

Taking $f(z) = z$ we get $(C = C_1 \cup \dots \cup C_\ell)$

$$A = \sum_{j=1}^{\ell} (2\pi i)^{-1} \int_{C_j} ((\xi - \lambda_j) + \lambda_j)(\xi I - A)^{-1} d\xi$$

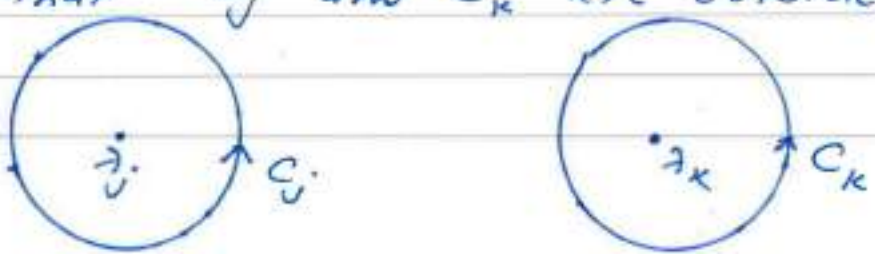
$$A = \sum_{j=1}^{\ell} (N_j + \lambda S_j) \qquad (3.28)$$

Where $N_j = (2\pi i)^{-1} \int_{C_j} (\xi - \lambda_j)(\xi I - A)^{-1} d\xi$ and (3.29)

$$S_j = (2\pi i)^{-1} \int_{C_j} (\xi I - A)^{-1} d\xi$$

Let us now prove $N_j \cdot N_k = 0$ if $j \neq k$
$S_j \cdot S_k = 0$ if $j \neq k$

Recall that C_j and C_k lie outside of each other



$$N_j N_k = (2\pi i)^{-2} \int_{C_j} \int_{C_k} (\zeta - \lambda_j)(z - \lambda_k)(\zeta I - A)^{-1}(zI - A)^{-1} \, d\zeta \, dz$$

$$= (2\pi i)^{-2} \int_{C_j} \int_{C_k} \frac{(\zeta - \lambda_j)(z - \lambda_k)}{\zeta - z} \left((zI - A)^{-1} - (\zeta I - A)^{-1} \right) d\zeta \, dz$$

Using Resolvant identity.

ζ runs along C_j.

z runs along C_k.

The first integral is

$$\int_{C_k} (z - \lambda_k)(zI - A)^{-1} \, dz \int_{C_j} \frac{(\zeta - \lambda_j) \, d\zeta}{\zeta - z}$$

But z lies outside C_j so the integral $= 0$ by Cauchy's thm.

Similarly

$$\int_{C_j} (\zeta - \lambda_j)(\zeta I - A)^{-1} d\zeta \int_{C_k} \frac{z - \lambda_k}{\zeta - z} \, dz = 0$$

Since ζ lies outside C_k

In exactly the same way $S_j S_k = 0$ if $j \neq k$

Now we show N_j is nilpotent and S_j is a projection.

It would be useful to go over the material on page 104 - 105.

As there we employ two Concentric Circles C_j, C_j' (see the description on p. 104)

Performing the SAME Computations

$$N_j^2 = (2\pi i)^{-2} \int_{C_j} \int_{C_j'} (\zeta - \lambda_j)(z - \lambda_j)(zI - A)^{-1}(\zeta I - A)^{-1} \, d\zeta \, dz$$

Again use Resolvant id but-

now

$$N_j^2 = (2\pi i)^{-1}(-1) \int_{C_j} \int_{C_j'} \frac{(\xi-\lambda_j)(z-\lambda_j)}{(\xi-z)} \left((zI-A)^{-1} - (\xi I-A)^{-1}\right) d\xi\, dz$$

z runs over C_j : ξ runs over C_j'

$$\frac{1}{2\pi i} \int_{C_j'} \left(\frac{\xi-\lambda_j}{\xi-z}\right)(z-\lambda_j)(zI-A)^{-1} d\xi =$$

$$(z-\lambda_j)^2 (zI-A)^{-1} \quad (\because z \text{ lies inside } C_j')$$

The other integral will be zero.

So

$$N_j^2 = \frac{1}{2\pi i} \int_{C_j} (z-\lambda_j)^2 (zI-A)^{-1} dz$$

Similarly

$$N_j^k = \frac{1}{(2\pi i)} \int_{C_j} (z-\lambda_j)^k (zI-A)^{-1} dz \qquad (3.30)$$

Ex: Prove: $S_j^2 = S_j$

Note that in (3.30) if we take k = algebraic Multipicity of λ_j then since

$$(zI-A)^{-1} = (\det(zI-A))^{-1} \boxed{Adj(zI-A)}$$

Holomorphic

all poles except λ_j lie outside C_j

The numerator $(z-\lambda_j)^k$ in (3.30) Cancels out the pole λ_j if $k \geq$ alg. mult of λ_j

and we conclude from (3.30)

$$N_j^k = 0 \quad \text{if} \quad k \geq \text{alg. Mult of } \lambda_j$$

Exercise: Check that S_j and N_k commute for all j, k $(k \neq j)$

(Examine the double integral for $S_j N_k$ over C_j, C_k and C_j, C_j' if $k = j$)

We have proved:

Thm 3.10: There exists a decomposition

$$A = N + S$$

N is nilpotent

S is diagonalizable

$SN = NS$

$N = \sum N_j$; $S = \sum S_j$

N_j, S_j given by (3.29)

That N is nilpt is clear since each N_j is so and $N_1, \ldots, N_\ell$ commute.

S_j $(j = 1, 2, \ldots, \ell)$ commute and each of these is diagonalizable

Notation S is from the terminology SEMISIMPLE

Claim: N, S are polynomials in A.

Proof: Let $m(z)$ = Min poly of A

$$\frac{m(\lambda) - m(\mu)}{\lambda - \mu} = \sum_{i,j} \alpha_{ij} \lambda^i \mu^j$$

for some constants α_{ij}

Assume $\lambda, \mu \notin \sigma(A)$ and then we get — setting $\mu = A$

$$m(\lambda)(\lambda I - A)^{-1} = \sum \alpha_{ij} \lambda^i A^j$$

$$\therefore (\lambda I - A)^{-1} = \sum \frac{\alpha_{ij} \lambda^i A^j}{m(\lambda)} \quad ; \quad \lambda \notin \sigma(A)$$

Integrating over C_k we get

$$S_k = \frac{1}{2\pi i} \int (\lambda I - A)^{-1} d\lambda = \sum \frac{\alpha_{ij} A^j}{2\pi i} \int_{C_k} \frac{\lambda^i d\lambda}{m(\lambda)}$$

RHS is clearly polynomial in A.

The claim immediately follows !

Now Suppose $A = \tilde{N} + \tilde{S}$
Where $\tilde{N}$ is nilpotent
$\tilde{S}$ is diagonalizable
$\tilde{N}\tilde{S} = \tilde{S}\tilde{N}$

THEN
$\tilde{N} = N$; $\tilde{S} = S$

Well, $A\tilde{N} = \tilde{N}^2 + \tilde{S}\tilde{N} = \tilde{N}^2 + \tilde{N}\tilde{S} = \tilde{N}(\tilde{N}+\tilde{S}) = \tilde{N}A$
Similarly $\tilde{S}A = A\tilde{S}$
So Both $\tilde{S}$, $\tilde{N}$ Commute with A.

Hence $\tilde{S}$, $\tilde{N}$ Commute with S, N since the latter two are polynomials in A
$\therefore$ $S, \tilde{S}, N, \tilde{N}$ mutually Commute

$S - \tilde{S} = \tilde{N} - N$
RHS nilptt, LHS diagonalizable
The only diag. matrix that is nilptt is Zero Matrix (Exercise)
So $S = \tilde{S}$; $N = \tilde{N}$.
Let us summarize all these as the IMPORTANT

Thm 3.11 (The S-N decomposition)

Given, $A \in M_n(\mathbb{C})$, there exist UNIQUE matrices $S, N \in M_n(\mathbb{C})$ such that
(i) $A = S + N$
(ii) $SN = NS$
(iii) N is nilpotent, S is diagonalizable
(Proof)
Further, S, N are polynomials in A

For use of S-N decompositions in the analytic theory of ODEs see for example Po-Fang Hseih and Y. Sibuya, Basic Theory of ordinary differential equations, Springer 1999.

We now close this Chapter with a brief discussion of Real Canonical Forms.
We shall restrict ourselves to matrices $M_n(\mathbb{R})$ which are diagonalizable over $\mathbb{C}$
For a more comprehensive account see p 126 ff of M. W. Hirsch and S. Smale: Diff. Equations, dynamical systems and lin. alg. Acad. press 1974

We consider a system of ODEs

$$\frac{dy}{dt} = Ay \quad ; \quad A \in M_n(\mathbb{R}) \qquad (3.32)$$

Assume A has non real eigen values. and say $\lambda = \alpha + i\beta$ $(\alpha, \beta \in \mathbb{R})$ is one of them
Then $\alpha - i\beta$ is also an eigen value.

Now we know $e^{\lambda t} v$: (v is an eigen vector Corresp. to λ) is a solution of the ODE but this solution is Complex valued.
$Re(e^{\lambda t} v)$ is a solution of the ODE namely

$$e^{\alpha t}((\cos\beta t)X - (\sin\beta t)Y) : v = X + iY$$
$$X, Y \in \mathbb{R}^n$$

Likewise $Im(e^{\lambda t} v)$ is also a sol. namely
$e^{\alpha t}((\sin\beta t)X + (\cos\beta t)Y)$.

Together we got the $(n \times 2)$ matrix

$$e^{\alpha t}[X \; Y]\begin{bmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{bmatrix} \qquad (3.33)$$

The Columns of (3.33) are $Re(e^{\lambda t}v)$, $Im(e^{\lambda t}v)$ respectively.

The REAL Canonical form provides us with an elegant formalism for carrying this out in general.

If $\lambda \in \mathbb{C}$, $(\lambda \notin \mathbb{R})$ is an eigen value of $A \in M_n(\mathbb{R})$ ← Imp!
and $v \in \mathbb{C}^n$ is the corresponding eigen vector
then

Claim: (i) $Re\, v$, $Im\, v$ are $\mathbb{R}$-lin. indep.

If the eigen value is in $\mathbb{R}$ then it has an eigen vector in $\mathbb{R}^n$.

Let $\lambda_1, \ldots, \lambda_k, \alpha_{k+1} \pm i\beta_{k+1}, \ldots, \alpha_m \pm i\beta_m$ be n eigen values (Counting Multiplicities)
So that $n = k + 2(m-k)$

Assume v is an eigen vector with eigen value
$\alpha + i\beta \quad (\beta \neq 0)$
then the Complex Conjugate $\bar{v}$ is an eigen vector with eigen value $\alpha - i\beta$
$v, \bar{v}$ are $\mathbb{C}$-linearly indep.

$X = Re\, v, \quad Y = Im\, v$

$c_1 X + c_2 Y = 0$ for $c_1, c_2 \in \mathbb{R}$

$\Rightarrow \quad c_1 (v + \bar{v}) + c_2 (-i)(v - \bar{v}) = 0$

$\Rightarrow \quad c_1 - ic_2 = 0; \quad c_1 + ic_2 = 0 \Rightarrow c_1 = c_2 = 0.$

So the claim (i) is established.

Assume A is diagonalizable over $\mathbb{C}$
$v_1, \ldots, v_k$ eigen vectors with respective eigen values $\lambda_1, \ldots, \lambda_k$
(WLOG. $v_1, \ldots, v_k \in \mathbb{R}^n$)
and $w_j = x_j + iy_j$ eigen vector in $\mathbb{C}^n$

with eigen value $\alpha_j + i\beta_j$ $(j = K+1, \ldots, m)$ (3.34)

Then, $v_1, \ldots, v_K, X_1, Y_1, X_2, Y_2, \ldots, X_m, Y_m$ is a basis for $\mathbb{R}^n$.

Suppose $\exists\ \gamma_1, \ldots, \gamma_K, c_1', c_1'', \ldots, c_m', c_m'' \in \mathbb{R}$

s.t: $\gamma_1 v_1 + \ldots + \gamma_K v_K + c_1' X_1 + c_1'' Y_1 + \ldots + c_m' X_m + c_m'' Y_m = 0$

Write $X_1 = \frac{1}{2}(w_1 + \bar{w}_1)$: $Y_1 = \frac{1}{2i}(w_1 - \bar{w}_1)$ etc;

and as before we conclude

$\gamma_1 = \ldots = \gamma_K = 0$; $\quad c_1' = c_1'' = \ldots = c_m' = c_m'' = 0$

Let us now find the matrix of the linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ wrt the basis (3.34).

$$A w_\ell = (\alpha_\ell + i\beta_\ell)(w_\ell)$$

$$A X_\ell + i A Y_\ell = (\alpha_\ell + i\beta_\ell)(X_\ell + i Y_\ell)$$

$$\therefore A X_\ell = \alpha_1 X_\ell - \beta_\ell Y_\ell \qquad \ell = K+1, \ldots, m$$

$$A Y_\ell = \beta_1 X_\ell + \alpha_1 Y_\ell.$$

(Similarly for AX_j, $AY_j \ldots$)

We see that matrix of $A : \mathbb{R}^n \to \mathbb{R}^n$ wrt (3.34) assumes the form

$$R = \begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_K & & & \\ & & & \alpha_{K+1} & \beta_{K+1} & \\ & & & -\beta_{K+1} & \alpha_{K+1} & \\ & & & & & \alpha_{K+2} \quad \beta_{K+2} \\ & & & & & -\beta_{K+2} \quad \alpha_{K+2} \\ & & & & & & \ddots \\ & & & & & & & \alpha_m & \beta_m \\ & & & & & & & -\beta_m & \alpha_m \end{bmatrix}$$

R is called the real canonical form of A.

Thus $P^{-1}AP = R$ for a nonsingular P

Ex Compute $\exp(tR)$

First Compute $\exp t \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$.

Before closing this Chapter we give one application of real Canonical form

Thm 3.12: The group $SO(n;\mathbb{R})$ is Connected (as a subset of $M_n(\mathbb{R})$)

proof: $A \in SO(n;\mathbb{R})$: $AA^T = A^TA = I_n$

So A is a normal matrix

A is Unitarily diagonalizable over $\mathbb{C}$.

Let R be its real Canonical form displayed above

Note: Eigen values are Unit Complex numbers $\exp i\theta_1, \ldots, \exp i\theta_n$

So that, if $\theta \neq 0$

$$\alpha = \cos\theta, \quad \beta = \sin\theta$$

Now the eigen value -1 must appear with even multiplicity in the Char. Eqn. and they can be paired off.

For a pair $-1, -1$ write

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos\pi & \sin\pi \\ -\sin\pi & \cos\pi \end{bmatrix}$$

$$= \begin{bmatrix} \cos t\pi & \sin t\pi \\ -\sin t\pi & \cos t\pi \end{bmatrix}_{t=1} \qquad (3.35)$$

Also $\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} = \begin{bmatrix} \cos t\theta & \sin t\theta \\ -\sin t\theta & \cos t\theta \end{bmatrix}_{t=1}$

So replacing Each α by $\cos t\theta$, β by $\sin t\theta$ in the displayed R in p126 we get a matrix R_t for $0 \leq t \leq 1$

Keep λ_j intact if $\lambda_j = 1$.

If $\lambda_j = -1$ use the trick (3.35)

The map $t \mapsto R_t$ is evidently Continuous

Exercise: Show that $R = P^{-1}AP$ for some orthogonal matrix P with det 1 when $A \in SO(n;\mathbb{R})$

Then $P R_t P^{-1} = A_t$ is a continuous path $[0,1] \to SO(n;\mathbb{R})$

At $t=0$,

$$P R_t P^{-1} = P R_0 P^{-1} = I.$$

At $t=1$: $P R_t P^{-1} = P\left(R_t\big|_{t=1}\right) P^{-1}$

$$= PRP^{-1} = A$$

So we have a path in $SO(n;\mathbb{R})$ connecting I and A.

Hint for the exercise:

The eigen vectors in $\mathbb{C}^n$ of A are orthonormal wrt std. hermitian product.

Take their length to be 2 for $w_{k+1}, \ldots, w_m$

We shall say no more on this.

Linear ODEs : Comparison and Oscillation Theorems. Sturm-Liouville problems

In this Chapter we shall confine ourselves to second order scalar DEs with variable coefficients

That is, equations of the form

$$y'' + p(x)y' + Q(x)y = 0 \qquad (4.1)$$

Many important equations of physics reduce to (4.1) notably the equatims of Legendre and Bessel.
A basic reference is the book of G.F. Simmons, Differential equations with applications and historical notes.
At a much more advanced level is the important work of Hsieh and Sibuya cited earlier (see p124)
An extensive yet accessible treatment is the book of G.D. Birkhoff and G.C. Rota

For eigen value problems, asymptotic behaviour of solutions etc; Vol I of Courant and Hilbert's Methods of Math. physics is highly recommended.
We begin with a simple result Concerning zeros of non trivial solutions of (4.1)

Thm 4.1: Assume $p(x)$, $q(x)$ are continuous real valued functions on an interval $I \subseteq \mathbb{R}$ and $y(x)$ is a non trivial solution of (4.1) on I
The zeros of $y(x)$ are isolated.
That is the set of zeros of $y(x)$ cannot have any limit point in I
Note that a non trivial solution is any solution other than the identically zero solution.

Suppose (ξ_n) is a sequence of zeros of $y(x)$ and that $\xi_n \to p \in I$. each $\xi_n \in I$, $(\xi_n \neq \xi_{n+1})$

Passing to a subseq. we may assume WLOG that (ξ_n) is a monotone sequence.

Assume $\xi_n < \xi_{n+1}$ for all $n \in \mathbb{N}$

By Continuity $y(p) = 0$ since $y(\xi_n) = 0 \ \forall n \in \mathbb{N}$

Now, by Rolle's thm, $\exists \ c_n \in (\xi_n, \xi_{n+1})$ such that $y'(c_n) = 0 \quad \forall n \in \mathbb{N}$

Now $\xi_n < c_n < \xi_{n+1}$ and $\xi_n \to p$

$\Rightarrow c_n \to p$

Again, Continuity of $y' \Rightarrow y'(p) = 0$

So $y(x)$ is the unique sol. of the IVP

$$y'' + P(x)y' + Q(x)y = 0$$
$$y(p) = 0, \ y'(p) = 0$$

But the identically zero function is also a solution of the SAME IVP

By Uniqueness then $y(x) \equiv 0$. But this is a Contradiction since $y(x)$ is assumed to be a nontrivial solution.

Thm 4.2 (Sturm Separation thm)

Suppose $y(x)$ and $z(x)$ are two Linearly Indep. Solutions of $y'' + P(x)y' + Q(x)y = 0$ then

Between two successive zeros of $y(x)$ there is a unique zero of $z(x)$

Thus the zeros of $y(x)$ and $z(x)$ interlace



Red crosses: Zeros of $y(x)$

Green Circles: Zeros of $z(x)$

A simple instance is of course the solutions $\sin x$ and $\cos x$ of $y'' + y = 0$.
The zeros of $\sin x$ and $\cos x$ interlace.

Note that in particular, $y(x)$ and $z(x)$ do not have a common zero!
This follows at once from the fact that $W(y(x), z(x)) \neq 0$ throughout I.
At a common zero the Wronskian would vanish.

Let a, b be two successive zeros of $y(x)$
Replacing y by $-y$ if necessary we may assume that $y(x) > 0$ on the open interval (a, b)
Assume $z(x)$ does not vanish in (a,b)
Again, replacing z by $-z$ if necessary we may assume $z(x) > 0$ for all $x \in (a, b)$.

Now $y(a) = 0$ and $y > 0$ on $(a,b) \Rightarrow y'(a) \geq 0$
$y'(a) \neq 0$ ($\because$ y is a non trivial solution
The zero sol. is the only one for which $y(a) = 0, y'(a) = 0$ can hold)
$\therefore$ $y'(a) > 0$. Similarly $y'(b) < 0$.

$$W(a) = y(a) z'(a) - z(a) y'(a)$$
$$= - z(a) y'(a) < 0 \quad ; \left(W(x) = W(y, z).\right)$$

While
$$W(b) = y(b) z'(b) - z(b) y'(b)$$
$$= - z(b) y'(b) > 0$$

So Wronskian $W(x)$ must vanish on (a,b) at some point which is a contradiction since y, z are linearly independent.

The Self adjoint form of (4·1):

Consider $y'' + p(x)y' + Q(x)y = 0$ (4·1)

put $\phi(x) = \exp \int_{x_0}^{x} p(s)ds$ (4·2)

Multiply (4·1) by (4·2) and we get an equivalent differential equation.

Note that $y''\phi + p(x)\phi y'$

$= \frac{d}{dx}(\phi(x)y')$

So the transformed equation is

$$\frac{d}{dx}(\phi(x)y') + q(x)\phi(x)y = 0 \qquad (4\cdot3)$$

Let us mult by $\phi(x)$:

$$\left(\phi(x)\frac{d}{dx}\right)\left(\phi(x)\frac{d}{dx}y\right) + q(x)(\phi(x))^2 y = 0. \qquad (4\cdot3)'$$

The transformation

$$\phi(x)\frac{d}{dx} = \frac{d}{dt} \qquad (4\cdot3)''$$

immediately suggests itself and we get

$$\frac{d^2}{dt^2}y + \tilde{Q}(x)y = 0 \qquad (4\cdot4)$$

where $\tilde{Q}(x) = Q(x)(\phi(x))^2$.

A note on (4·3)'': Let us define

$$t = \int_{x_0}^{x} \frac{du}{\phi(u)} \qquad (4\cdot5)$$

$\frac{dt}{dx} = \frac{1}{\phi(x)} > 0$ so that t is a differentiable strictly increasing function and its inverse is likewise strictly inc. diff...

If y is a function of x then for the composite function

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \phi(x)\frac{dy}{dx}$$

i.e $$\left(\phi(x)\frac{d}{dx}\right)y = \frac{d}{dt}y$$

which is what is meant by (4·3)".

In view of this one may confine oneself to differential equations of the form (4·3) or even (4·4)

Def: Equation (4·3) is called the self adjoint-form of (4·1)

Examples: (1) The Legendre Equation

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0 \qquad (4\cdot6)$$

It can be written as

$$\frac{d}{dx}\left((1-x^2)y'\right) + p(p+1)y = 0 \qquad (4\cdot6)'$$

Note that (4·6) is not of the form (4·1)
One has to first divide (4·6) by $(1-x^2)$ to bring it to the form $y'' + P(x)y' + Q(x)y = 0$
Thus $I = (-1, 1)$ is the interval on which (4·6) is defined

Same Comment applies to the next one (4·7)

(2) Tchebycheff's DE: $$(1-x^2)y'' - xy' + p^2y = 0 \qquad (4\cdot7)$$

Check that the self adjoint form is

$$\frac{d}{dx}\left(\sqrt{1-x^2}\,y'\right) + \frac{p^2y}{\sqrt{1-x^2}} = 0 \qquad (4\cdot7)'$$

(3) The Bessel's DE:

This is the equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0. \qquad (4.8)$$

The equation arises in the study of vibrations of a circular membrane clamped along the rim

More details are available on my NPTEL Videos (online) and the slides available on my website Fourier Analysis and Applications (Slides)

The self adjoint form of (4.8) is

$$(xy')' + \left(x - \frac{p^2}{x}\right)y = 0 \qquad (4.8)'$$

(4) The Hermite's Equation:

$$y'' - 2xy' + \lambda y = 0 \qquad (4.9)$$

The self adjoint form is (Mult by $\exp(-x^2)$)

$$\frac{d}{dx}\left(e^{-x^2}y'\right) + \lambda e^{-x^2}y = 0 \qquad (4.9)'$$

For more on this see the NPTEL Videos and Slides of my Fourier Analysis course (Slides available on my website and also links to videos)

(5) Find the Self adjoint form of the Laguerre Equation

$$xy'' + (1-x)y' + \alpha y = 0 \qquad (4.10).$$

We now prove

Thm 4.3 (Sturm's Comparism-Thm)

Suppose $q(x)$ and $r(x)$ are continuous functions on the interval $I \subseteq \mathbb{R}$ and $q(x) < r(x)$ for $x \in I$

Then,

between two successive zeros of a non trivial solution of $y'' + q(x)y = 0 \qquad (4.11)$

there is at least one zero of every solution of

$$z'' + r(x) z = 0. \qquad (4.12)$$

Proof: Clearly we may assume that the solution $z(x)$ of (4.12) under consideration is non trivial.

The theorem asserts that (non trivial) solutions of (4.12) oscillate more rapidly than those of (4.11). Look at the case when $q(x)$, $r(x)$ are constant and this would make the thm appear natural and the result as one that could be anticipated.

Let ξ_1, ξ_2 be two successive zeros of a non trivial solution $y(x)$ of $y'' + q(x)y = 0$.
We may assume $y(x) > 0$ on (ξ_1, ξ_2)

Again, $y(\xi_1) = 0 \Rightarrow y'(\xi_1) \neq 0$
and as before $y'(\xi_1) > 0$.

Similarly $y'(\xi_2) < 0$.

Assume a non trivial sol. $z(x)$ of (4.12) does not vanish on (ξ_1, ξ_2). Suppose $z(x) > 0$ on (ξ_1, ξ_2)

Define

$$\phi(x) = y(x) z'(x) - z(x) y'(x) = W(y, z)$$

$$\phi'(x) = yz'' - zy'' = yz(q - r) < 0 \text{ on } (\xi_1, \xi_2)$$

~~$\phi(x) = \phi(\xi_1) + \int_{\xi_1}^{x} \phi'$~~

$$\phi(\xi_1) = -z(\xi_1) y'(\xi_1) < 0 \Rightarrow \phi(\xi) < 0$$

and $\phi' < 0 \Rightarrow \phi$ is monotone dec.

$\Rightarrow \phi(x) < 0$ throughout $[\xi_1, \xi_2]$

In particular

$$\phi(\xi_2) \not< 0$$

As before, $y'(\xi_1) > 0$
$y'(\xi_2) < 0$.
See page 131.

But $\phi(\xi_2) = -z(\xi_2) y'(\xi_2) > 0$

($\because y > 0$ on (ξ_1, ξ_2) and $y(\xi_2) = 0$)

Contradiction.

The Airy's differential Equation:

This is the 2nd order ODE

$$y'' + xy = 0 \qquad (4.13)$$

(often it is taken as $y'' - xy = 0$ (4.13)' but the transf. $x \mapsto -x$ converts (4.13) into (4.13)') Extremely imp in Geometric Optics and other parts of Math. physics.

Thm 4.4: Every non trivial solution of (4.13) has infinitely many zeros on the positive real axis $(0, \infty)$.

If $\xi_1, \xi_2, \xi_3, \ldots$ is the sequence of zeros with $\xi_1 < \xi_2 < \xi_3 < \ldots$ then

$$\lim_{n \to \infty} (\xi_{n+1} - \xi_n) = 0.$$

Proof: Let us apply Sturm's Comparism-Thm to (4.13) and the ODE

$$u'' + k^2 u = 0 \qquad (4.14)$$

on the interval $[k^2, \infty)$

Between two successive zeros of $\sin kx$ (nontrivial Sol of (4.14)) there must be a zero of every non trivial Sol. of (4.13).

This Completes the proof of the first part

Let $\epsilon > 0$ be arbitrary. Select $k \in \mathbb{N}$ such that $\frac{2\pi}{k} < \epsilon$. Then the distance between Successive zeros of $\sin kx$ is $\frac{2\pi}{k} < \epsilon$

Now, we know $\xi_n \to \infty$ as $n \to \infty$

So $\xi_n > k^2$ for all $n \geq n_0$.

and $\xi_{n+1} - \xi_n < \epsilon$ for $n \geq n_0$.

To see this note that ξ_n, $\xi_n + \frac{2\pi}{k}$ are Successive zeros of $u(x) = \sin k(x - \xi_n)$

and so $\xi_{n+1} \in (\xi_n, \xi_n + \frac{2\pi}{k})$

Ex: Every non trivial solution of the Bessels equation $x^2y'' + xy' + (x^2 - p^2)y = 0$ (4.15) has infinitely many zeros in $(0, \infty)$

The substitution $x = e^t$ transforms the ODE into $\frac{d^2y}{dt^2} + (e^{2t} - p^2)y = 0$ (4.16)

on the interval $t \geq \frac{1}{2}\log(k^2 + p^2)$

$e^{2t} - p^2 \geq k^2$ so we Compare (4.16) with the equation

$$u'' + k^2 u = 0 \qquad (4.17)$$

(4.17) has a sequence of zeros going off to $+\infty$ and so does (4.16)

If $T_1, T_2, T_3, \ldots$ are the zeros of (4.16) then $e^{T_1}, e^{T_2}, e^{T_3}, \ldots$ are the zeros of (4.15).

We shall now prove a useful result due to A. Wintner providing a sufficient condition for the existence of infinitely many zeros.

Thm 4.5 (Wintner) Let $q(x)$ be a nonnegative Continuous function on (a, ∞) such that

$$\int_a^\infty q(t)\,dt = +\infty \qquad (4.18)$$

Then every non trivial solution of $y'' + q(x)y = 0$ has infinitely many zeros on (a, ∞)

proof: Suppose not. That is $y(x)$ is a solution with only finitely many zeros on (a, ∞)

WLOG (replacing y by $-y$ if Necessary) we can assume

$$y(x) > 0 \quad \text{for all} \quad x \geq x_0.$$

The ODE gives $y'' \leq 0$ on $[x_0, \infty)$

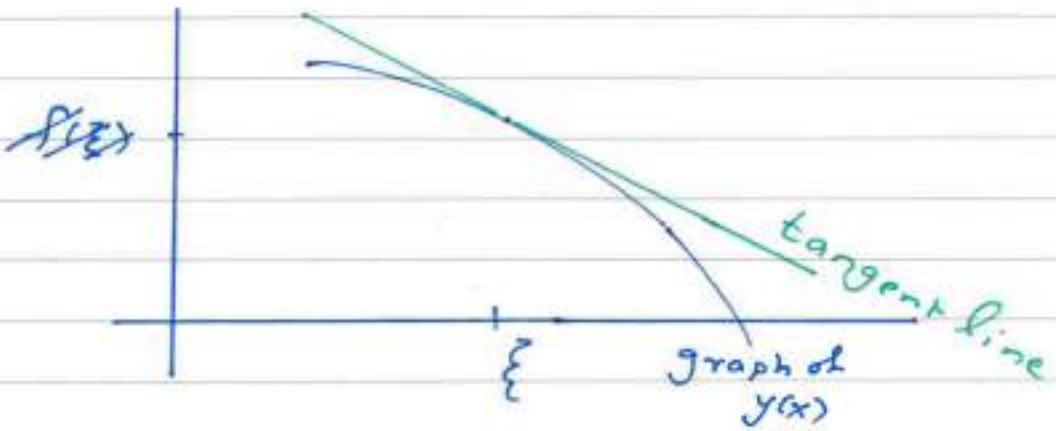
$$y'(x) = y'(\xi) + \int_{\xi}^{x} y''(s)\,ds \leq y'(\xi) \qquad (4.19)$$

for any choice $\xi \in [t_0, \infty)$

$$\therefore\ y(x) \leq y'(\xi)(x-\xi) + y(\xi) \quad \text{for } x \geq \xi \qquad (4.20)$$

Assume $y'(\xi) < 0$ for some ξ. $\therefore\ \xi \geq x_0$

Then (4.20) says that the graph of the solution $y(x)$ lies below the line through $(\xi, y(\xi))$ and tangent to the graph of $y(x)$



But $y'(\xi) < 0 \Rightarrow$ the tangent line cuts the x-axis

$\Rightarrow$ graph of $y(x)$ also cuts the x-axis at a point $\geq x_0$. Contradiction

The assumption $y'(\xi) < 0$ is wrong

Thus, we must have $y'(x) \geq 0$ for all $x \geq x_0$.

$\therefore$ (i) y' is monotone dec. and nonneg on $[x_0, \infty)$

So (ii) y is monotone increasing and positive on $[x_0, \infty)$

$y'(x)$ has a limit as $x \to \infty$

$y(x)$ has a limit or $\to +\infty$ as $x \to +\infty$

Now the ODE $y'' + q(x)y = 0$ gives

$$y'(x) = y'(x_0) - \int_{x_0}^{x} q(s)y(s)\,ds$$

We conclude $\int_{x_0}^{\infty} q(s)y(s)\,ds$ Converges

But $0 < y(x_0) \leq y(x)$ for $x \geq x_0$.

$\therefore \quad 0 < \int_{x_0}^{\infty} y(x_0) q(s)\,ds \leq \int_{x_0}^{\infty} y(s) q(s)\,ds$

$\Rightarrow \int_{x_0}^{\infty} q(s)\,ds$ Converges.

This is a Contradiction

Wintner's Thm has been tremendously generalized by Einer Hille - See his lectures on ODEs Addism Wesley 1967 p 456 ff.

Ex: Discuss the solutions of

$$y'' + \frac{\rho y}{x^2} = 0 \qquad (4.21)$$

ρ is a real Constant.

Is it true that non trivial sol. have infinitely many zeros?

In this case $q(x) = \frac{\rho}{x^2}$; $\int_{x_0}^{\infty} q(s)\,ds < \infty$

Can you say Conditions given in Wintner's Thm are Sufficient but not necessary?

Note: (4.21) is a Simple Cauchy Euler equation. Look at the Cases

$\rho = \frac{1}{4}$, $\rho < 1/4$ and $\rho > 1/4$

Explicitly find Solutions

We shall close this discussim here and take up next

Two point Boundary Value problems

Greens Functions

Boundary Value problems - Green's function

So far we have been looking at initial value problems for systems of ODEs.

Now we turn to a very different problem - Two point boundary value problem (BVP) for a scalar linear ODE.

To keep the discussion simple we shall confine ourselves to second order ODEs

Let us begin with the simplest case:

$$y'' + p(x)y = 0 \text{ or } f(x) \qquad (4.22)$$

Subject to $y(a) = 0;\ y(b) = 0$ (4.22)'

p, f are continuous on the closed bounded interval $[a,b]$. Often $f(x) = 0$.

$$V = \{ y \in C^2[a,b] \ / \ y(a) = y(b) = 0 \}$$

which is a vector subsp. of $C^2[a,b]$

$$Ly = y'' + p(x)y \ : \ y \in V$$

$L : V \to C[a,b]$ is then a linear transf.

The problem amounts to solving

$$Ly = f \text{ for given } f \in C[a,b]$$

seeking $y \in V$.

If $\ker L$ is non trivial the problem obviously will not have a unique solution.

For example with $p(x) = 1$, $a = 0$, $b = \pi$

$\sin x$ is in the kernel of $Ly = y'' + y$

Ex: Show that if $p(x) \leq 0$ on $[a,b]$ then $\ker L$ is trivial

Well, let $y \in \ker L$ namely

$$y'' + p(x)y = 0 : \quad y(a) = 0,\ y(b) = 0$$

Mult by y and integrate over $[a,b]$

$$-\int_a^b y'^2 + \int \rho y^2 = 0 \quad : \text{ boundary terms arising out of integration by parts}$$

vanish thanks to the BC : $y(a) = 0 ; y(b) = 0$.

$\rho \leq 0$ throughout $[0, b]$

$\Rightarrow y \equiv 0$ (Remember we are looking at real valued solutions here)

The higher dimensional analogue of this problem is that of solving

$$\Delta u + \rho(x) u = f \quad \text{on } \Omega \qquad (4.23)$$

$$u|_{\partial\Omega} = 0 \qquad (4.23)'$$

Ω is a given open set in $\mathbb{R}^n$ with smooth boundary and we shall assume Ω is bounded
This problem is called the Dirichlet problem

How do these problems arise ?

(1) Consider the problem of studying vibrations of a membrane Ω clamped along the boundary $\partial\Omega$
The displacement function $u(x,t)$ satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0 \qquad (4.24)$$

Let us look for special solutions (Standing waves)

$$u(x,t) = v(x) \exp(ikt) \qquad (4.24)'$$

We readily check that v satisfies the equation

$$\Delta v + \alpha^2 v = 0 \qquad (4.25)$$

α is a constant (What is it in terms of c, k ?)
Equation (4.25) is called the reduced wave equation or the Helmholtz equation

If the membrane has variable density $\rho(x)$ then in place of (4.25) we would get

$$\Delta v + \alpha^2 \rho(x) v = 0 \qquad (4.25)'$$

Since the membrane is clamped along the boundary we supplement (4.25)' with the boundary cond.

$$v|_{\partial\Omega} = 0 \qquad (4.25)''$$

In the one dimensional case (of a vibrating string) we get exactly (4.22) - (4.22)'
The one dimensional problem provides us with an excellent paradigm for the study of BVP.

It is not difficult to show (and is usually done in Coursen on partial differential equations) that Solving the equation $\Delta u + \rho(x) u = f : u|_{\partial\Omega} = 0$ is equivalent to Solving

$$\Delta u + \rho(x) u = 0 \text{ with } u|_{\partial\Omega} = g$$

(for a different g) and vice - versa.
We shall not provide the details here. Instead we show how these problems arise out of Complex analysis

A celebrated result in the theory of one Complex variable is the Riemann Mapping thm:

Thm 4.6: If Ω is a bounded simply connected domain in $\mathbb{C}$ then $\exists$ a biholomorphic map

$$\phi : \Omega \to \{ |z| < 1 \}.$$

That is any bounded simply connected domain is biholomorphic to the open unit disc in $\mathbb{C}$.
We shall not provide all the details - Only an outline on how a BVP arises out of this.

Assume $0 \in \Omega$ so that $|\phi(0)| < 1$

By Composing with a Möbius map we may assume

$$\phi(0) = 0$$

Since ϕ is a biholo: Origin is a simple zero of $\phi(z)$ whereby

$\psi(z) = \frac{\phi(z)}{z}$ is holomorphic on Ω and $\psi(0) \neq 0$

Since Ω is simply connected $\log(\psi(z))$ is holo. on Ω (one fixes a value of $\log(\psi(0))$

$\operatorname{Re}(\log \psi(z)) = \log|\psi(z)|$ is thus harmonic on Ω

If we can determine this harmonic function we would have gone a long way towards establishing the existence of ϕ.

Riemann was motivated by physical considerations while dealing with the problem of finding the harmonic function $\operatorname{Re}(\log \psi(z))$.

i.e $\operatorname{Re}(\log \phi(z) - \log z)$

Now $|\psi(z)| = |z|$ along $\partial\Omega$

so that $\operatorname{Re}(\log(\psi(z)) = \log|z|$ along $\partial\Omega$

Thus the problem at hand is to find u such that

$$\Delta u = 0 \quad \text{on } \Omega \quad \text{and}$$
$$u|_{\partial\Omega} = -\log|z|$$

Namely, solving a specific BVP for Ω

For more details, consult the book

J. Rauch, Partial differential equations
Springer Verlag, Indian Reprint 2010

Specifically go to p 173

(Circular harmonics)

Another noteworthy application of eigen value problems is towards reassembling a shattered fresco at Overtari Chapel in Padua – See the citations [23], [24] in M. Fornasier, R. Ramlau and G. Teschke, A comparison of Joint Sparsity and total variation minimization algorithms in a real life art restoration problem: https://www.researchgate.net/publication/220390697

The Green's function

Let $Ly = y'' + q(x)y$ operation on
$V = \{ y \in C^2[a,b] / \; y(a) = y(b) = 0 \}$
Assume $\ker L = \{0\}$
Let y_1, y_2 be the two linearly indep solutions of the IVP

$u'' + q(x)u = 0$ with respective initial Conditions (i) $u(a) = 1, \; u'(a) = 0$ and
(ii) $u(a) = 0, \; u'(a) = 1$

These two solutions form a basis for the set of all solutions of the ODE $u'' + q(x)u = 0$.
$W(y_1, y_2) \equiv 1$ (Abel - Liouville formula)
Now to invert the map
$L : V \to C[a,b]$
let us use the Method of variation of parameters to solve

$y'' + q(x)y = f(x)$ on $[a,b]$
(Boundary Cond. will be incorporated later)
We seek the solution in the form

$$v_1 y_1 + v_2 y_2 \qquad (4.26)$$

Where
$$\left.\begin{aligned} v_1' y_1 + v_2' y_2 &= 0 \\ v_1' y_1' + v_2' y_2' &= f(x) \end{aligned}\right\} \qquad (4.26)'$$

This gives at once

$$v_1(x) = -\int_a^x f(t) y_2(t)\, dt \text{ and} \qquad (4.27)$$

$$v_1(x) = \int_a^x f(t) y_1(t)\, dt$$

Whereby a particular Sol. is

$$y(x) = \int_a^x K(x,t) f(t)\, dt \qquad (4.27)'$$

Where $K(x,t) = y_1(t) y_2(x) - y_2(t) y_1(x)$; $t \leq x$
Define $K(x,t) = 0$ if $t \geq x$
Observing that $K(x,t)$ is Continuous

on the Square $[a,b] \times [a,b]$.

Now the most general sol. of the ODE is

$$y = c_1 y_1(x) + c_2 y_2(x) + \int_a^b K(x,t) f(t)\, dt \qquad (4.28)$$

c_1, c_2 have to be chosen such that the Bounday Conditions

$y(a) = 0$: $y(b) = 0$ hold.

at a we get from (4.28)

$$0 = c_1 y_1(a) + c_2 y_2(a) \; ; \text{ but } y_2(a) = 0$$

$y_1(a) = 1 \Rightarrow c_1 = 0$.

Next, at b the condition is

$$0 = c_2 y_2(b) + \int_a^b K(b,t) f(t)\, dt \qquad (4.29)$$

Now observe that $y_2(b) \neq 0$ because if that were so, $y_2(b) = 0$, $y_2(a) = 0$ and $y_2(x)$ would be a nontrivial element in Kernel L which is a Contradiction !

From (4.29) we can uniquely determine c_2 as

$$c_2 = -\frac{1}{y_2(b)} \int_a^b K(b,t) f(t)\, dt \qquad (4.29)'$$

Incorporating these in (4.28) we get

$$y(x) = -\frac{y_2(x)}{y_2(b)} \int_a^b K(b,t) f(t)\, dt + \int_a^b K(x,t) f(t)\, dt$$

$$= \int_a^b G(x,t) f(t)\, dt \quad \text{where} \qquad (4.30)$$

the function $G(x,t)$ is given by

Where (4.31)

$$G(x,t)\,y_2(b) = \begin{cases} y_1(b)\,y_2(t)\,y_2(x) - y_2(b)\,y_1(x)\,y_2(t) & (t \le x) \\ y_1(b)\,y_2(t)\,y_2(x) - y_2(b)\,y_1(t)\,y_2(x) & (t \ge x) \end{cases}$$

Observe that

$$G(x,t) = G(t,x) \quad \text{on } [a,b]\times[a,b] \qquad (4.32)$$

Definition: $G(x,t)$ above is called the Green's function for the BVP

$u'' + p(x)u = f(x)$; $u(a) = 0$: $u(b) = 0$.

RHS: $f(x)$ is a given continuous fun. on $[a,b]$

Let $R : C[a,b] \to C[a,b]$ be given by

$$Rf = \int_a^b G(x,t)\,f(t)\,dt$$

Ex: Prove that R is a Compact linear map. That is if U is the closed unit ball in $C[a,b]$ then $R(U)$ is preCompact in $C[a,b]$

Hint: Use Ascoli Arzela theorem.

Exercise: Check directly using (4.27)' on page 144 and (4.28) on p 145 Rf is twice diff. and satisfies

$$L(Rf) = f \qquad (4.33)$$

In otherwords we have found a right inverse of the map $L : V \to C[a,b]$

Let $L^2[a,b]$ be the usual Hilbert Space of Square integrable functions on $[a,b]$ (as is defined in standard text books in analysis)

$C[a,b]$ is dense in $L^2[a,b]$. Recall that $L^2[a,b]$ has metric given by

$$d(f,g) = \left(\int_a^b |f(t) - g(t)|^2\,dt \right)^{\frac{1}{2}}$$

Assume for simplicity that functions are real valued (we shall not dwell in any detail so this is mainly motivational). For $f, g \in L^2[a,b]$

$$\langle f, g \rangle = \int_a^b f(t)\, g(t)\, dt$$

(In case of complex valued functions one has to put a bar - complex conjugation - on $g(t)$)

Exercise: Show that $V = \{ y \in C^2[a,b] \,/\, y(a) = 0,\ y(b) = 0 \}$ is dense in $L^2[a,b]$

So the map L is defined on a dense vector subspace of $L^2[a,b]$ and is linear

In functional analysis one says L is a densely defined linear operator on $L^2[a,b]$ and V is the domain of L

Exercise: Check that for $y, z \in V$

$$\langle Ly, z \rangle = \langle y, Lz \rangle \qquad (4.33)$$

Thus L is a symmetric densely defined linear operator on $L^2[a,b]$

Note that we avoid saying L is self adjoint operator on $L^2[a,b]$ since there is a subtle distinction between symmetry and self adjointness. We shall not say anything more on this.
See for example a complete yet self contained discussion in the book

Renardy and Rogers

The symmetry of L is related to the symmetry of the Green's function but we shall not be able to describe this in detail. - well outside the scope of this course

Exercise: Let us consider the map

$R: C[a,b] \to C[a,b]$ given by

$$Rf = \int_a^b G(x,t) f(t)\,dt$$

$G(x,t)$ is the Green's function for the BVP.

Show that

$$\|Rf\|_{L^2} \leq C \|f\|_{L^2}$$

Where $\|f\|_{L^2} = \left(\int_a^b |f(t)|^2 dt \right)^{1/2}$

So that R is a continuous lin. transf.

Use density of $C[a,b]$ in $L^2[a,b]$ to show that R extends continuously as a linear map from $L^2[a,b]$ to itself. We continue to denote the extended map by R

Prove: If B is the closed unit ball in $L^2[a,b]$ then $R(B)$ is compact in $L^2[a,b]$.

Hint: First show that even if $f \in L^2[a,b]$ but not in $C[a,b]$, the image Rf lies in $C[a,b]$ so $R(B) \subset C[a,b]$

Recall that if $f_n \to f$ uniformly on $[a,b]$ then $f_n \to f$ in $L^2[a,b]$.

Show that if $f, g \in L^2[a,b]$ then

$$\langle Rf, g \rangle = \langle f, Rg \rangle$$

You need the fact that G is symmetric.

Thus $R: L^2[a,b] \to L^2[a,b]$ is a Compact-Self adjoint operator on a Hilbert Space

We shall not get into functional analytic aspects of BVP.

In many physical problems a parameter enters into the BVP. For example the parameter α in the reduced wave equation

$$\Delta V + \alpha^2 V = 0 \quad \text{(see page 141)}$$

equation (4.25)

The one dimensional analogue of this vibrating string clamped at its ends; $\rho(x)$ is the density of the string. The BVP reads:

$$\left.\begin{aligned} y'' + \lambda \rho(x) y &= 0 \\ y(a) = 0 &: \; y(b) = 0 \end{aligned}\right\} \quad (4.34)$$

Recall that we assumed that the operator

$$L = \frac{d^2}{dx^2} + \rho(x) : V \to C[a,b]$$

had trivial kernel. Now this operator will depend on λ and we shall have to assume

$$\text{kernel of } \frac{d^2}{dx^2} + \lambda\rho(x) \text{ is trivial}$$

This leads us to the following

Definition: A complex number λ is said to be an eigen value for the BVP (4.34) if there is a non trivial solution $y(x)$ satisfying (4.34). The nontrivial sol is called Eigenfunction corresp to λ

Consider the simplest case $\rho(x) \equiv 1$ (uniform string) and $a = 0$, $b = \pi$

$$y'' + \lambda y = 0$$

gives $y(x) = c_1 \cos x\sqrt{\lambda} + c_2 \sin x\sqrt{\lambda}$

$y(0) = 0 \Rightarrow c_1 = 0 \quad \therefore \; y(x) = c_2 \sin\sqrt{\lambda}\, x$

$y(\pi) = 0$ and $y(x)$ non trivial

$$\Rightarrow \sin\sqrt{\lambda}\,\pi = 0$$

$$\Rightarrow \lambda = 0, 1, 4, 9, \ldots$$

Ex: Show that $\lambda = 0$ does not give a non trivial solution

The eigen values form a discrete seq $\to \infty$
we shall prove this in general.

Thm 4.7: Assume $\rho : [a,b] \to \mathbb{R}$ is real valued, continuous and positive almost everywhere. Then, the eigen values of the BVP

$$y'' + \lambda \rho(x) y = 0; \quad y(0) = 0, \; y(1) = 0$$

are all real.

pf: Suppose not. There is $\lambda = \alpha + i\beta : \alpha, \beta \in \mathbb{R}$ with a non trivial sol. $y'' + \lambda \rho y = 0$

$y(0) = 0, \; y(1) = 0$: Obviously then y is also complex: $y = u + iv$; $u = \operatorname{Re} y$, $v = \operatorname{Im} y$

$$u'' + \rho(\alpha u - \beta v) = 0$$
$$v'' + \rho(\beta u + \alpha v) = 0$$
$$u(0) = 0, \; u(1) = 0; \quad v(0) = 0, \; v(1) = 0$$

Mult first eqn by v second by u and subtract-

$$u''v - v''u - \rho\beta(v^2 + u^2) = 0$$

Integrate over $[0,1]$ and perform an int. by parts we get

$$\beta \int_0^1 \rho\,(u^2 + v^2) = 0.$$

$\rho > 0$ a.e $\Rightarrow \beta = 0$ since $u^2 + v^2 > 0$ a.e.

Thm 4.8 (Orthogonality of Eigenfunctions)
If λ, μ are distinct eigen values with corresp. eigen functions $y(x)$, $z(x)$ then

$$\int_a^b y(x) z(x) \rho(x)\, dx = 0.$$

proof: $y'' + \lambda \rho(x) y = 0$: $y(a) = 0 = y(b)$

$z'' + \mu \rho(x) z = 0$: $z(a) = 0 = z(b)$

Mult first by z, second by y subtract.
Perform one integration by parts:

$$(\lambda - \mu) \int_a^b y(x) z(x) \rho(x)\, dx = 0.$$

Result Follows.

Let $\lambda \in \mathbb{C}$ and $y_1(x,\lambda)$, $y_2(x,\lambda)$ be the two solutions of the IVP

$$y'' + \lambda \rho(x) y = 0 \quad \text{on } [a,b]$$

with respective Initial Conditions

$y(a) = 1$ and $y(a) = 0$

$y'(a) = 0$ $\qquad$ $y'(a) = 1$

Evidently $y_1(x,\lambda)$, $y_2(x,\lambda)$ are analytic wrt λ. In fact for fixed $x \in [a,b]$ They are entire functions of λ

Exercise: They are certainly of class C^1 in the variables $\mathrm{Re}\,\lambda$, $\mathrm{Im}\,\lambda$. Use the Variational eqn and Check Cauchy Riemann equations hold.

Now we seek a ~~general~~ sol.

$$y(x,\lambda) = c_1 y_1(x,\lambda) + c_2 y_2(x,\lambda)$$

Satisfying the BC $\quad y(a,\lambda) = 0$

$y(b,\lambda) = 0$

c_1, c_2 not both Zero. Thus $c_1 = 0$ and

$$0 = c_2 y_2(b,\lambda)$$

$c_2 \neq 0$ would mean λ must be a Zero of the Entire function

$$\lambda \mapsto y_2(b,\lambda)$$

Since the Zeros of an entire function are isolated we immediately Conclude that

In order for the BVP

$$y'' + \lambda \rho(x) y = 0; \quad y(a) = 0 = y(b) \qquad (4.85)$$

it is Nec. and Sufficient that λ be a root of the entire function $y_2(b,\lambda)$

Thm 4.9: The BVP (4.85) has only Countably many eigen values.

Exercise: Find the Green's function for the BVP $y'' = -9f$

$y(0) = 0 = y(1)$

and verify that it is symmetric

Ans: $G(x,t) = \begin{cases} (1-x)t & \text{if } t \le x \\ (1-t)x & \text{if } x \le t \end{cases}$

Variational Characterization of eigen values: The Ritz - Rayleigh principle

This principle is extremely important in quantum mechanics for determining bound states and their energies approximately. References will be provided later.

We begin with a finite dimensional situation which is basically some results in linear algebra. Incidentally, we shall give a variational proof of the Spectral theorem. See also my NPTEL videos and slides of Fourier analysis Course in my website

Thm 4.10 (Variational Characterization of eigen values)

Let A be a $(n \times n)$ real symmetric matrix and $Q(x)$ be the associated quadratic form:

$$Q(x) = x^T A x : \quad x \in \mathbb{R}^n$$

The Rayleigh quotient is by def

$$Q(x) / \|x\|^2 \qquad (4.36)$$

(It would be more logical to denote the quadratic form by $Q_A(x)$ but we shall suppress A if there is only one matrix in the discussion)

$$\lambda_1 = \inf_{x \neq 0} \frac{Q(x)}{\|x\|^2} \qquad (4.37)$$

is the smallest eigen value of A.

$$\lambda_1 = \inf_{x \in \Sigma} Q(x) : \quad \Sigma = \{ x \in \mathbb{R}^n / \|x\| = 1 \}$$

(4.37)'

Inf is attained at an eigen vector x_1 with eigen value λ_1

$\lambda_2 = \inf \{ Q(x) / x \in \Sigma : x \perp x_1 \}$
is the next larger eigenvalue ($\geq \lambda_1$)
and the inf is attained at an eigen vector x_2

$\lambda_3 = \inf \{ Q(x) / x \in \Sigma, \ x \perp x_1, \ x \perp x_2 \}$
is the next eigenvalue ($\geq \lambda_2$) and the inf is attained at an eigen vector x_3.

There is an orthonormal basis consisting of Eigen vectors of A. (SPECTRAL THEOREM)

It will be clear from the argument that the result with trivial modifications hold for hermitian matrices $A \in M_n(\mathbb{C})$ i.e $A = A^*$
Orthonormality of eigen vectors is then wrt standard hermitian product in $\mathbb{C}^n$.

Proof. Since Σ is compact, $\lambda_1 = \inf_{x \in \Sigma} Q(x)$ exists and is attained at x_1
$\|x_1\| = 1$ by Construction
Let $h \neq 0$ be arb. in $\mathbb{R}^n$
Look at $Q\left((x_1 + h) \|x_1 + h\|^{-1} \right) \geq \lambda_1$
Introduce a real parameter t:

$$Q\left((x_1 + th) \|x_1 + th\|^{-1} \right) \geq \lambda_1$$

Equality holds when $t = 0$.

$\therefore (x_1 + th)^T A (x_1 + th) - \lambda_1 \|x_1 + th\|^2 \geq 0$
$x_1^T A x_1 - \lambda_1 = 0$ So we are left with

$$t(h^T A x_1 + x_1^T A h) + t^2 h^T A h - (2t h^T x_1 + t^2 \|h\|^2)\lambda_1 \geq 0$$

Divide by t ($t \neq 0$) and let $t \to 0$
Of course we need to consider the two cases
$t > 0$ and $t < 0$.
Accordingly we get two inequalities:

$h^T A x_1 + x_1^T A h \geq 2\lambda_1 h^T x_1$ and
$h^T A x_1 + x_1^T A h \leq 2\lambda_1 h^T x_1$

$\Rightarrow$ noting that $h^T A x_1 = x_1^T A h$ (both sides are 1×1 matrices AND A is real symmetric) that

$$h^T A x_1 = \lambda_1 h^T x_1$$

$$\therefore h^T (A x_1 - \lambda_1 x_1) = 0$$

$\therefore \langle h, A x_1 - \lambda_1 x_1 \rangle = 0$. Since $h \in \mathbb{R}^n$ was arbitrary we conclude

$$A x_1 = \lambda_1 x_1 \text{ as desired.}$$

λ_1 is an eigen value with eigen vector x_1

We now look at $\lambda_2 = \inf_{x \in \Sigma,\ x \perp x_1} Q(x)$

attained at $x_2 \in \Sigma$ s.t $x_2 \perp x_1$

(Compactness !)

Again, let $h \in \mathbb{R}^n$, $h \neq 0$ AND $h \perp x_1$

then $\dfrac{x_2 + th}{\|x_2 + th\|} \in \Sigma$ and $\perp x_1$

$$\therefore Q\left(\frac{x_2 + th}{\|x_2 + th\|}\right) \geq \lambda_2.$$ Equality holds if $t = 0$

$$\therefore (x_2 + th)^T A (x_2 + th) \geq \lambda_2 \|x_2 + th\|^2$$

As before $x_2^T A x_2 - \lambda_2 \|x_2\|^2 = 0$ and we are left with

$$t(h^T A x_2 + x_2^T A h) \geq 2t\lambda_2 h^T x_2 + t^2 (\text{something})$$

Divide by $t \neq 0$ and let $t \to 0$.

We get as before

$$h^T (A x_2 - \lambda x_2) = 0 \quad : \quad h \perp x_1$$

We cannot conclude directly that $A x_2 = \lambda x_2$ due to the constraint $h \perp x_1$. However for $h \in \mathbb{R}^n$ arbitrary, write

$$h = h' + h'' : \quad h'' \perp x_1$$

$$h' = c_1 x_1$$

$$x_1^T(Ax_2 - \lambda_2 x_2) = x_2^T Ax_1 - \lambda_2 \langle x_1, x_2 \rangle$$
$$= \lambda_1 \langle x_1, x_2 \rangle - \lambda_2 \langle x_1, x_2 \rangle = 0$$

$\therefore$ $h'^T(Ax_2 - \lambda_2 x_2) = 0$. We already have
$h''^T(Ax_2 - \lambda_2 x_2) = 0$

Adding, we get $h^T(Ax_2 - \lambda_2 x_2) = 0 \quad \forall\ h \in \mathbb{R}^n$
and $Ax_2 = \lambda_2 x_2$

For the next stage, λ_3 is the inf attained at x_3 and we would get

$h^T(Ax_3 - \lambda_3 x_3) = 0$: $h \perp x_1$, $h \perp x_2$

For general $h \in \mathbb{R}^n$

$h = c_1 x_1 + c_2 x_2 + h''$: $h'' \perp x_1$, $\not\equiv$ $h'' \perp x_2$

we have as above

$x_1^T(Ax_3 - \lambda_3 x_3) = 0$
$x_2^T(Ax_3 - \lambda_3 x_3) = 0$ AND
$h''^T(Ax_3 - \lambda_3 x_3) = 0$
$\Rightarrow h^T(Ax_3 - \lambda_3 x_3) = 0 \quad \forall\ h \in \mathbb{R}^n$

and again $Ax_3 = \lambda_3 x_3$

The process continues until we have n - orthonormal vectors $x_1, \ldots, x_n$ after which it terminates (how? what is $\{ x \in \Sigma \mid x \perp x_1, \ldots, x \perp x_n \}$)

<u>Exercise</u> Suppose A is a $n \times n$ real symmetric matrix and $x \in \Sigma$

Can it happen that $Q(x) = \lambda$: λ is an eigenvalue of A and yet x is <u>not</u> an eigenvector?

Well, if $\lambda = \lambda_1$, then Necessarily x is an eigenvector. A little thought will convince you that if $Q(x) = \lambda_n$ (largest eigenvalue) then again x is Necessarily an eigenvector

So the remaining case is $n \geq 3$

For $n = 3$, assume that A has 3 distinct

eigen values $\lambda_1 < \lambda_2 < \lambda_3$

We shall show that $\exists\ x \in \Sigma$ such that

$$Q(x) = \lambda_2$$

and x is NOT an eigen vector.

First take the unit vectors x_1, x_3 that are eigen vectors corresp to λ_1, λ_3 respectively

$$x_1 \perp x_3$$

put $x = x_1 \cos\theta + x_3 \sin\theta$ (unit vector)

$$Q(x) = \lambda_1 \cos^2\theta + \lambda_3 \sin^2\theta$$

When $\theta = 0$, $Q(x) = \lambda_1$

When $\theta = \pi/2$, $Q(x) = \lambda_3$

By intermediate value thm, there is a θ for which $Q(x) = \lambda_2$. The corresp.

$$x = x_1 \cos\theta + x_3 \sin\theta$$

is NOT an eigen vector since it lies in the plane spanned by x_1, x_3

Whereas $x_2 \perp \text{Span}(x_1, x_3)$

Also x is not a multiple of x_1 or x_3 since $\theta \neq 0, \pi/2$.

So for a 3×3 matrix with 3 distinct eigenvalues (symm)

we shall always find a $x \in \Sigma$ s.t. $Q(x) = \lambda_2$

x not an eigen vector.

We now look at the Case of BVP:

$$y'' + \lambda \rho(x) y = 0$$

$$y(a) = 0; \quad y(b) = 0$$

$\rho(x) > 0$ almost everywhere on $[a, b]$

and ρ is continuous on $[a, b]$ (real valued)

For simplicity we shall assume

$$a = 0 \quad \text{and} \quad b = 1$$

The analogue of the Rayleigh quotient is the expression

$$Q[y] = \frac{\int_0^1 (y'(x))^2 dx}{\int_0^1 (y(x))^2 \rho(x) dx} \qquad (4.38)$$

Observe that the conditions we have imposed on ρ give us an inner product on $C[0,1]$:

$$\langle\langle y, z \rangle\rangle = \int_0^1 y(x) z(x) \rho(x) dx \qquad (4.39)$$

(In case the functions are complex valued, replace $z(x)$ by $z(\overline{x})$)

The expression (4.38) suggests that we must look at the set Σ consisting of continuous piecewise diff. functions y such that

(Cont.)

$$\int_0^1 (y(x))^2 \rho(x) dx = 1 \quad ; \quad \boxed{y(0) = y(1) = 0}$$

Σ incorporates BC

Is

However there is a "small" problem. The set Σ is ~~most~~ Compact in $L^2_\rho[0,1]$? We shall investigate this below.

L^2_ρ is the L^2-Space wrt the measure $\rho(x)dx$ on $[0,1]$ (You could simply take the metric completion* of $C[0,1]$ wrt the metric induced by (4.39))

So the $\inf_{y \in \Sigma} Q[y]$ need not be attained!

Since $Q[y] \geq 0$ the infimum μ of course exists. Let (y_n) be a sequence in Σ such that

$$Q[y_n] \to \mu$$

(*) $\|y\|_\rho^2 = \int_0^1 \rho(x) (y(x))^2 dx$ defines a norm on $C[0,b]$ but the metric Space is NOT Complete. Every incomplete Metric Sp has a Completion.

The problem is that we need a convergent subsequence.

To begin with assume that each $y_m \in C^1[0,1]$

Then WLOG we may assume

$$Q[y_m] \leq \mu + 1 \quad \text{for all } m \in \mathbb{N}$$

Next

$$y_m(t) = y_m(s) + \int_s^t y'(\xi)\, d\xi$$

$$\therefore |y_m(t) - y_m(s)| \leq (t-s)^{1/2} \sqrt{\mu+1} \quad ; \quad t \geq s$$

So that the seq (y_m) is equicontinuous

Note that the result holds even if (y_m) is Cont. and piecewise C'.

Taking $s = 0$ we also get unif. boundedness

So there exists a subseq converging uniformly

Problem is that we do not know about convergence of (y_m') ...

Let us imitate the procedure used for matrices

Let us ASSUME that (y_m') converges to y (in some sense atleast in $L^2[0,1]$)

So that $Q[y] = \mu$

Let ϕ be of class C' and

$$Q[y + t\phi] \geq \mu$$

$$\therefore \int_0^1 (y' + t\phi')^2 \geq \mu \int_0^1 \rho(x)\,(y(x) + t\phi(x))^2\, dx$$

to deal with the term $\int_0^1 y'\phi'$ we need to

integrate by parts ! we can assume ϕ has support in $(0,1)$ so that bdy terms drop out

But how to transfer the derivative to y'

We know nothing about y'' and the validity is suspect.

Thm 4.11: Let Σ be the class of continuous piecewise C' functions on $[0,1]$
$\rho(x)$ cont and $\rho > 0$ almost everywhere on $[0,1]$

Suppose $\inf_{y \in \Sigma,\ \|y\|_\rho = 1} Q[y]$ is attained at some u (i)

AND $u \in C^2[0,1]$ then u satisfies the (ii)

$$\text{BVP} \quad u'' + \lambda \rho(x) u = 0$$
$$u(0) = 0, \quad u(1) = 0$$

λ is the value of the infimum

$$\|y\|_\rho^2 = \int_0^1 (y(x))^2 \rho(x)\, dx$$

$Q[y]$ is as described on p 157 (Equation (4.38))

Assumption (iii) is the smoothness or regularity of the minimizer.

proof is fairly easy since two major assumptions (i) - (ii) have been made.

Let $\phi \in C_c^\infty(0,1)$ smooth function with compact support in $(0,1)$.

Then

$$Q[u + \epsilon\phi] \geq \lambda \|u + \epsilon\phi\|_\rho^2$$

$$\int_0^1 (u' + \epsilon\phi')^2 \geq \lambda \int_0^1 (u + \epsilon\phi)^2 \rho$$

Now $\int_0^1 u'^2 = \lambda \quad : \quad \int_0^1 u^2 \rho = 1$

So we are left with the following

$$2\epsilon \int_0^1 u'\phi' - 2\lambda\epsilon \int_0^1 u\phi\rho + \epsilon^2 (\text{something}) \geq 0$$

Divide by 2ϵ and let $\epsilon \to 0$. As before Consider the cases $\epsilon > 0$, $\epsilon < 0$ separately and we get

$$\int_0^1 (u'\phi' - \lambda u \phi \rho) = 0$$

perform an integration by parts in $\int_0^1 u'\phi'$

(Here we need $u \in C^2$!)

$$\int_0^1 (u'' + \lambda u \rho)\phi = 0.$$

Now since ϕ was arbitrary we conclude

$$u'' + \lambda u \rho = 0$$

Note that by def of Σ, u vanishes at 0 and 1 (Σ incorporates the BC)

Dirichlet Principle: The assertion that the Minimization problem has a solution of class C^2 (in the class Σ) is known as the Dirichlet principle.

See Courant. R: Dirichlet's principle, Conformal mapping and minimal Surfaces. Springer 1950.

Particularly p 24 ff as well as the preface/introduction for a detailed account of the history of this and a proof (in the two dimensional case)

We shall not prove the validity here. That is we shall not prove that the assumptions (i) - (iii) on p159 are true.

Nodes: If u is an eigenfunction with eigen value λ namely

$$u'' + \lambda \rho(x) u = 0 \quad \text{on } [0,1]$$

$$u(0) = 0 ; \quad u(1) = 1$$

The nodes of u are the Zeros of u lying in the interval $(0,1)$

We know that there can only be finitely many nodes thanks to Thm 4.1.

Thm 4.12 Each eigen value is simple namely to each eigen value the set of eigen functions corresp is one dimensional

(for the BVP $u'' + \lambda \rho u = 0$; $u(0)=0$, $u(1)=0$)

As a consequence the nodes are well defined depend only on λ

The last clause follows because if u_1, u_2 are eigenfunctions with the same eigen value λ then

$$u_1 = c u_2 \text{ for some } c \neq 0$$

so the zeros of u_1, u_2 are identical.

To prove the theorem observe that if u_1, u_2 are two linearly indep eigenfunctions with SAME eigen value λ then $\{u_1, u_2\}$ is a basis for the set of all solutions of $u'' + \lambda \rho(x) u = 0$.

Now, in particular the solution y of the IVP $u'' + \lambda \rho(x) u = 0$, $u(0) = 1$, $u'(0) = 0$ must be given by $y = c_1 u_1 + c_2 u_2$

but this is impossible since $y(0) = 1$

RHS $= 0$ when $x = 0$.

Imp. Remark: To keep the size of this within limits we are not discussing other BC

(a) $u'' + \lambda \rho(x) u = 0$

$u'(0) = 0$; $u'(1) = 0$

(b) Periodic BC: Solutions are required to satisfy $u(0) = u(1)$; $u'(0) = u'(1)$

These are discussed in most books See for example

H. F. Weinberger: First Course in PDEs with Complex Variables and transform Methods (p 124) Dover NY 1995

The result Thm 4.12 is false in higher dimensions

For example $\Delta u = \lambda u$ in a domain Ω

with $u|_{\partial\Omega} = 0$

Thm 4.13: Let λ be the smallest eigen value of the BVP (we know it is ≥ 0) since $\rho(x) > 0$ a.e on $[0,1]$.

The eigenfunction $u(x)$ does not vanish on $(0,1)$

proof: Suppose $u(x)$ is the eigen function and $u(x)$ vanishes at $x_0 \in (0,1)$

We know

$$\int_0^1 (u'(x))^2 dx = \lambda \quad : \quad \int_0^1 (u(x))^2 \rho(x)\, dx = 1$$

Note that replacing u by $|u|$ neither of the integral changes in value

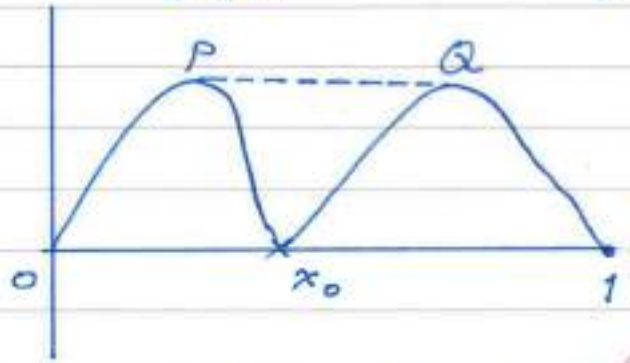
We shall now find a $\tilde{u}$ such that

$$\int_0^1 (\tilde{u}(x))^2 \rho(x) dx > 1 \quad \text{and}$$

$$\int_0^1 (\tilde{u}'(x))^2 dx < \lambda$$

which would contradict the fact that

$$\lambda = \inf_{y \neq 0} \frac{Q[y]}{\|y\|_\rho^2}$$



graph of $|u|$

Q need not be a peak. Make PQ horizontal

It is clear that $|u|$ must have maxima in $(0, x_0)$ and $(x_0, 1)$ respectively. Let P, Q be the peak points WLOG height $P \leq$ ht of Q.

Join P, Q by a line segment to obtain the graph of another function $\tilde{u}$

Obviously for this $\tilde{u}$ we have $\int_0^1 \tilde{u}^2 \rho > 1$

While $(\tilde{u}')^2$ must be smaller than u'^2

This sketch suffices for a convincing argument as

We now show that the nth eigenfunction has precisely $(n-1)$ nodes namely $(n-1)$ simple zeros in the fundamental interval $(0,1)$

Recall that the zeros of a non trivial solution

$$y'' + P(x)y' + Q(x)y = 0$$

(P, Q are continuous functions on an interval I) cannot have limit points in I.

Further a zero (if any) must be simple namely if $y(x_0) = 0$: $x_0 \in I$ (Thm 4.12 p 161)

then $y'(x_0) \neq 0$

Because $y'(x_0) = 0$ would mean $y(x)$ is a sol. of the IVP $y'' + P(x)y' + Q(x)y = 0$

$$y(x_0) = 0, \quad y'(x_0) = 0$$

But trivial sol is also a sol. of this IVP.

$\therefore y(x) \equiv 0$ Contradiction

Now Consider $y'' + \lambda \rho(x) y = 0$

$\rho(x) \geq 0$ on $I = [0,1]$ and $\rho > 0$ a.e

Let $\xi_1(\lambda), \xi_2(\lambda) \ldots$ be the successive zeros of a non trivial sol. $y(x,\lambda)$: $y(0,\lambda) = 0$, $y'(0,\lambda) = 1$ *

We can extend ρ to $[0,\infty)$ appropriately and regard the solutions as being defined on $[0,\infty)$

$$\xi_1(\lambda) < \xi_2(\lambda) < \ldots$$

The functions $\lambda \mapsto \xi_1(\lambda)$ are continuous - in fact they will be diff wrt λ

Well, let us prove Continuity / diff pointwise.

Fix λ_0 and $\xi_k(\lambda_0)$ is the kth zero of a non trivial sol. $y(x, \lambda_0)$ of the ODE

$$y'' + \lambda_0 \rho(x) y = 0.$$

Since $\left.\frac{\partial y}{\partial x}\right|_{\substack{\lambda = \lambda_0 \\ x = \xi_k(\lambda_0)}} \neq 0$ (the zero being simple)

the Implicit function theorem is applicable

(*) Unambiguously fixes $y(x,\lambda)$. We shall assume

 This in what follows (tacitly).

around the "seed solution" $(\xi_k(\lambda_0), \lambda_0)$
of the equation $y(x,\lambda) = 0$.

For the Implicit-function thm see for example Chapter 9 of Rudins principles of Math. Analysis. However, all we really need here is a special case for function of two real variables. A neat and elegant argument using basic analysis of functions from $\mathbb{R} \to \mathbb{R}$ is available in J. C. Burkill: First Course in Mathematical analysis, Cambridge Univ. Press, 1991 pp 166-168.

Let us also recall Sturm's Comparison thm with a different proof (!)
Let $y(x,\lambda)$ and $z(x,\mu)$ be non trivial solutions of

$$y'' + \lambda \rho(x) y = 0$$
$$z'' + \mu \rho(x) z = 0 : \qquad \lambda < \mu$$

Both $y(x,\lambda)$, $z(x,\mu)$ vanish at $x = 0$
$y(\xi, \lambda) = 0$ for $\xi > 0$ and $y(x,\lambda) > 0$ on $(0,\xi)$ and $z \geq 0$ on $[0,\xi]$
Assume ~~(xx~~ Mult first eqn by z
Second by y
Subtract and integrate by parts over $[0, \xi]$

$$z(\xi) y'(\xi) + (\lambda - \mu) \int_0^\xi \rho y z = 0$$

Now $y'(\xi) < 0$ and so $z(\xi) y'(\xi) \leq 0$
while $(\lambda - \mu) \int_0^\xi \rho y z \not\leq 0$

Recall $\rho > 0$ a.e ; y, z pos. a.e
Thus $z \geq 0$ leads to a contradiction
$\therefore$ z must change sign in $(0,\xi)$
All we have assumed is $\rho > 0$ a.e and not everywhere!!

We assume $\rho(x)$ is extended beyond 1 so that $\rho(x) > 0$ a.e and $\rho(x)$ is bounded and $\geq m > 0$ on $[x_0, \infty)$ for some m, $x_0 \in \mathbb{R}$.

Thus each non trivial sol. $y(x, \lambda)$ of $y'' + \lambda \rho(x) y = 0$ with $\lambda > 0$ has infinitely many zeros in $(0, \infty)$

Let $\phi_1(x), \phi_2(x), \phi_3(x), \ldots$ be the list of eigenfunctions with Eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$

$$\phi_j'' + \lambda_j \phi_j \rho = 0$$
$$\phi_j(0) = 0 \; ; \; \phi_j(1) = 0.$$

Let $\xi_1(\lambda_j), \xi_2(\lambda_j), \ldots$ be the list of zeros of ϕ_j in ascending order in $(0, \infty)$

Thus,

$\xi_1(\lambda_1) = 1$ (the eigenfunction ϕ_1 has no zeros in $(0,1)$)

Now we know (Thm (4.8) p 150) that

$$\int_0^1 \phi_j(x) \phi_k(x) \rho(x) \, dx = 0 \quad \text{if } j \neq k \qquad (4.40)$$

(Orthogonality of eigenfunctions)

WLOG $\phi_1 > 0$ on $(0,1)$

Thus ϕ_2 must vanish in $(0,1)$

It cannot maintain the same sign on $(0,1)$ thanks to (4.40) and $\rho > 0$ a.e on $(0,1)$.

In other words $0 < \xi_1(\lambda_2) < 1$

We show that $\phi_2(x)$ vanishes exactly once in $(0,1)$. Suppose NOT.

Then

$0 < \xi_1(\lambda_2) < \xi_2(\lambda_2) < 1 = \xi_1(\lambda_1) < \xi_2(\lambda_1)$; $\lambda_2 > \lambda_1$

Continuity of $\lambda \mapsto \xi_2(\lambda)$

now gives that $\exists \; \lambda_0 \in (\lambda_1, \lambda_2)$

Such that $\xi_2(\lambda_0) = 0$.

That is to say, there is a solution $y(x, \lambda_0)$ of the ODE $y'' + \lambda_0 g(x) y = 0$ which is not identically zero.

$$y(0, \lambda_0) = 0 \; ; \; y'(0, \lambda_0) = 1$$

$y(x, \lambda_0)$ vanishes at $\xi_1(\lambda_0), \; \xi_2(\lambda_0) = 1, \ldots$

But the condition $\xi_2(\lambda_0) = 1$ immediately says $y(x, \lambda_0)$ is then eigen function with eigen value $\lambda_0 < \lambda_2$ which is a Contradiction since $\lambda_0 > \lambda_1$ and $\lambda_1, \lambda_2, \lambda_3, \ldots$ is the list of all eigen values.

Next we show ϕ_3 has Exactly TWO Zeros in $(0,1)$

Well ϕ_2 vanishes at $0, 1$ and $\xi = \xi_1(\lambda_2)$

So ϕ_3 must vanish once in $(0, \xi)$ and once in $(\xi, 1)$

So ϕ_3 has at least two Zeros in $(0,1)$.

Again if ϕ_3 has 3 or more Zeros then

$$0 < \xi_1(\lambda_3) < \xi_2(\lambda_3) < \xi_3(\lambda_3) < 1$$

while $\xi_2(\lambda_2) = 1 < \xi_3(\lambda_2)$

$\therefore \exists \; \lambda_0 \in (\lambda_2, \lambda_3)$ such that

$$\xi_3(\lambda_0) = 1$$

That is $\exists$ a Sol. $y(x, \lambda_0)$ of

$$y'' + \lambda_0 g y = 0 :$$

$$y(0, \lambda_0) = 0$$

$$y'(0, \lambda_0) = 1$$

AND $y(x, \lambda_0)$ vanishes at $\xi_3(\lambda_0) = 1$

$\Rightarrow y(x, \lambda_0)$ is an eigen fun with eigen value $\lambda_0 \lneq \lambda_3$ Contradiction since there is no eigen value in (λ_2, λ_3) ...

This suffices for a proof.

Remarks: For the BVP

$$\Delta u + \lambda \rho(x) u = 0 \quad \text{on } \Omega$$
$$u|_{\partial\Omega} = 0$$

Higher dim. analogue: Ω is an open connected set in $\mathbb{R}^n$ with smooth boundary

N is again the set of zeros of an eigen function with eigen value λ.

A theorem of R. Courant asserts that $\Omega - N$ has at most k connected components if λ is the k-th eigen value

See Courant - Hilbert Methods of Math. phy. vol I, p 452 Indian Reprint 2008

Method of Ritz for Computing approx Eigen values

$$I[y] = \int_0^1 (y'(x))^2 dx \cdot \left(\int_0^1 (y(x))^2 \rho(x) dx \right)^{-1} \qquad (4.41)$$

problem is to Minimize $I[y]$

Idea is to select a Countable collection of lin. indep piecewise smooth functions

$u_1, u_2, u_3, \ldots$

such that $\mathbb{R}$-lin span $(u_1, u_2, u_3, \ldots)$ is dense in the space of all piecewise twice cont-diff functions wrt norm

$$\left(\int_0^1 y'^2 + \int_0^1 \rho y^2 \right)^{\frac{1}{2}} \qquad (4.42)$$

$V_n = \mathbb{R}$-lin span $(u_1, \ldots, u_n)$

First Consider

$I(c_1 u_1 + \ldots + c_n u_n)$ as a function of $c_1, \ldots, c_n$ and minimize this ratio of two quadratic forms in $c_1, \ldots, c_n$.

For fixed n, let $I(c_1 u_1 + \dots + c_n u_n)$ attain its minimum at $c_1^\circ, \dots, c_n^\circ$

Call $c_1^\circ u_1 + \dots + c_n^\circ u_n = \phi_n$

$I[\phi_n]$ decreases monotonically to the first eigen value λ_1.

Assume $m > n$ and $I(c_1 u_1 + \dots + c_m u_m)$ aquire its min at $c_1^{\circ\circ}, \dots, c_m^{\circ\circ}$

In particular setting $c_{n+1}, \dots, c_m = 0$

we infer

$$I[c_1^{\circ\circ} u_1 + \dots + c_m^{\circ\circ} u_m] \leq I[c_1 u_1 + \dots + c_m u_m] \text{ for all } c_1, \dots, c_m$$

and in particular $(m > n)$

$$I[c_1^{\circ\circ} u_1 + \dots + c_m^{\circ\circ} u_m] \leq I[c_1 u_1 + \dots + c_n u_n] \text{ for all } c_1, \dots, c_n$$

$$\therefore I[\phi_m] \leq \inf_{c_1, \dots c_n} I[c_1 u_1 + \dots + c_n u_n]$$

$$\therefore I[\phi_m] \leq I[\phi_n] \quad \text{if} \quad n < m$$

The fact that $I[\phi_m] \to \lambda_1$ follows from the Dirichlet principle. This means:

The $\inf_{u \in \Sigma} I[u]$ is attained at some $u \in C^2$

Now use the fact that linear span of $u_1, u_2, u_3, \dots$ is dense. Let $\epsilon > 0$ (wrt norm 4.42)

Then

$$\inf_{u \in \Sigma} I[u] + \frac{\epsilon}{2} > I[\tilde{u}] \text{ for some } \tilde{u} \in \Sigma$$

Now $|I[\tilde{u}] - I[c_1 u_1 + \dots + c_n u_n]| < \epsilon/2$

for certain $c_1, \dots, c_n \in \mathbb{R}$ and $n \in \mathbb{N}$

$$\therefore \inf_{u \in \Sigma} I[u] + \epsilon > I[\phi_n] \text{ for some } n$$

and then

$$\inf_{u \in \Sigma} I[u] + \epsilon > I[\phi_m] \quad \forall \quad m > n$$

A good choice of $w_1, w_2, w_3, \ldots$ is the set of piecewise linear functions with rational slopes and rational nodes...

As an example we take a familiar case See the book J. Dettman: Math. Methods in physics and engineering Dover 1988 pp 161-162 For an indepth account on Calculus of variations the book of R. Weinstock is highly recommended Calc. variations with applications to physics and Engg. Dover NY 1974

Consider $y'' + \lambda y = 0 : y(0) = 0; y(1) = 0$

We know the first eigen value is $\lambda_1 = \pi^2$

Take

$$u_1(x) = \begin{cases} x & \text{if } 0 \le x \le \frac{1}{2} \\ 1-x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

Rayleigh quotient $I[u_1] = 12$

Take

$u_2(x) = x(1-x)$. Rayleigh quotient is 10.

Next, take

$u = c_1 u_1 + c_2 u_2$; u_1, u_2 as above

$$I[u] = \int_0^1 u'^2 \Big/ \int_0^1 \rho u^2 \qquad (\rho = 1)$$

$$= \frac{c_1^2 + \frac{1}{3} c_2^2 + \frac{1}{2} c_1 c_2}{\frac{1}{12} c_1^2 + \frac{1}{30} c_2^2 + \frac{5}{96} c_1 c_2}.$$

Minimize over c_1, c_2 and we get the value 9.94 which is a good approx to $\pi^2 = 9.87$

Example: Let us use this to find the first zero of the non trivial solution of

Airy's equation: $Ai(x)$ satisfies

$$y'' + xy = 0$$
$$y(0) = 0, \quad y'(0) = 1$$

Put

$$u(t) = Ai(t\xi)$$
$$\therefore u'' = \xi^2 Ai''(t\xi)$$
$$\therefore u'' + \xi^3 t u = \xi^2 (Ai''(t\xi) + t\xi\, Ai(t\xi)) = 0$$

$u(0) = 0; \quad u(1) = Ai(\xi)$

Now select ξ to be the first positive zero of $Ai(x)$. Then

$$u'' + \xi^3 t u = 0$$
$$u(0) = 0 : u(1) = 0$$

So ξ^3 is the first eigen value of the BVP

$$u'' + \lambda t u = 0 \quad ; \rho(t) = t$$
$$u(0) = 0; \quad u(1) = 0$$

Take $u_1(x), u_2(x)$ as on p169 and then

$$I[c_1 u_1 + c_2 u_2] = \frac{A c_1^2 + 2B c_1 c_2 + C c_1^2}{\alpha c_1^2 + 2\beta c_1 c_2 + \gamma c_2^2}$$

$$A = \frac{1}{24}, \quad B = \frac{5}{192}, \quad C = \frac{1}{60}$$

$$\alpha = 1, \quad \beta = \frac{1}{2}, \quad \gamma = \frac{1}{3}.$$

Now minimize over c_1, c_2 and we get-

$$\text{Min } I[c_1 u_1 + c_2 u_2] = 19.92 \text{ (approx)}$$

So $\xi^3 = 19.92$

$$\xi = 2.71079 \text{ (approx)}$$

Ex: Use MATHEMATICA & Compute $Ai(\xi)$ and see how close is it to zero.

Lemma 4.14 (Simultaneous reduction of two Quadratic forms)

Suppose A, B are two real symm matrices such that B is positive definite

$= 0$

$\det(A - \lambda B)$ is called the Char. Equation of the pair (A, B) and its roots eigenvalues

$$\lambda_1 = \min_{x \neq 0} \frac{x^T A x}{x^T B x}$$

is the smallest eigenvalue and it is attained at an eigenvector v_1, namely

$$(A - \lambda_1 B) v_1 = 0; \quad v_1 \neq 0.$$

The next eigenvalue λ_2 is

$$\min_{x \neq 0,\ x \perp Bv_1} \frac{x^T A x}{x^T B x}$$

and is attained at an eigenvector v_2

$$(A - \lambda_2 B) v_2 = 0; \quad v_2 \neq 0$$

By construction $v_2 \perp Bv_1$

We get this way $v_1, \ldots, v_n$ such that

$v_1, \ldots, v_n$ all non zero

$v_i \perp Bv_j$ if $i \neq j$

$$\langle Av_i, v_j \rangle = \langle (A - \lambda_i B) v_i, v_j \rangle + \lambda_i \langle Bv_i, v_j \rangle$$

$$= \lambda_i \delta_{ij}$$

Ex: Check $v_1, \ldots, v_n$ is a Basis for $\mathbb{R}^n$

The quadratic forms $x^T A x$, $x^T B x$ when expressed in coord wrt $v_1, \ldots, v_n$ are simultaneously diagonal.

Proof exactly parallels the proof of Spectral thm given on pp. 153-155

$\Sigma = \{ x \in \mathbb{R}^n \mid x^T B x = 1 \}$ is compact

$\lambda_1 = \inf_{\Sigma} x^T A x$ attained at v_1

Now imitate the

perturbation argument given earlier.

See F. Gantmacher: Matrix Th. Vol I p 326 ff for applications to the theory of small oscillations. Material covered so far is of paramount importance in Fourier analysis, PDE, theory of Compact Selfadjoint Op. in a Hilbert space and quantum mechanics, approximation theory and variational principles.

In addition to the refs cited so far It is important to consult the two volumes by L. Rayleigh's Theory of Sound, Dover publishes (1945)

Rayleigh quotients appear in section 88 p 109 ff (see examples on p 112 vol I).

In section 9.2 (p 118) Rayleigh discusses eigenfunction expansions

Fourier coeff are called normal coordinates

See the interesting discussion in section 142 on # of zeros of finite lin. comb. of eigenfunctions (called normal functions)

Rayleigh's two volumes is one of the most important reference for the physics of vibrations

Chapter 14 of the book

S. Sternberg: A Mathematical Companion to Quantum Mechanics, Dover 2020

discusses applications to computation of lowest eigenvalue of the Hamiltonian Op. for the hydrogen atom (pp 156-157) and subsequently on p 158 for the hydrogen molecule. Sternberg discusses imp. provides historical details and many imp refs are cited (see C. A. Coulson's Valence cited in the bibliography)

Coda: proof of existence of eigenvalues and eigenfunctions in the case when $\rho : [0,1] \to \mathbb{R}$ is continuous and $\rho > 0$ throughout $[0,1]$

Here we can give a more elementary argument based on Sturm's Comparison thm

See G. F. Simmons: Differential Equations with applications and historical notes, 2nd Ed. McGraw Hill 1972.

First, let $m^2 = \inf \{ \rho(x) / x \in [0,1] \}$

$M^2 = \sup \{ \rho(x) / x \in [0,1] \}$

$0 < m^2 \leq M^2$ $\quad (m, M \geq 0)$

Clearly the case $m = M$ is completely elementary

For $\lambda > 0$, $y(x,\lambda)$ denotes the unique solution of the IVP:

$$y'' + \lambda \rho(x) y = 0$$
$$y(0) = 0, \quad y'(0) = 1$$

This solution will be denoted by $y(x,\lambda)$

WLOG we may assume $\rho(x)$ is defined and continuous on $\mathbb{R}$ with $\inf \rho \geq m^2$ and $\sup \rho \leq M^2$

Now, by Sturm's Comparison thm Comparing $y(x,\lambda)$ with $\sin m\sqrt{\lambda}\, x$ we immediately see that

$y(x,\lambda)$ has infinitely many zeros $0 < \xi_1(\lambda) < \xi_2(\lambda) < \xi_3(\lambda) < \dots$

On the other hand

$\sin M\sqrt{\lambda}\, x$ has its first positive zero at $\dfrac{\pi}{M\sqrt{\lambda}} > 1$ if λ is small positive.

Comparing $y(x,\lambda)$ and $\sin M\sqrt{\lambda}\,x$ we see that if λ is small positive $y(x,\lambda)$ cannot vanish on $(0,1]$ and so $\xi_1(\lambda) > 1$

However if λ is large positive the solution $\sin m\sqrt{\lambda}\,x$ has its first zero located at $\frac{\pi}{m\sqrt{\lambda}} < 1$

and so by Sturm's thm $y(x,\lambda)$ must have a zero in $(0,1)$ as well. Thus

$\xi_1(\lambda) < 1$ if λ is large pos.

By Intermediate value thm and Continuity of $\lambda \mapsto \xi_1(\lambda)$ we see that there must be a $\lambda = \lambda_1$ for which $\xi_1(\lambda_1) = 1$

Thus

$y(x,\lambda_1)$ vanishes at $x = 0$

$x = \xi_1(\lambda_1) = 1$

and non vanishing on $(0,1)$

In otherwords

λ_1 is the first eigen value with eigen function $y(x,\lambda_1)$

Note that we can also get an estimate of λ_1 in terms of M and m and I shall leave this to the student.

Now we let $\lambda > \lambda_1$. Again $\sin M\sqrt{\lambda}\,x$ has its two positive zeros at

$\frac{\pi}{M\sqrt{\lambda}}$, $\frac{2\pi}{M\sqrt{\lambda}}$

So if $\frac{2\pi}{M\sqrt{\lambda}} = 1$, $\sin M\sqrt{\lambda}\,x$ has exactly two zeros in $(0,1]$

Whereby $y(x,\lambda)$ must have its

Second zero $\xi_2(\lambda) \geq 1$

On the other hand $\sin m\sqrt{\lambda}\, x$ has its positive zeros at

$$\frac{\pi}{m\sqrt{\lambda}}, \frac{2\pi}{m\sqrt{\lambda}}, \ldots$$

So if $\frac{2\pi}{m\sqrt{\lambda}} = 1$, $\sin m\sqrt{\lambda}\, x$ has two zeros in $(0,1]$

Hence by Sturm's thm $y(x,\lambda)$ must have atleast two zeros in $(0,1]$

That is $\xi_2(\lambda) \leq 1$

So $\exists\ \lambda_2$ such that $\lambda_2 > \lambda_1$ and

$$\xi_2(\lambda_2) = 1$$

namely

$y(x,\lambda_2)$ satisfies $y'' + \lambda_2 \rho(x) y = 0$

$y(0) = 0$, $y(\xi_2(\lambda_2)) = 0$

namely $y(1) = 0$

Thus y is an eigenfunction with eigenvalue λ_2 and this is evidently the 2nd eigenvalue.

The procedure continues and we construct the successive eigenvalues and eigenfunctions for the BVP

$$y'' + \lambda \rho(x) y = 0; \quad y(0) = 0, \ y(1) = 0$$

Note that the condition $\rho > 0$, ρ cont. on $[0,1]$ is slightly restrictive but the above proof comes at a significantly "lower price". In contrast the validity of Dirichlet principle is highly nontrivial.

Part II

Nonlinear Systems:

In this second part of the course we commence with the study of nonlinear systems of ODEs

We shall break this into several Chapters ...

V - Nonlinear Systems (Generalities)

We shall mostly confine ourselves to autonomous systems namely,

$$\frac{dx}{dt} = f(x) \qquad (5.1)$$

$f : \Omega \to \mathbb{R}^n$ is a smooth function defined on an open connected set Ω in $\mathbb{R}^n$.
The fundamental existence uniqueness thm of chapt 1 applies and to each point $p \in \Omega$
$t \longmapsto \phi_t(p)$ denotes the unique solution of the IVP

$$\dot{x} = f(x), \quad x(0) = p \qquad (5.2)$$

The initial time will always be taken to be zero.
Of course the solution is only defined on an interval I containing the origin
the size of I could depend upon p.

We first introduce some terminology:

Def 5-1 (Rest points or equilibrium points)
In the above context,
A point $p \in \Omega$ is said to be a rest point of (5.1) if $f(p) = 0$.
Rest point = equilibrium points.

The first step in understanding the behaviour of solutions of (5.1) is to locate the rest points.
Notice that,
A point $p \in \Omega$ is a rest point iff the constant function

$$x(t) \equiv p \quad \text{for all } t \in \mathbb{R}$$

satisfies the differential equation (5.1)
This is immediate.

Let us look at some simple examples:

(i) For the one dimensional system : $\Omega = \mathbb{R}$

$$\frac{dy}{dt} = y^2$$

The origin is the only rest point.

(ii) For the one dimensional case

$$\frac{dy}{dt} = k y(1-y) \qquad (K>0 \text{ const}) \quad (5.3)$$

Or more complicated:

$$\frac{dy}{dt} = \frac{k y(1-y)}{(5+\sin^2 y)} \qquad (5.4)$$

It would be rather awkward to attempt solving (5.4) using the "method of sep. var".
Writing out explicit solutions of (5-3) would be tedious!

Let us study the behaviour of solutions of (5-3) without explicitly solving (5-3)
The example appears in Verhulst's model for a one dimensional species ecological system

See for example p3 of J.D. Murray, Mathematical Biology, vol I, Springer 2002 (third Edition)

Let us take an Initial condition

$$y(0) = c \quad \text{for (5-3)}$$

If $c = 0$ or 1 namely if the IC is at a rest point then the solution is obviously the Constant Solution $y(t) \equiv 0$ or $y(t) \equiv 1$

Setting aside these two cases let us next assume $0 < c < 1$

$y(t)$ = population of the species at time t

So Initial population $\in (0,1)$.

$y'(0) = kc(1-c) > 0$

So the solution $y(t)$ is strictly increasing at the origin and

$$y'(t) = k\,y(t)(1-y(t))$$

$\Rightarrow$ $y(t)$ will monotonically increase with t as long as $y(t) < 1$

Que: Can it happen that $y(t_0) = 1$ for some $t_0 > 0$?

Well, then $y(t)$ and the constant function 1 both satisfy the ODE $\dot{y} = ky(1-y)$ and both have the same IC at $t = t_0$.

So by uniqueness

$$y(t) \equiv 1 \text{ for all } t \in \mathbb{R}$$

which is a contradiction.

Thus, $y(t)$ is FORCED to remain in the open interval $(0,1)$ and in particular is confined in the compact set $[0,1]$

By the general principles of Chapt I, life span of solution is infinite. That is to say, the maximal domain of existence is $\mathbb{R}$.

So $\lim_{t\to\infty} y(t) = \ell$ exists. Likewise $\lim_{t\to-\infty} y(t) = m$ exists.

But $y' = ky(1-y)$ (5.5)

$\Rightarrow$ $\lim_{t\to\infty} y'(t)$ exists. Use MVT to show that $\lim_{t\to\infty} y'(t) = 0$

$\therefore$ Equation (5.5) now implies

$$0 = k\ell(1-\ell)$$

$\therefore \ell = 1$ ($\ell \neq 0$ since $y(t)$ monotone increasing; $y(0) > 0$)

Ex: prove that $m = 0$.

We now come to the very important Conclusion that ALL Solutions $y(t)$ with Initial values $y(0) \in (0,1)$ approach the equilibrium State namely the rest point 1.

Exercise: prove that if the IC $y(0) > 1$ then again the solution $y(t)$ exists for all $t \in \mathbb{R}$ and $\lim_{t \to \infty} y(t) = 1$

Look up Murray's book cited above to understand this in the light of population dynamics in a One Species ecology.

We shall prove that in general for any ~~System~~ System (5.1), if a Solution $x(t)$ has a limit as $t \to +\infty$ then this limit must be a rest point

In Chapter Zero we looked at the Case of the Simple pendulum which is really a planar System

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell} \sin x_1 \qquad (5.6)$$

Equilibrium points: $(0,0)$, $(\pm\pi, 0)$, $(\pm 2\pi, 0)$ ...
However, Since $x_1(t)$ is the angular displacement wrt the mean position
the equilibrium points $(\pi, 0)$ and $(-\pi, 0)$ are identical.
The equilibrium points $(0,0)$ and $(2\pi, 0)$ are identical etc;

So there are really only two equilibrium points.

Let us prescribe IC: $x_1(0) = 0$, $x_2(0) = c$
We have seen that for small values of c the solution $(x_1(t), x_2(t))$ has the property that- it is $x_1(t)$ is periodic

Recall from Chapt 0 that for a sol. $(x_1(t), x_2(t))$

$$\frac{d}{dt}\left(\frac{x_2^2}{2} - \frac{g}{\ell}\cos x_1\right) = 0$$

Or $$\frac{x_2^2}{2} - \frac{g}{\ell}\cos x_1 = \frac{c^2}{2} - \frac{g}{\ell} \qquad (5.7)$$

if IC prescribed are $x_1(0) = 0$
$x_2(0) = c$

$$\boxed{x_2^2 = c^2 - \frac{4g}{\ell}\sin^2\left(\frac{x_1}{2}\right) \qquad (5.7')}$$

Law of Conservation of Energy (!)

Note that if $c^2 < \frac{4g}{\ell}$ the pendulum exhibits oscillatory motion
with $\theta_0 = \sup x_1(t) < \pi$
and $x_1(t) = \theta_0$ for some $t = t_0$
Namely, the pendulum reaches the max. amplitude at $t = t_0$, at this time $\dot{x}_2(t_0) = 0$ (its velocity becomes zero) and then swings back ...

However if $c^2 > 4g/\ell$
then $\inf x_2^2 \neq 0$. That is $|\dot{x}_1(t)| \geq m > 0$ for all $t \in \mathbb{R}$

Assume $\dot{x}_1(0) = x_2(0) = c > 0$
then $\dot{x}_1(t) = x_2(t) > 0$ for all times

The amplitude $x_1(t)$ keeps increasing unboundedly: $x_1(t) \geq mt$ for all t
The pendulum ascends to the topmost position to angle π and swings beyond, performing a circular motion ...

The quality of motion is different for

$c^2 < 4g/\ell$ and $c^2 > 4g/\ell$

Que: What happens if $c^2 = 4g/\ell$?
with $c > 0$.
Well, equation (5·7)' now assumes a simple form:

$$x_2^2 = \frac{4g}{\ell} \cos^2\left(\frac{x_1}{2}\right)$$

Or $x_2 = \sqrt{\frac{4g}{\ell}} \cos\left(\frac{x_1}{2}\right)$ since we are assuming $x_2(0) = c > 0$.
That is

$$\frac{dx_1}{dt} = \left(\frac{4g}{\ell}\right)^{\frac{1}{2}} \cos\left(\frac{x_1}{2}\right) \qquad (5\cdot8)$$

Exercise: Solve (5·8) explicitly: $x_1(0) = 0$.
Use your solution to check that

$$\lim_{t\to\infty} x_1(t) = \pi \;;$$

$$\lim_{t\to\infty} x_2(t) = 0$$

In this case the solution $(x_1(t), x_2(t))$ has a limit as $t \to \infty$ and this limit is the REST POINT $(\pi, 0)$.
This is one of the rare cases of the pendulum equation being solvable via elementary methods.

Let us calculate the value of the energy (5.7) for the case $x_1(0)=0$, $x_2(0) = c = \left(\frac{4g}{\ell}\right)^{\frac{1}{2}}$

$$\text{The value} = \left(\frac{x_2^2}{2} - \frac{g}{\ell} \cos x_1 \right)\Big|_{t=0}$$

$$= \frac{c^2}{2} - \frac{g}{\ell} = \frac{g}{\ell}.$$

Note that the point $(\pi, 0)$ is a critical point of the energy function

$$\phi(x_1, x_2) = \frac{x_2^2}{2} - \frac{g}{\ell} \cos x_1 \qquad (5.7)$$

Def 5.2: (Critical points and Critical values)
We give the GENERAL DEF.

Let Ω be an open subset of $\mathbb{R}^n$
$F: \Omega \to \mathbb{R}^m$ be a smooth map.
A point $p \in \Omega$ is said to be a critical point of F if the derivative matrix

$$DF(p) \qquad (5.9)$$

has rank strictly less than m

That is to say the derivative $DF(p) : \mathbb{R}^n \to \mathbb{R}^m$ fails to be surjective

Remember that the derivative $DF(p)$ is a linear transf: $\mathbb{R}^n \to \mathbb{R}^m$

A point $p \in \Omega$ which is not a critical point is simply referred to as a non critical point!

A point $q \in \mathbb{R}^m$ is called a critical value if $q = F(p)$ for some critical point $p \in \Omega$

A value $q \in \mathbb{R}^m$ which is not a critical value is called a REGULAR value.

Note: $q \in \mathbb{R}^m$ is a REGULAR value iff $F^{-1}(q)$ does not contain ANY critical point.

Even if $F^{-1}(q)$ contains ONE critical point, then q is a critical value.

By convention, if $F^{-1}(q)$ is Empty then q is a regular value.

Coming back to the pendulum:

The two cases considered $c^2 < 4g/\ell$, $c^2 > 4g/\ell$

Corresponds to the cases when

Energy $< g/\ell$

Energy $> g/\ell$ respectively.

So we see that when the energy crosses a critical value $(= g/\ell)$ the qualitative behaviour of the trajectories (solutions) is seen to undergo a dramatic change

Energy $< g/\ell \longrightarrow$ Oscillatory motion.

Pendulum ascends from zero amplitude to a max value $\theta_0 < \pi$, its velocity becomes zero at finite time $t = t_0 (= T/4)$ after which it swings back.

Energy $> g/\ell \longrightarrow$ Pendulum exhibits circular motion; It reaches the topmost-point $x_1(t_0) = \pi$ at some finite time t_0 with nonzero velocity and swings past it exhibiting a circular motion.

These two sets of qualitatively different motions seperated by a VERY special

trajectory ! The case wherein

$$\lim_{t\to\infty} x_1(t) = \pi \quad ; \quad \lim_{t\to\infty} \dot{x}_1(t) = 0 .$$

As the pendulum ascends, it loses just the "right amount of energy" slowing down, taking infinite amount of time to reach the pinnacle !

Let us now establish a simple general result-

Thm 5.1: Suppose $y(t)$ is a solution of the system of ODEs $\frac{dy}{dt} = f(y)$

($f: \Omega \to \mathbb{R}^n$ smooth ; Ω open in $\mathbb{R}^n$) with maximal interval of existence (α, β). Assume $\lim_{t\to\beta-} y(t)$ (resp or $\lim_{t\to\alpha+} y(t)$) exists

Say $p \in \Omega$.

Then (i) $\beta = +\infty$ (resp $\alpha = -\infty$)

(ii) p is a rest point.

proof (ii) Assume the Contrary namely, $f(p) \neq 0$

$|f(p)| = 2\mu > 0$ say.

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y(s))\, ds \quad : \quad t_0 \in (\alpha, \beta)$$

$$\langle y(t) - y(t_0),\ f(p) \rangle = \int_{t_0}^{t} \langle f(y(s)),\ f(p) \rangle\, ds$$

$$\therefore \left| \int_{t_0}^{t} \langle f(y(s)),\ f(p) \rangle\, ds \right| \leq |f(p)|\,(|y(t)| + |y(t_0)|)$$

$$\leq 2M\,|f(p)|$$

$$M = \sup_{t \geq t_0} |y(t)| \quad (\text{finite since } \lim_{t\to\beta-} y(t) \text{ exists})$$

But,

$$\langle f(y(s)),\ f(p) \rangle \to |f(p)|^2$$

$$\therefore \langle f(y(s)),\ f(p) \rangle > \tfrac{1}{2}\,|f(p)|^2 \quad : \quad s \geq t_0 .$$

for some $t_0 \in (\alpha, \beta)$

$$\therefore \frac{1}{2}|f(p)|^2 (t - t_0) \leq \int_{t_0}^{t} \langle f(y(s)), f(p) \rangle ds$$

$$\leq 2M |f(p)|$$

$$\therefore |f(p)|^2 (t - t_0) \leq 4M |f(p)|$$

Now by (i), $\beta = +\infty$ (we shall provide a proof)
So letting $t \to +\infty$ we arrive at a contradiction since $|f(p)| > 0$ (by assumption)

To see that $\beta = +\infty$ simply use the fundamental ideas laid down in Chapt 1 Concerning life span of solutions.

Well, suppose $\beta < +\infty$.

$\lim_{t \to \infty} y(t)$ exists $= p \in \Omega \Rightarrow \exists\ t_0 \in (\alpha, \beta)$

such that $y(t) \in \bar{B}_\epsilon(p) \subset \Omega$; $t \geq t_0$

($\bar{B}_\epsilon(p)$ compact)

Thus $(t, y(t))$ is constrained to stay inside the compact subset $[t_0, \beta] \times \bar{B}_\epsilon(p) \subset \mathbb{R} \times \Omega$

But we know if the lifespan is finite $(t, y(t))$ must exit every compact subset of $\mathbb{R} \times \Omega$ as $t \to \beta -$. Proof is complete.

First Integrals. Invariant Sets:

First we define the notion of invariant sets. We shall assume that the solutions of the ODE $\dot{y} = f(y)$ are defined on whole of $\mathbb{R}$ (for any choice of initial conditions in Ω)

Def 5.3: A non empty subset $S \subset \Omega$ is said to be forward (backward) invariant if for $p \in S$, the unique solution

$t \mapsto \phi_t(p)$ of the IVP $\dot{y} = f(y)$, $y(0) = p$

lies in S for $t \geq 0$ (resp $t \leq 0$)

That is to say for $t \geq 0$, the point $\phi_t(p)$ of the trajectory through p lies in S.

For backward invariance the condition is that $\phi_t(p) \in S$ for $t \leq 0$.

S is said to be invariant if it is both forward as well as backward invariant.

That is to say $p \in S \Rightarrow \phi_t(p) \in S$ for all $t \in \mathbb{R}$

A simple example: A rest point is obviously an invariant set.

Ofcourse Ω is trivially an invariant set.

It is easy to see that

If S_1, S_2 are invariant then $S_1 \cap S_2$ and $S_1 \cup S_2$ are both invariant

We shall regard the empty set as being invariant

Are arbitrary unions and intersections of invariant sets again invariant?

Discuss these properties for forward / backward invariant sets.

Def 5.4 (First Integrals) A global first integral for the system

$$\dot{y} = f(y)$$

($f : \Omega \to \mathbb{R}^n$ smooth) is a smooth function $\phi : \Omega \to \mathbb{R}$ such that

(i) $\nabla \phi \neq 0$ almost everywhere on Ω

(ii) ϕ is constant along solutions namely $\frac{d}{dt}(\phi(y(t)) \equiv 0$ for all solutions $y(t)$ of our system of ODEs.

(i) excludes trivialities such as constant functions.

Example: For the pendulum

$$\frac{dx_1}{dt} = x_2; \quad \frac{dx_2}{dt} = -\frac{g}{\ell}\sin x_1$$

the energy $\phi(x_1, x_2) = \frac{x_2^2}{2} - \frac{g}{\ell}\cos x_1$

is a first integral. Note that $\nabla\phi$ vanishes at $(0,0)$ and $(\pm\pi, 0)$

This is why we did not demand $\nabla\phi \neq 0$ everywhere in Ω. The gradient $\nabla\phi$ is allowed to vanish on a small subset of Ω.

Example: Lotka-Volterra system (see p 60 of these slides) Governing two species Ecology.

See p 79 ff of the book of J.D. Murray: Mathematical Biology, Vol-I, Springer 2002 for details.

This is the planar system

$$\frac{dx}{dt} = ax - bxy \qquad x, y > 0$$

$$\frac{dy}{dt} = -cy + exy \qquad (a, b, c, e > 0 \text{ constants}) \qquad (5.10)$$

To find a first integral we employ a simple trick

$$\frac{dy}{dx} = \frac{-y(c - ex)}{x(a - by)}$$

$$\left(\frac{a - by}{y}\right)dy + \frac{(c - ex)}{x}dx = 0$$

Gen sol. of this provides a first integral:

$$\phi(x,y) = a\log y + c\log x - by - ex \qquad (5.11)$$

Exercise: Check directly that (5.11) is a first integral for (5.10)

Note: If ϕ is a first integral of $\dot{y} = f(y)$ then $\nabla\phi \cdot f(y) \equiv 0$

and conversely (provided $\nabla\phi \neq 0$ almost everywhere)

We have pointed out (see p60) that the map ϕ above is a proper map from the open first-quadrant in $\mathbb{R}^2$ to $\mathbb{R}$.

Assume that $F: \Omega \to \mathbb{R}$ is a first-integral for the system $\frac{dy}{dt} = f(y)$

Then: For each $c \in \mathbb{R}$, $F^{-1}(c)$ is an invariant set

Well, assume $F^{-1}(c) \neq \phi$ and $p \in F^{-1}(c)$

$t \longmapsto \phi_t(p)$ is the unique sol. with Initial Cond. p.

$F(\phi_t(p))$ is Constant in time

$$F(\phi_t(p)) = F(\phi_0(p)) = F(p) = c$$

for all $t \in \mathbb{R}$

$\therefore \phi_t(p) \in F^{-1}(c)$ for all $t \in \mathbb{R}$

So $p \in S = F^{-1}(c) \Rightarrow \phi_t(p) \in S$ for all $t \in \mathbb{R}$

Exercise: Suppose $F: \Omega \to \mathbb{R}$ is a first-integral for the system $\dot{y} = f(y)$

Then the set

$$S_{a,b} = \{ y \in \Omega / \; a < F(y) < b \}$$

is an invariant set.

First integrals are Conserved quantities. The fact that F is Const. along solutions is an example of a Conservation Law (like the law of Conservation of energy for the pendulum)

Thus first integrals Can be thought of as some kind of generalization of the notion of "Energy"

Let us now take up a slightly different example

$$\frac{dx}{dt} = y + x(1-x^2-y^2) \qquad (5.12)$$

$$\frac{dy}{dt} = -x + y(1-x^2-y^2)$$

Claim: The set $S = \{(x,y) \in \mathbb{R}^2 \,/\, x^2+y^2 < 1\}$ is an invariant set

Mult. first equation by x, second by y and add.

$$x\dot{x} + y\dot{y} = +(x^2+y^2)(1-x^2-y^2)$$

So for a solution $(x(t), y(t))$:

$$\frac{d}{dt}\left((x(t))^2 + (y(t))^2\right)$$

$$= 2\left((x(t))^2 + (y(t))^2\right)\left(1 - (x(t)^2 + y(t))^2\right)$$

Or putting $\rho(t) = (x(t))^2 + (y(t))^2$ we get—

$$\frac{d\rho}{dt} = 2\rho(1-\rho) \qquad (5.13)$$

So if $\rho(0) = (x(0)^2 + y(0)^2) < 1$ namely if the initial point $(x(0), y(0)) \in S$ and $\neq (0,0)$

then $\dot{\rho}(0) = 2\rho(0)(1-\rho(0)) > 0$

ρ increases with time

(The case $(x(0), y(0)) = (0,0)$ is trivial. Origin is a rest point)

$\rho(t)$ is strictly increasing in a nbd of $t=0$.

Now $\rho(t_0) = 1$ for some $t = t_0$

$\Rightarrow \rho(t) \equiv 1$ Using uniqueness for IVP with IC $\rho(t_0) = 1$ for the ODE (5.13)

and that leads to a contradiction since ρ is strictly inc. at $t=0$.

Thus $\rho(t)$ can never attain the value 1

So $0 < \rho(t) < 1$ for all times

$\Rightarrow (x(t), y(t)) \in S$ for all times.

We shall see that the system (5.12) does not admit first integrals.

Note that $x(t) = \sin t$
$y(t) = \cos t$
satisfies the system (5.12) so the circle
$\partial S = \{(x,y) \in \mathbb{R}^2 / x^2+y^2=1\}$
is the image of this solution
Thus ∂S is an invariant set.

Exercise: Show that $T = \{(x,y) \in \mathbb{R}^2 / x^2+y^2>1\}$ is an invariant set for (5.12)

Def 5.5: Periodic Solutions

A solution $y(t)$ of the autonomous system

$$\frac{dy}{dt} = f(y) \qquad (5.14)$$

($f: \Omega \to \mathbb{R}^n$ smooth) is said to be periodic if
(i) $y(t)$ is non constant
(ii) There exists $T>0$ such that
$y(0) = y(T)$

Now note that $y(t+T) = z(t)$ is also a solution of (5.14)
Further $z(0) = y(T) = y(0)$
So by Uniqueness, $z(t) \equiv y(t) \quad \forall\, t \in \mathbb{R}$
That is $y(t) = y(t+T) \quad \forall\, t \in \mathbb{R}$
So $t \mapsto y(t)$ is a periodic function in the usual sense.
The set of all periods is an infinite cyclic subgroup of $(\mathbb{R}, +)$ and the positive generator of this subgroup is called the period of the solution

Thm 5.2: Let $y(t)$ be a periodic solution of

$$\dot{y} = f(y)$$

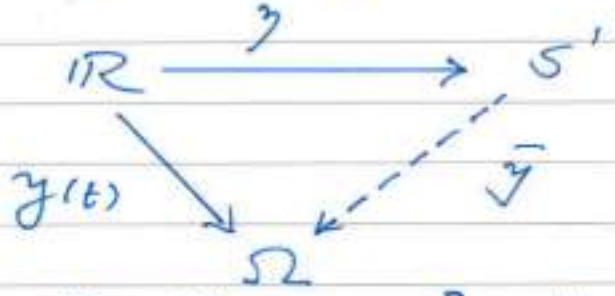
($f: \Omega \to \mathbb{R}^n$ smooth)

Then the image $\{ y(t) / t \in \mathbb{R} \}$ is homeomorphic to the circle $S^1 = \{ z \in \mathbb{C} / |z| = 1 \}$

proof: Let T be the period of $y(t)$ (the minimal period)

$\gamma: \mathbb{R} \to S^1$ given by $\gamma(s) = \exp\left(2\pi i \frac{s}{T}\right)$

$\gamma(s_1) = \gamma(s_2)$ iff $s_1 - s_2 = nT$; $n \in \mathbb{Z}$



Define $\bar{y}: S^1 \to \Omega$ via the prescription

$$\bar{y}\left(\exp 2\pi i s/T\right) = y(s)$$

Now $\exp\left(2\pi i \frac{s_1}{T}\right) = \exp\left(2\pi i \frac{s_2}{T}\right)$

$\iff s_1 - s_2 = nT$; $n \in \mathbb{Z}$

$\iff y(s_1) = y(s_2)$ (Recall that T is the Minimal period)

By the familiar property of quotient maps (See J.L. Kelley, General topology, Springer p. 95)

$\bar{y}: S^1 \to \Omega$ is continuous. It is evidently injective and S^1 being compact, $\bar{y}$ is a homeomorphism onto its image.

In particular, for a two dimensional system

$$\dot{x} = P(x,y)$$
$$\dot{y} = Q(x,y) \qquad (5.15)$$

: P, Q smooth on Ω

Ω open in $\mathbb{R}^2$

The image of a periodic solution is a Jordan Curve.

Assume that Ω is a Convex open set in $\mathbb{R}^2$; $P, Q : \Omega \to \mathbb{R}$ smooth and (5.15) has a periodic solution. Its image γ is a Jordan Curve lying in Ω

Interior of γ is then $\subset \Omega$

(Recall the Jordan Curve theorem)

The interior of γ encloses a rest point.

This is seen in the following examples

(i) Harmonic Oscillator

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1$$

(ii) $\dot{x} = y - x(1-x^2-y^2)$ (eqn (5.12))

$$\dot{y} = -x - y(1-x^2-y^2)$$

(iii) Simple pendulum

$$\dot{x} = y, \quad \dot{y} = -\frac{g}{\ell} \sin x$$

In (ii) the system has a UNIQUE periodic solution $(\sin t, \cos t)$

Que: Is it in general true? For a Convex Ω with a periodic solution of the ODE (5.15) does Image enclose a rest point?

YES.

One can prove this using Stokes' theorem for example. We shall infact prove a much stronger version using the Brouwer's fixed point thm.

Note that Int γ is an invariant set in Ω (I repeat, Int γ is the interior of γ in the Sense of Jordan Curve thm).

This invariance follows at once from appealing to the Uniqueness Clause in the existence Uniqueness

-Thm of Chapt 1.
We shall prove in a later Chapter -that if there is an invariant set D homeomorphic to -the closed unit ball in $\mathbb{R}^n$ -then D must contain a rest point.

We now discuss periodic solutions of the Lotka-Volterra System (5.10)

However before taking -that up we must discuss some general notions in Calculus -that are frequently employed in -the -theory of autonomous systems.

Some Calculus: Let Ω be an open set in $\mathbb{R}^n$ a smooth function $f: \Omega \to \mathbb{R}$ should be regarded as some physical quantity such as energy -for instance (called observables) and only -those notions -that are coordinate invariant are of interest.

So we begin by defining Coordinate transf:

Def 5.6: A Coordinate transf is a smooth bijective map $F: \Omega_1 \to \Omega_2$ (Ω_1, Ω_2 open in $\mathbb{R}^n$) such -that $DF(x)$ is nonsingular -for each $x \in \Omega_1$.
This ensures -that $F^{-1}: \Omega_2 \to \Omega_1$ is also smooth
(See W. Rudin, principles, Chapt 9)

If f is an observable on Ω_2 namely $f: \Omega_2 \to \mathbb{R}$ is a smooth ~~map~~ function -then $f \circ F$ is an observable on Ω_1.

Infact f and $f \circ F$ are really -the same entities in different (equivalent) Coordinate systems. For instance one may express a function of (x, y, z) in spherical polar coord. Or Cylindrical Coord or say ellipsoidal Coord (!) depending upon ones convenience

These Coordinate systems are discussed in Classical texts on mathematical physics - See p 226 of R. Courant and D. Hilbert, Methods of Math phy vol I (Indian reprint 2008). These coord. are of use in several applications (See G. Dassios, Ellipsoidal harmonics, Cambridge University press)

The notion of a Critical point is certainly invariant. Suppose $p \in \Omega_2$ is a Critical point of $f : \Omega_2 \to \mathbb{R}$ Then $F^{-1}(p) = q \in \Omega_1$ is a Critical point of $f \circ F : \Omega_1 \to \mathbb{R}$

Follows at once from Chain Rule. Exercise. prove this.

In elementary Calculus Courses one encounters the notion of a the Hessian $\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right]$

This is a real symmetric matrix.

We now examine what happens to the Hessian at a Critical point when we perform a Coordinate transformation

Elementary Calculus tells us that the point p is a local Minimum if the Hessian is positive def. (and p is critical)

We of course expect this property of being a local min a Coordinate invariant namely,

if $p \in \Omega_2$ is a local Min for $f : \Omega_2 \to \mathbb{R}$ then $q = F^{-1}(p)$ would be a local Min for $f \circ F : \Omega_1 \to \mathbb{R}$

So the Hessian of $f \circ F$ at $F^{-1}(p)$ would also be positive definite (?) Let us check this algebraically. We shall infact prove More!

Thm 5.3: Suppose $F : \Omega_1 \to \Omega_2$ is a Coord transf (Also known as a diffeomorphism)

Assume $p \in \Omega_2$ is a Critical point of $f : \Omega_2 \to \mathbb{R}$

Then

(i) $q = F^{-1}(p) \in \Omega_1$ is a Critical pt of $f \circ F : \Omega_1 \to \mathbb{R}$

AND

(ii) The Hessian of f at p is Congruent to the Hessian of $f \circ F$ at q.

Recall from linear algebra:

Two real symmetric matrices A, B are said to be Congruent if there exists a non singular matrix P such that $B = P^T A P$: $P \in GL_n(\mathbb{R})$

Check: Congruence is an equivalence relation.

The Spectral Thm says every real Symm matrix is Congruent to a diagonal Matrix. Prove: Every real symmetric matrix is Congruent to a diagonal matrix with diagonal entries $+1, -1$ or zero.

# of zeros that appear = nullity of A.

Proof of the Theorem: (i) was an exercise.

proof of (ii) For simplicity of notation let us assume $p = q = 0$.

Let $y = F(x)$: $(y_1, \ldots, y_n)$ Coord in Ω_2

$(x_1, \ldots, x_n)$ Coord in Ω_1

Hessian of f at origin: $\left[\frac{\partial^2 f}{\partial y_i \partial y_j}(0) \right]$

Now, $\frac{\partial f}{\partial y_i} = \sum_{t=1}^{n} \frac{\partial \tilde{f}}{\partial x_t} \frac{\partial x_t}{\partial y_i}$: $\tilde{f} = f \circ F : \Omega_1 \to \mathbb{R}$

$$\frac{\partial^2 f}{\partial y_j \partial y_i} = \sum_{s,t=1}^{n} \frac{\partial^2 \tilde{f}}{\partial x_s \partial x_t} \frac{\partial x_s}{\partial y_j} \frac{\partial x_t}{\partial y_i} + \boxed{\sum_{t=1}^{n} \frac{\partial \tilde{f}}{\partial x_t} \frac{\partial}{\partial y_j}\left(\frac{\partial x_t}{\partial y_i} \right)} \qquad (5.13)$$

Now at the origin, $\frac{\partial \tilde{f}}{\partial x_t} = 0$ for $t = 1, 2, \ldots, n$

So the stuff inside the box in RHS of (5.13) Vanishes (!)

So we are left with

$$\frac{\partial^2 f}{\partial y_i \partial y_j}(0) = \sum_{s,t=1}^{n} \frac{\partial^2 \tilde{f}}{\partial x_s \partial x_t}(0) \frac{\partial x_s}{\partial y_j} \frac{\partial x_t}{\partial y_i}$$

This last equation clearly reveals that the real symm. matrices $\left[\frac{\partial^2 f}{\partial y_i \partial y_j}(0)\right]$ and $\left[\frac{\partial^2 \tilde{f}}{\partial x_s \partial x_t}(0)\right]$ are Congruent. Proof is Complete.

Sylvester's law of Inertia: This is an important theorem in linear algebra that we state here. For a proof see S. Lang: Linear Algebra, 2nd Ed. Addison-Wesley, 1972 (pp 237-238)

Thm 5.4 (Sylvester's law of Inertia) If A, B are Congruent real symmetric matrices then

Pos. Eigen values of A = # of Pos. Eigen values of B
Neg. Eigen values of A = # of Neg. Eigen values of B
of Zero Eigen values of A = # of Zero eigen values of B
(nullity A) (nullity of B)

$\nu_+(A)$ = # of pos Eigen values, $\nu_-(A)$ = # of Neg. Eig. Val.
$\nu_0(A)$ = Nullity of A

$(\nu_+(A), \nu_-(A), \nu_0(A))$ is called the Inertia triple of A.

Congruent Matrices have the SAME Inertia triple.

In view of this, at a Critical point p the inertia triple of the Hessian of f (at p) has a well defined meaning indep of Coordinates.

Def: A critical point p of a C^∞ function $f: \Omega \to \mathbb{R}$ is said to be non degenerate if the Hessian of f at p is non singular

It is clear that non-degeneracy is a Coordinate invariant notion.

We now state (without proof) the following imp result:

Thm 5.5 (Morse lemma): Suppose $f: \Omega \to \mathbb{R}$ is a C^∞ function and $p \in \Omega$ is a nondegenerate critical point of f

Then there exists a nbd U of p and a nbd V of 0 in $\mathbb{R}^n$ and a coordinate transf $F: U \to V$:

$F(p) = 0$ (i.e F is bijective smooth, DF nonsing throughout U)

Such that

$f \circ F^{-1} : V \to \mathbb{R}$ is the quadratic map.

$$y \mapsto y_1^2 + \dots + y_k^2 - y_{k+1}^2 - \dots - y_n^2$$

k = # of pos. Eigen values of Hessian of f at p.

$n-k$ = # of neg Eigen values of Hessian of f at p.

Proof is available in many books. For example*

M. W. Hirsch: Differential topology, Springer 1976. (See p145 second printing)

Let us look at the case of pendulum equation and the Lotka Volterra systems:

For the pendulum:

$$\phi(x_1, x_2) = \frac{x_2^2}{2} - \frac{g}{\ell} \cos x_1$$

Hessian is non degenerate at both rest points $(0,0)$, $(\pi, 0)$

Check that the inertia triple

for $(0,0)$ is $(2,0,0)$ (2 pos. Eigen values)

for $(\pi, 0)$ is $(1,-1,0)$ (1 pos and 1 neg eigen value)

Ex. Discuss the Critical pt of (5.11) (Lotka-Volterra eqn)

* We cite Hirsch since we have earlier cited his other books on DEs...

Back to differential Equations:

Thm 5.6: Suppose S is an invariant set for $\dot{y} = f(y)$ ($f: \Omega \to \mathbb{R}^n$ smooth, Ω open in $\mathbb{R}^n$) such that S is homeomorphic to a circle.

(i) If S does not contain any rest points then S is the image of a periodic sol. That is for any $p \in S$, the IVP with IC $y(0) = p$ is periodic and its image is S.

(ii) If S contains one or more rest points then for each $p \in S$ as Initial Cond, then $\gamma(t)$, the solution, has $\mathbb{R}$ as maximal interval of existence and $\lim_{t \to \pm\infty} \gamma(t)$ exist and is a rest point on S

The limits are equal if there is exactly one rest point on S

and distinct if there are atleast two rest points on S (of course we assume $\gamma(t)$ nonconst)

In (ii) the invariant set S is called a homoclinic cycle if there is only one rest point on S. In case of two or more rest points lying on S, we say S is a heteroclinic cycle.

Postponing the proof, let us illustrate the use of this For the Lotka Volterra system we had the first integral

$$\phi(x,y) = a \log y + c \log x - by - ex \qquad (5.11)$$

Critical point : $\left(\frac{c}{e}, \frac{a}{b}\right)$ is also a rest point

It is readily verified that the unique rest point happens to be a local Maximum of the first integral $\phi(x,y)$.

We can of course work with $-\phi$ and then $\left(\frac{c}{e}, \frac{a}{b}\right)$

is a local Min of $-\phi$

By Morse lemma there is a coordinate transf-

$$(x, y) = F(u, v)$$

Mapping a nbd of $(0,0)$ diffeomorphically onto a nbd of $\left(\frac{c}{e}, \frac{a}{b}\right)$ such that-

$$(\phi \circ F)(u, v) = u^2 + v^2 \qquad (5.14)$$

The level sets $(\phi \circ F)^{-1}(c_0)$ of (5.14) are all circles surrounding $(0,0)$ (in the nbd in question)

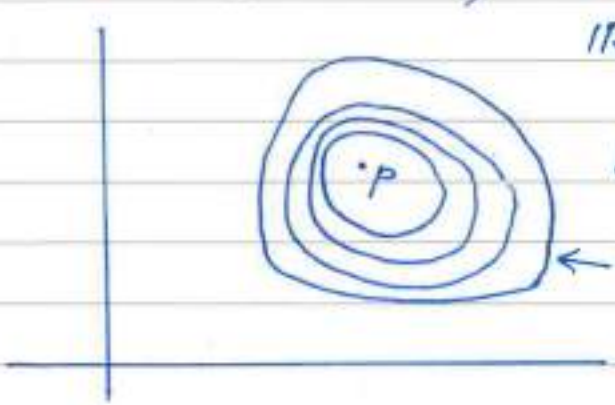
Likewise the level sets $\phi^{-1}(c_0)$ are diffeo. images of a circle and so is a Jordan curve in (x,y) plane containing $\left(\frac{c}{e}, \frac{a}{b}\right)$ in its interior

But the level sets $\phi^{-1}(c_0)$ are invariant sets for the Lotka-Volterra system (5.10) and these do not contain rest points if $c_0 \neq -\phi\left(\frac{c}{e}, \frac{a}{b}\right)$ (the actual value of the Min of $-\phi$)

So by Thm 5.6, the level sets $-\phi^{-1}(c_0)$ are all periodic trajectories (images of periodic sol) of the Lotka-Volterra system (5.10)

Here the value c_0 is close to the Critical value $-\phi\left(\frac{c}{e}, \frac{a}{b}\right)$ since the Morse lemma is local in nature

As such the validity of Morse lemma is in the nbd of the critical point in question.



It is of interest to know if ALL the level sets of $-\phi$ are closed curves like the ones depicted. This will require one more theorem from Calculus (Morse theory) as we shall presently discuss. The answer is Yes due to two simple observations

(i) There is only one Critical point of ϕ in the first quadrant.

(ii) The function $\phi : \mathbb{R}^2_+ \to \mathbb{R}$ is a proper Map.

See P. Glendinning: Stability, Inst. & Chaos. CUP 1994 (p. 132)

Before this let us also look at the pendulum Equation

The first integral is

$$\phi(x_1, x_2) = \frac{x_2^2}{2} - \frac{g}{\ell}\cos x_1$$

$$\phi(0,0) = -\frac{g}{\ell} \quad (\text{Min}) \qquad : \qquad \phi(\pi, 0) = g/\ell$$

Assume $c_0 > -\frac{g}{\ell}$ and close to $-g/\ell$

Then by Morse lemma, the level sets $\phi^{-1}(c_0)$ are closed curves (homeomorphic copy of the Circle) enclosing the rest point $(0,0)$ inside

So for c_0 close to $-g/\ell$ and $> -g/\ell$ the solutions are periodic

In terms of IC, let $x_1(0) = 0$

$$x_2(0) = c$$

So that ϕ takes the value $\frac{c^2}{2} - \frac{g}{\ell} = c_0$

The requirement $c_0 > -\frac{g}{\ell}$ and close to $-g/\ell$

Now translates to the Condition that the Initial Velocity c be small in magnitude

For such c, the pendulum exhibits osc. motion

$x_1(t)$ and $x_2(t)$ are periodic with period T

$$x_1(T/4) = \sup_{t \in \mathbb{R}} x_1(t) = \theta_0 < \pi$$

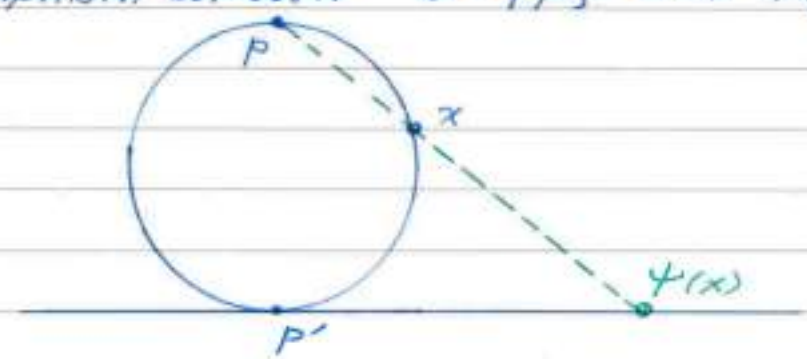
The results obtained earlier via analysis have now been re-established via topological methods.

Remark: It is of interest to know the period of the solutions of the Lotka Volterra system. It would depend on initial values. See the book J. Grasman, Asymptotic methods for relaxation oscillators, Springer Verlag 1987 p 72 ff for an in depth discussion of these matters.

Let us now turn to the proof of Thm 5.6 (Stated p 199)

It is clear that S being compact invariant, if the IC is from S, the solution is constrained to lie in S and so $\mathbb{R}$ is the maximal interval of existence.

Recall that the Sterographic projection establishes a homeomorphism between $S - \{p\}$ and $\mathbb{R}$



We shall denote this by $\psi : S - \{p\} \to \mathbb{R}$

For $x \in S - \{p\}$, $\psi(x)$ is the intersection of the ray $\overrightarrow{px}$ with the real axis which is tangent to S at the point p' antipodal to p (figure is clear)

Ex: Prove that ψ is a homeomorphism

Now let $a \in S$ and $\gamma : \mathbb{R} \to S$ be the unique Sol. of $\dot{y} = f(y)$ with IC $y(0) = a$

We shall first show γ cannot be injective. Suppose γ is injective.

Take a seq $t_n \to +\infty$ s.t $\gamma(t_n) \to$ to a point $p \in S$
(this is possible by Compactness of S)

Claim: $\lim_{t\to\infty} \gamma(t) = p$ and that would imply p is a rest point. Contradiction since in (i) we are assuming S does not contain rest pts.

Now it may happen $\gamma(t_0) = p$ for some t_0 in which case, by injectivity,

$$\gamma(t) \neq p \quad \text{for} \quad t > t_0.$$

So $\psi \circ \gamma$ is an injective Cont. map $(t_0, \infty) \to \mathbb{R}$
So is either strictly increasing or strictly decreasing
Now $\psi(\gamma(t_n)) \to \infty$

$$\Rightarrow \lim_{t\to\infty} (\psi\circ\gamma)(t) = +\infty$$

$\Rightarrow \gamma(t) \to p$ as $t \to +\infty$. Job is done!

So $\exists\ t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$ s.t $\gamma(t_1) = \gamma(t_2)$
But by Thm 5.2 $\gamma(t)$ is periodic
Well $\gamma(t + t_2 - t_1)$ and $\gamma(t)$ both satisfy the ODE and both take the same value at $t = t_1$. By Uniqueness

$$\gamma(t + t_2 - t_1) = \gamma(t) \text{ for all } t \in \mathbb{R}.$$

We now show that $\gamma(\mathbb{R}) = S$.
Suppose not. Let $\gamma(\mathbb{R}) \subset S - \{q\}$ for some $q \in S$
now $S - \{q\}$ is homeomorphic to an open interval $J \subseteq \mathbb{R}$
Let $T > 0$ be the Minimum period of γ
So that $\gamma(\mathbb{R}) = \gamma([0,T]) = J_0$ Compact interval in $\mathbb{R}$

$$\tilde{=} [0,1]$$

$[0,T] \xrightarrow{\eta} S^1$ (Circle) ; η quotient map
$\eta(0) = \eta(T)$
η injective on $[0,T)$

γ : $[0,T] \to J_0$; $\bar{\gamma}$: $S^1 \to J_0$

$\bar{\gamma}$ is uniquely defined, is bijective and

Continuous since γ is cont (property of quotients see p. 192)

$\therefore \bar{\gamma} : S' \to [0,1] = J_0$ is a homeomorphism and this is a Contradiction. Proof of (i) is Complete.

Proof of (ii): If S Contains one or more rest points, and $\gamma : \mathbb{R} \to S$ is a non Const Solution, the Same Stereographic proj argument on p 203 reveals that $\lim_{t \to \pm\infty} \gamma(t)$ must exist

This The two limits would agree if there is a unique rest point on S otherwise these would be distinct.

Let us now return back to the case of the pendulum

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{g}{\ell} \sin x_1$$

$$x_1(0) = 0; \quad x_2(0) = C > 0$$

Let us look at the Critical case $C^2 = 4g/\ell$

$$\frac{x_2^2}{2} - \frac{g}{\ell} \cos x_1 = \frac{C^2}{2} - \frac{g}{\ell} = \frac{g}{\ell}$$

$$\therefore \frac{x_2^2}{2} = \frac{2g}{\ell} \cos^2\left(\frac{x_1}{2}\right)$$

$$\therefore x_2 = 2\sqrt{\frac{g}{\ell}} \cos\left(\frac{x_1}{2}\right)$$

$$\therefore \frac{dx_1}{dt} = 2\sqrt{\frac{g}{\ell}} \cos\left(\frac{x_1}{2}\right) \quad ; x_1(0) = 0.$$

$$\text{Sec}\left(\frac{x_1}{2}\right)(t) = \cosh \sqrt{\frac{g}{\ell}}\, t$$

as $t \to +\infty$, $x_1(t) \to \pi$; $x_2(t) \to 0$

$x_1(t)$ is an odd function of t

$\therefore x_1(t) \to -\pi$ as $t \to -\infty$

So the trajectory $(x_1(t), x_2(t)) \to (\pm\pi, 0)$ as $t \to \pm\infty$

Sketch the graph and show that this is a Homoclinic cycle.

Note: $(\pi, 0)$ and $(-\pi, 0)$ are identical pts.

We now show that EVERY solution of the Lotka Volterra system (except the Const. Sol. Corresp. to the equilibrium point) is periodic.

Well, for each Sol $(x(t), y(t))$

$\phi(x(t), y(t)) \equiv$ const c ; $-\phi$ is first Int. (5.11)

$(x(t), y(t)) \in \phi^{-1}(c)$. We have to show that ALL level sets are homeomorphic to the circle (being invariant, Thm 5.6 on p199 now implies that $(x(t), y(t))$ is a periodic solution)

We know that for values c close to the global min $\left(\frac{c}{e}, \frac{a}{b}\right)$ Morse lemma gave us that $\phi^{-1}(c)$ is a closed curve homeomorphic to a circle (See p200). We now need to establish this for all level sets

$-\phi(x,y) = a\log y + c\log x - by - ex$

note that $\phi(0,0) = +\infty$

There are no critical values in the interval

$$\left(\phi\left(\frac{c}{e}, \frac{a}{b}\right), +\infty\right)$$

↑ Global Min. Prove that this is a global Min *

Now we shall appeal to a fundamental result in Morse-Theory:

Thm 5.7: Let $f: \Omega \to \mathbb{R}$ be a proper smooth function Let $a, b \in \mathbb{R}$ Such that $f^{-1}(a)$, $f^{-1}(b)$ nonempty and $f^{-1}[a,b]$

(*) Prove that if $\psi: I \to \mathbb{R}$: I open interval is smooth, $\psi'(t)$ vanishes exactly once in I and at this point ψ is a local Min. Then this pt is a Global Min

Result is false in two variables. But $\phi = \psi_1(x) + \psi_2(y)$ take advantage of this!

does not contain any critical value of f

Then $f^{-1}(a)$, $f^{-1}(b)$ are diffeomorphic
See M.W. Hirsch: Differential topology 1976 p.153.

It follows at once from this that all level curves $\phi^{-1}(c)$ for ϕ given by

$$\phi(x,y) = -a\log y - c\log x + by + ex \qquad (5.11)'$$

(the first integral of Lotka-Volterra system) are homeomorphic to the circle (except the constant one with c as critical value)

Note: The level curves lie in $\mathbb{R}^2_+$ since ϕ is defined only on $\mathbb{R}^2_+$.

Thus All non constant solutions of the Lotka-Volterra equation are periodic solutions.

Ex: The Duffing's Equation:

$$x'' + x \pm x^3 = 0$$

Convert it into a first order system:

$$\dot{x}_1 = x_2 \qquad (5.15)$$
$$\dot{x}_2 = -x_1 \mp x_1^3$$

First integral: $\phi(x_1, x_2) = x_1^2 + x_2^2 \pm \frac{1}{2} x_1^4$

Note that the rest point for the $-$ case is $(0,0)$
Whereas for the plus case
there are 3 rest points $(0,0)$; $(1,0)$ and $(-1,0)$
We shall discuss this latter one

$\phi(x_1, x_2) = x_1^2 + x_2^2 - \frac{1}{2} x_1^4$. $(0,0)$ is a Min
$(1,0)$ and $(-1,0)$ are saddles (namely the Hessian has both pos. and neg eigenvalues. Non degenerate)

Critical values: $\phi(0,0) = 0$

$$\phi(\pm 1, 0) = \frac{1}{2}$$

$\phi^{-1}(\frac{1}{2})$ is a union of two parabolas (Sketch!)

$$x_2 = \pm \frac{1}{\sqrt{2}} (x_1^2 - 1) \qquad (5.16)$$

$\phi : \mathbb{R}^2 \to \mathbb{R}$ is not a proper map! Why? *

Let Ω be the lens shaped region between these parabolas

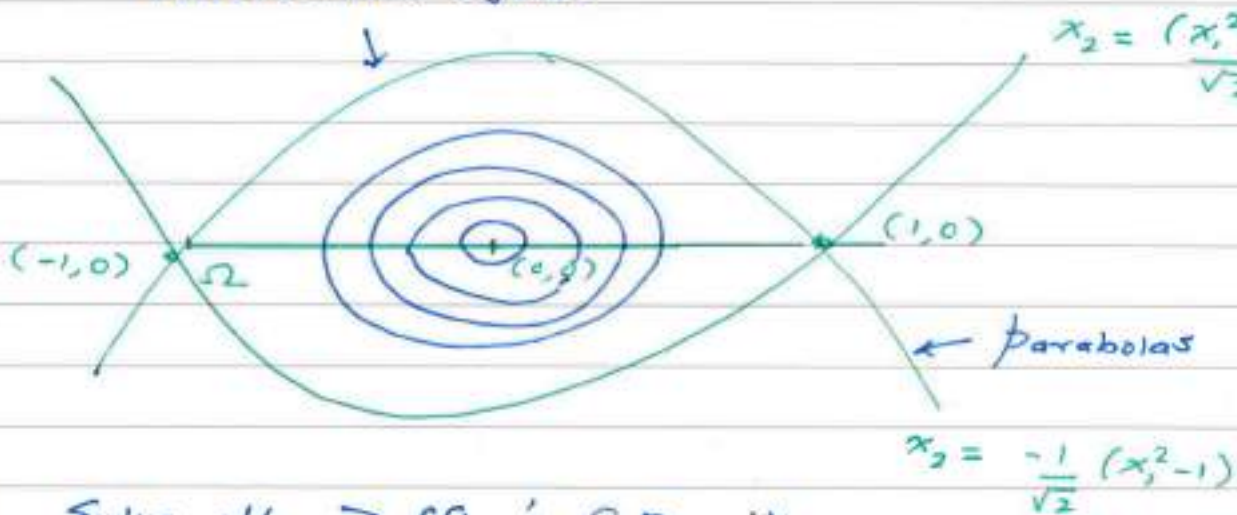
$\phi : \Omega \to \mathbb{R}$ is then a proper map.

and the level sets within Ω are all mutually diffeo. except of course $\phi^{-1}(0) = \{(0,0)\}$.

All Solutions of the Duffing's eqn (5.15) with IC $(x_1(0), x_2(0)) \in \Omega$ with the exception of the Const. Sol., are periodic.

Note that there is a heteroclinic cycle formed by appropriate arcs of (5.16) (Sketch).

Heteroclinic cycle.



Ex: Solve the Duffing's Eqn with

IC: $x_1(0) = 0 \quad x_2(0) = \frac{1}{\sqrt{2}}$

Do $\lim_{t\to\infty} x_1(t)$, $\lim_{t\to\infty} x_2(t)$ exist; limit as $x \to -\infty$? illustrate thm 5.1 (p 185)

Ex: Discuss $\dot{x}_1 = x_2 \quad \dot{x}_2 = x_1 - x_1^3$

Vibrations of a buckled beam. See S. Wiggins, Global bifurcations and Chaos p173. Springer 1988

$\phi(x_1, x_2) = x_2^2 - \frac{1}{2}(x_1^2 - 1)^2 + \text{const} \cdots$

Discuss: $\dot{x}_1 = x_1^2 - x_2^2$; $\dot{x}_2 = 2x_1x_2$

Find a first integral. Level sets are circles each of which is a homoclinic cycle.

Discuss: $\dot{x} = 1 - x^2 + y^2$; $\dot{y} = 2xy$

Discuss the nature of the level curves.

Ovals of Cassini: $P = (a,0)$, $Q = (-a,0)$: $a > 0$

$X = (x,y)$

Ovals of Cassini are loci of points X such that

$$\text{dist}(X,P) \cdot \text{dist}(X,Q) = \text{Const.}$$

Sketch the ovals of Cassini.

Well, $\phi(x,y) = ((x-a)^2 + y^2)((x+a)^2 + y^2)$

and the ovals of Cassini are level sets of ϕ

Locate the critical points of ϕ and determine the critical value.

Illustrate Thm 5.7.

Note that $(0,0)$ is a critical point and a^4 is a critical value.

$(a,0)$ and $(-a,0)$ are also critical points.

Are there any others? Can you sketch the level curves using Morse lemma, symmetry etc;

Can you use physical intuition - from say electrostatics level curves could be thought of as lines of const- potential (Equipotential surface sliced by xy plane ...)

We shall not discuss the solutions in detail here.

We must now turn to the proof of theorem 5.7.

This is a nice application of ODE theory.

First, some simple observations. Recall that the unique solution of $\dot{y} = f(y)$ with IC

$$y(0) = p$$

was denoted by $\phi(t,p)$. It is often

denoted by $\phi_t(p)$ thereby providing a map
$\phi_t : \Omega \to \Omega$.

(Recall we are assuming that solutions have $\mathbb{R}$ as maximal interval of existence)

Each $\phi_t : \Omega \to \Omega$ $(t \in \mathbb{R})$ is a diffeomorphism
$\phi_0 = id_\Omega$ (identity map on Ω) and

$$\phi_{s+t} = \phi_s \circ \phi_t \qquad (5.17)$$

From the last two it follows at once that each ϕ_t is bijective and $\phi_t^{-1} = \phi_{-t}$

The differentiability of ϕ_t simply follows from the fact that the solution of the IVP

$\dot{y} = f(y)$; $y(0) = p$

is smooth wrt initial condition p.
So it follows at once that each ϕ_t is a diffeo. of Ω onto itself.

We shall formalize these notions in a later Chapter.

To prove (5.17) simply notice that-

$$\frac{d}{dt}(\phi_{t+s}(p)) = \frac{d}{d(s+t)}\phi_{t+s}(p) = f(\phi_{s+t}(p))$$

$$\frac{d}{dt}\left(\phi_t(\phi_s(p))\right) = f(\phi_t(\phi_s(p)))$$

So both $t \mapsto \phi_{t+s}(p)$
$t \mapsto \phi_t(\phi_s(p))$

satisfy the ODE. They take the same value $\phi_s(p)$ when $t = 0$. By uniqueness, we get (5.17). Proof is complete.

$\{\phi_t\}_{t \in \mathbb{R}}$ is thus a one parameter group of diffeomorphisms of Ω

Note: This is called the flow generated by the ODE. We are assuming for each $p \in \Omega$ the solution $\phi_t(p)$ exists for all $t \in \mathbb{R}$.

We need a couple of simple results

Lemma 5.8: Suppose $f: \Omega \to \mathbb{R}$ is a smooth proper map and $[a,b]$ contains no critical values

Then,

$\exists\ \epsilon > 0$ such that the open set $f^{-1}((a-\epsilon, b+\epsilon))$ does not contain any critical point.

This in particular implies that on

$G = f^{-1}((a-\epsilon, b+\epsilon))$

the field $\frac{\nabla f}{\|\nabla f\|}$ is meaningful and smooth

Proof: Suppose no such ϵ exists.

Then for each $n \in \mathbb{N}$ there exists a

$p_n \in f^{-1}\left((a-\frac{1}{n}, b+\frac{1}{n})\right)$ s.t. $\nabla f(p_n) = 0$

We now have $(p_n) \in f^{-1}[a-1, b+1]$; $n \in \mathbb{N}$

Since f is proper, (p_n) has a subseq which converges to a point $p \in \Omega$

$\therefore\ f(p_n) \to f(p)$

$\nabla f(p_n) = 0$ for all n

By continuity of ∇f, we conclude $\nabla f(p) = 0$

Now $f(p_n) \in (a - \frac{1}{n}, a + \frac{1}{n})$

$\therefore\ a - \frac{1}{n} \le f(p_n) \le a + \frac{1}{n}$

letting $n \to \infty$, $a \le f(p) \le b$ $\therefore p \in f^{-1}[a,b]$

This is a contradiction since $[a,b]$ contains no critical values but $\nabla f(p) = 0$.

The second fact we need is a basic Calculus result

(A smooth version of Urysohn lemma)

Lemma 5.9 If G is open in $\mathbb{R}^n$ and K is a compact subset of G, there exists $\tilde{K}$ compact s.t. $K \subset \tilde{K} \subset G$

and $\phi: \mathbb{R}^n \to \mathbb{R}$ smooth s.t. $\phi|_K \equiv 1$

$\phi(x) = 0$ if $x \notin \tilde{K}$

Proof of Theorem 5.7:

So $f: \Omega \to \mathbb{R}$ is a proper smooth function such that $[a,b]$ does not contain any critical value then $f^{-1}(a)$, $f^{-1}(b)$ are diffeomorphic

(see pp 205-206)

Let $\epsilon > 0$ such that the open set

$$G = f^{-1}(a-\epsilon, b+\epsilon)$$

does not contain any critical points.

Let $\rho(x)$ be a smooth function such that

$\rho(x) \equiv 0$ outside G

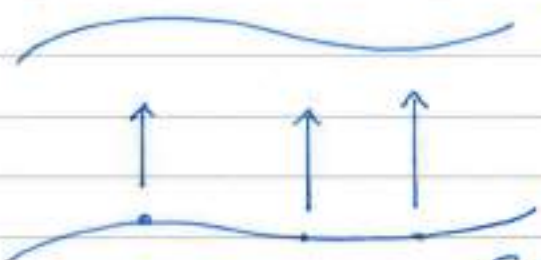
$\rho(x) \equiv 1$ in a nbd of $f^{-1}[a,b]$

(the smooth Urysohn function)

Let $$X(x) = -\frac{\rho(x)\,.\,\nabla f(x)}{\|\nabla f(x)\|^2} \qquad (5.18)$$

Note that $\nabla f(x) \neq 0$ in $\bar{G}$; outside of $\bar{G}$ it may be zero but $\rho(x) \equiv 0$ there and (5.18) is meaningful ($=0$ outside G)

On $K = f^{-1}[a,b]$ $\quad X(x) = \dfrac{\nabla f(x)}{\|\nabla f(x)\|^2}$



$X(x)$ has rest points outside the compact set

$$K_\epsilon = f^{-1}[a-\epsilon, b+\epsilon]$$

So if $p \in K_\epsilon$ the solution $\phi(t,p)$ of the ODE

$$\dot{x} = X(x) \;:\; x(0) = p$$

Cannot exit K_ϵ and so must remain within K_ϵ

It follows that the maximal interval of existence of all solutions of

$$\dot{x} = X(x)$$

is the whole of $\mathbb{R}$.

In other words for each $t \in \mathbb{R}$ we have a map $\underset{t}{\Phi} : \Omega \to \Omega$ given by

$$\underset{t}{\Phi}(p) = \phi(t, p)$$

Yielding a one parameter family of diffeomorphisms forming a group (see p 209)

Now, for any $q \in \Omega$

$$\frac{d}{dt} f(\underset{t}{\Phi}(q)) = \frac{d}{dt} f(\phi(t,q))$$

$$= \nabla f(\phi(t,q)) \cdot \frac{d}{dt} \phi(t,q)$$

$$= \nabla f \cdot X \Big|_{\phi(t,q)}$$

$$= -\rho(x) \frac{\nabla f \cdot \nabla f}{\|\nabla f\|^2} \Big|_{\phi(t,q)}$$

$$= -\rho(\phi(t,q))$$

Now let $a \leq c_1 < c_2 \leq b$ and $q \in f^{-1}(c_2)$

As long as $\phi(t,q)$ remains in a nbd N of $f^{-1}([a,b])$ where $\rho \equiv 1$,

we shall have $\rho(\phi(t,q)) = 1$ and

$\frac{d}{dt}(f(\underset{t}{\Phi}(q)) = -1$ for all such times.

say $0 \leq t < T$

Thus $\frac{d}{dt}(f(\underset{t}{\Phi}(q)) = -1$ on this interval

Integrating from 0 to t

$$\therefore \quad f(\underset{t}{\Phi}(q)) - f(\underset{0}{\Phi}(q)) = -t$$

$$\therefore \quad f(\underset{t}{\Phi}(q)) = f(q) - t = c_2 - t \leq c_2 \leq b$$

Also $c_2 - t \geq a$ if $t \leq c_2 - a$

Certainly, if t lies between 0 and $c_2 - a$

$f(\underset{t}{\Phi}(q)) \in [a,b]$

and $\underset{t}{\Phi}(q) \in N$ and our analysis would be valid.

In otherwords T is atleast $c_2 - a$.

In particular, taking $t = c_2 - c_1$ we get

$$f\left(\underline{\Phi}_{c_2-c_1}(q)\right) = c_2 - (c_2 - c_1) = c_1$$

$\therefore$ $\underline{\Phi}_{c_2-c_1}$ is a diffeomorphism that takes points of $f^{-1}(c_2)$ to points of $f^{-1}(c_1)$

To show that it is surjective observe that $\underline{\Phi}_{c_1-c_2}$ maps $f^{-1}(c_1)$ to $f^{-1}(c_2)$ and is the inverse of $\underline{\Phi}_{c_2-c_1}$. To see that $\underline{\Phi}_{c_1-c_2}$ maps $f^{-1}(c_1)$ to $f^{-1}(c_2)$ proceed as before. Start with $q \in f^{-1}(c_1)$ and take q as IC for the system of ODEs $\dot{x} = X(x)$

$$\frac{d}{dt} f\left(\underline{\Phi}_t(q)\right) = -1 \text{ holds for } T' \leq t \leq 0$$

namely for all such t with $\underline{\Phi}_t(q) \in f^{-1}[0,b] \subset N$

Integrating we again get

$$f\left(\underline{\Phi}_t(q)\right) = f(q) - t = c_1 - t \geq c_1 \geq a$$

Now $\underline{\Phi}_t(q) \in f^{-1}[a,b]$ as long as $c_1 - t \leq b$

i.e $t \geq c_1 - b$

namely on the interval $[c_1 - b, 0]$

and for $t = c_1 - c_2$

$$f\left(\underline{\Phi}_t(q)\right) = c_1 - (c_1 - c_2) = c_2$$

i.e $\underline{\Phi}_{c_1-c_2}(q) \in f^{-1}(c_2)$ as asserted.

We have shown that for any pair $c_1, c_2 \in [a,b]$ the level sets $f^{-1}(c_1)$ and $f^{-1}(c_2)$ are diffeomorphic via $\underline{\Phi}_{c_2-c_2}$

Proof of Sylvester's law of Inertia

For Completeness we provide here a proof of Sylvester's law of inertia

We first show the following: (Preliminary Version)

Let A be a real symmetric matrix and P_1, P_2 be non singular matrices such that

$P_1^T A P_1$ and $P_2^T A P_2$ are both diagonal namely, (5.19)

$$y^T P_1^T A P_1 y = y_1^2 + \dots + y_k^2 - y_{k+1}^2 - \dots - y_r^2$$
$$z^T P_2^T A P_2 z = z_1^2 + \dots + z_\ell^2 - z_{\ell+1}^2 - \dots - z_\mu^2$$

We prove $k = \ell$, $r = \mu$

proof by Contradiction. Suppose not. Assume $k < \ell$

$$V_1 = \{ P_1 y \,/\, y \in \mathbb{R}^n,\ y_1 = \dots = y_k = 0 \}$$

V_1 is the image under P_1 of a $(n-k)$ dim subsp

so $\quad \text{Dim } V_1 = n - k$

$V_2 = \{ P_2 z \,/\, z_{\ell+1} = \dots = z_n = 0 \}$. Again

$\text{Dim } V_2 = \ell$

$$\text{Dim } V_1 + \text{Dim } V_2 = n - k + \ell \geq n$$

$\therefore \quad V_1 \cap V_2 \neq (0)$

pick a non zero vector $x \in V_1 \cap V_2$

$$x^T A x = y^T P_1^T A P_1 y \qquad (x = P_1 y : y \neq 0)$$
$$= -y_{k+1}^2 - \dots - y_n^2$$

Now by def of $x = P_1 y$, we have $y_1 = \dots = y_k = 0$

and so at least one of $y_{k+1}, \dots, y_n$ is $\neq 0$

$\therefore \quad x^T A x > 0$

Again, $x = P_2 z$ for some $z \neq 0$, $z_{\ell+1} = \dots = z_n = 0$

$$x^T A x = z^T P_2^T A P_2 z = z_1^2 + \dots + z_\ell^2 > 0$$

$\therefore \quad x^T A x > 0$ and $x^T A x < 0$. Contradiction

Hence $l = k$

Now recall that if S is a real symmetric matrix then for $i \neq j$, the (i,j)th entry s_{ij} of S is given by $2s_{ij} = \dfrac{\partial^2}{\partial y_i \partial x_j}(y^T S y)$

It now follows from (5.19) that $P_1^T A P_1$: $P_2^T A P_2$ are both diagonal matrices of rank $k+\lambda$, $l+\mu$ respectively

But rank $(P^T A P)$ = rank A if P non singular

$\therefore k+\lambda = l+\mu$

$\Rightarrow \lambda = \mu$

Suppose B is a real symm matrix.

by Spectral thm B is Congruent to

$D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$

If $\lambda_1 > 0$.

Mult first row of D_1 by $1/\sqrt{\lambda_1}$,

Mult first col. of D_1 by $1/\sqrt{\lambda_1}$,

Then D is congruent to the diagonal matrix

$D_1 = \text{diag}(1, \lambda_2', \lambda_3', \ldots, \lambda_n')$

Proceeding thus, if $\lambda_2' < 0$. Mult 2nd row by $1/\sqrt{-\lambda_2}$ and second col. by $1/\sqrt{-\lambda_2}$ etc:

Ultimately we get that

B is Congruent to D_0 : D_0 has diagonal entries $+1, -1$ or 0

Evidently, since D is obtained via Spectral thm

Inertia triple of B = Inertia triple D

= Inertia triple of D_0.

and B cong D_0.

Let C be another real symm. Matrix.

Similarly C Cong. D_{00} (diag with Entries $1, -1$ or 0)

Inertia triple C = Inertia triple D_{00}

Assume now B and C are Congruent-
Then
D_0, D_{00} are Congruent:
$D_0 = P^T D_{00} P_0$.

(5.20)

$$y^T P_1^T D_{00} P_1 y = y^T D_0 y = y_1^2 + \dots + y_k^2 - y_{k+1}^2 - \dots - y_\lambda^2$$

$(k, \lambda-k, 0)$ = Inertia triple D_0 = Inertia triple B
Take P_2 = Id. Matrix

$$z^T (P_2^T D_{00} P_2) z = z_1^2 + \dots + z_\ell^2 - z_{\ell+1}^2 - \dots - z_\mu^2$$

$(\ell, \mu-\ell, 0)$ = Inertia triple D_{00}
= Inertia triple C.

But we then infer (from the Preliminary Version established on p 214)
that $k = \ell$, $\lambda = \mu$
Hence Inertia triple B = Inertia triple C

We now resume our study of ODEs with an
Example: (See L. Perko: Diff. Equations and dynamical Systems, 3rd Ed. (2001) p 231)
Consider the System

$$\dot{x} = -y + xz^2 \qquad (5.21)'$$
$$\dot{y} = x + yz^2 \qquad (5.21)''$$
$$\dot{z} = -z(x^2+y^2) \qquad (5.21)'''$$

Show that $f(x,y,z) = x^2+y^2+z^2$ is a first-integral
Show that the xy plane is an invariant set-
The xy plane meets $f^{-1}(1)$ along the equitorial Circle.

There are no rest points on $x^2+y^2=1$, $z=0$ and so this invariant set is the image of a periodic solution.

$(0,0,1)$ and $(0,0,-1)$ are two rest points lying on the unit sphere

$$S = \{(x,y,z) / x^2+y^2+z^2=1\}.$$

Now let

$$p \in S_+ = \{(x,y,z) / x^2+y^2+z^2=1,\ z>0\}$$

and $p \neq (0,0,1)$

Show that the solution $\phi(t,p)$ of the ODE with IC $p = (x_0, y_0, z_0)$ approaches the equator as $t \to \infty$ by spiraling around. Discuss the situation in the Southern hemisphere.

Sol: Most efficient way to deal with this is to introduce

$\rho(x,y,z) = z$ and study this along the solution $\phi(t, x_0, y_0, z_0)$: $p = (x_0, y_0, z_0)$

$\phi(t,p)$

$$\frac{d}{dt}\rho(\phi(t,p))$$

$$= \frac{dz}{dt} = -z(t)\left((x(t))^2 + (y(t))^2\right)$$

Notice that if $x(t)^2 + y(t)^2 = 0$ at some time t then $z(t)^2 = 1$ ($\because x^2+y^2+z^2=$ is a first integral)

$\therefore z(t) = 1$ (the sol. starts in S_+ and so cannot cross the equator which is a different trajectory) and then the sol. would be the const solution at the rest point $(0,0,1)$

$\therefore x(t)^2 + y(t)^2 > 0$

$$\therefore \frac{dz}{dt} < 0$$

So $z(t)$ is monotone decreasing and then $x(t)^2 + y(t)^2$ must be monotone increasing

So say $x(t)^2 + y(t)^2 \geq \alpha > 0$

$$\therefore \frac{dz}{dt} \leq -\alpha z$$

$$\therefore 0 \leq z(t) \leq z(0)\, e^{-\alpha t}$$

So the z-coordinate of the solution $\phi(t,p)$ decays to zero exponentially fast and the trajectory (as a set) comes closer and closer to the equator $x^2+y^2=1$, $z=0$.

The trajectory cannot have a definite limit because such a limit would have to be a rest point and we know there are none on the equator.

Exercise: Demonstrate the spiraling of the trajectory towards equator by employing spherical polar coordinates:

$$\begin{aligned} x(t) &= \cos\theta(t)\sin\phi(t) \\ y(t) &= \sin\theta(t)\sin\phi(t) \qquad (5\text{-}22) \\ z(t) &= \cos\phi(t) \end{aligned}$$

Compute using (5-22) and the ODE the expression

$$x\dot{y}-y\dot{x}$$

and then obtain $\dfrac{x\dot{y}-y\dot{x}}{x^2+y^2} = 1$

Thus $\dfrac{d\theta}{dt} = 1$ $\qquad (5\text{-}23)'$

You can also obtain an ODE for $\phi(t)$:

From (5-22)

$$\dot{z}(t) = -\sin\phi(t)\,\dot{\phi}(t)$$

From the ODE $\dot{z} = -z(x^2+y^2) = -z(1-z^2)$

$$\therefore (\sin\phi(t))\,\dot{\phi}(t) = (\cos\phi)\sin^2\phi$$

$$\therefore \dot{\phi}(t) = \cos\phi\sin\phi \qquad (5\text{-}23)''$$

One can integrate (5-23)' - (5-23)'' explicitly and obtain the actual trajectories.

Ex: At what angle does the solution cut the longitudes? Are the solution curves on the sphere Loxodromes? See D.J. Struick Lectures on Classical diff. geom 2nd Ed. p60.

Consider the system

$$\dot{x} = y + y(x^2+y^2)$$
$$\dot{y} = x - x(x^2+y^2)$$

Show that $f(x,y) = (x^2+y^2)^2 - 2(x^2-y^2)$ is a first integral.

There are three critical points $(0,0)$, $(1,0)$, $(-1,0)$. Determine the inertia triple of the Hessian at each of these critical points. These are also equilibrium points.

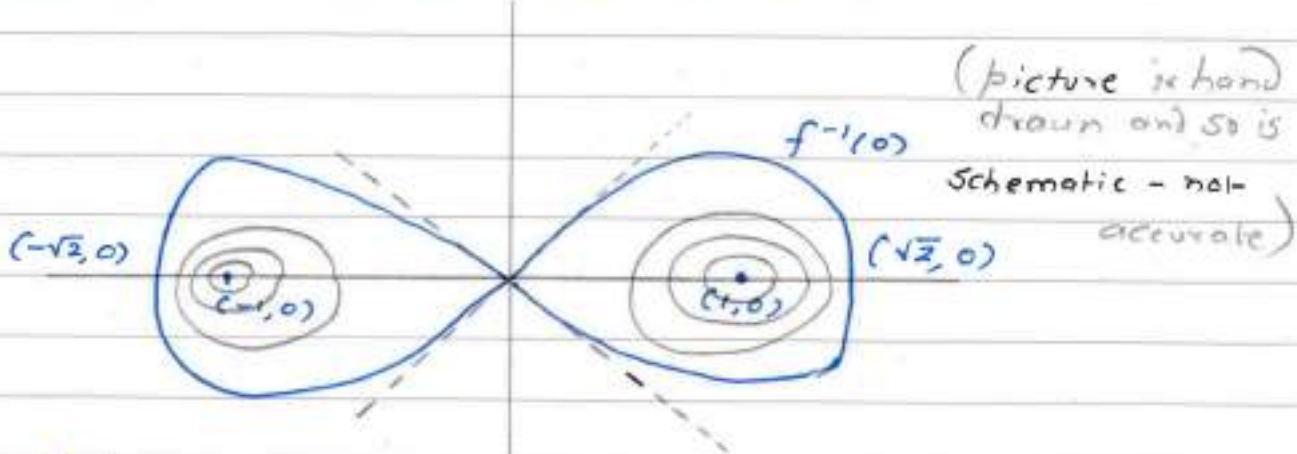
Ans: For $(0,0)$ inertia triple is $(1,1,0)$
For $(1,0)$ and $(-1,0)$ inertia triple is $(2,0,0)$

By Morse lemma for initial conditions close to $(\pm 1, 0)$ the trajectories are closed curves homeomorphic to circles.

The point $(0,0)$ is an example of a saddle we shall define this later.

By Morse lemma the level curves of $f^{-1}(c)$ for on nbd of $(0,0)$ look like (diffeo. to) level curves of x^2-y^2

Exercise: Sketch the level curves of f
Is f a proper map?
Draw first $f^{-1}(0)$ and $f^{-1}(-1)$



$f^{-1}(-1)$ is simply the union of the three critical points. Figure is supposed to be symm wrt x and y axes!

Example: Euler's equations for a spinning top.

This is the system of ODEs given by

$$A\dot{x}_1 = (B-C)x_2x_3$$
$$B\dot{x}_2 = (C-A)x_1x_3 \qquad (5.23)$$
$$C\dot{x}_3 = (A-B)x_1x_2$$

(x_1, x_2, x_3) is the instantaneous angular velocity and A, B, C are the principal moments of inertia (For details see A. Sommerfeld: Mechanics, Academic Press 1964 p139ff especially p.140)

Assume $A > B > C > 0$.

This system of ODEs offers two first integrals:

$$\phi = Ax_1^2 + Bx_2^2 + Cx_3^2$$
$$\psi = A^2x_1^2 + B^2x_2^2 + C^2x_3^2$$

These are functionally indep (there is no non trivial relation $F(\phi, \psi) \equiv 0$ between them. We shall make this notion precise later)

The level sets $\phi^{-1}(c_1)$, $\psi^{-1}(c_2)$ are both ellipsoids and assume that c are specified

$$x_1(0) = p_1, \quad x_2(0) = p_2, \quad x_3(0) = p_3$$

The solution $(x_1(t), x_2(t), x_3(t))$ is a curve in $\mathbb{R}^3$ which is (a certain parametrization of) the intersection of the two ellipsoids

$$Ax_1^2 + Bx_2^2 + Cx_3^2 = c_1 = Ap_1^2 + Bp_2^2 + Cp_3^2$$
$$A^2x_1^2 + B^2x_2^2 + C^2x_3^2 = c_2 = A^2p_1^2 + B^2p_2^2 + C^2p_3^2$$
$$(5.24)' - (5.24)''$$

Namely $(x_1(t), x_2(t), x_3(t))$ lies on the intersection of $(5.24)' - (5.24)''$.

The intersection $(5.24)' - (5.24)''$ is a fourth degree algebraic curve.

Let us define a function $\xi(t)$ given by

$$\xi(t)^2 = (x_1(t))^2 + (x_2(t))^2 + (x_3(t))^2 \qquad (5.25)$$

We can solve the 3 equations $(5.24)' - (5.24)''$ and (5.25)

Thus $(x_1(t))^2 = \dfrac{c_1(B+C) - c_2 - \xi BC}{(A-B)(C-A)}$

$(x_2(t))^2 = \dfrac{c_1(C+A) - c_2 - \xi CA}{(B-C)(A-B)}$ (5.26)

$(x_3(t))^2 = \dfrac{c_1(A+B) - c_2 - \xi AB}{(C-A)(B-C)}$

The Curve of intersection of (5.24)' - (5.24)" is parametrized by (5.26)

Consider a solution that starts in the first Orthant ($x_1>0$, $x_2>0$, $x_3>0$)

Since $A>B>C$, $\dot{x}_2<0$ while $\dot{x}_1>0$, $\dot{x}_3>0$ (initially)

So $x_2(t)$ decreases, $x_1(t)$, $x_3(t)$ increase and this situation prevails as long as $(x_1(t), x_2(t), x_3(t))$ does not Cross one of the Coordinate planes.

There exists a finite time T_0 such that $x_2(T_0)=0$

Suppose not - then $x_2(t)$ is monotone $\downarrow$ and bdd by 0

$\lim_{t\to+\infty} x_2(t)$ exists.

$x_1(t)$, $x_3(t)$ keep moving away from origin Zero and are bdd (Since the ellipsoids (5.24)' - (5.24)" are Compact) So $\lim_{t\to\infty}(x_1(t), x_2(t), x_3(t))$ exists and

This must be a rest point (z_1, z_2, z_3) with $z_1>0$, $z_3>0$

This is clearly a Contradiction.

So $\exists$ a first time instant $T>0$ s.t. $x_2(T)=0$

Now look at the set of functions: For $T\le t\le 2T$

$$y_1(t) = x_1(2T-t)$$
$$y_2(t) = -x_2(2T-t) \qquad (5.27)$$
$$y_3(t) = x_3(2T-t)$$

$(y_1(t), y_2(t), y_3(t))$ satisfies the ODE and agrees with $(x_1(t), x_2(t), x_3(t))$ at $t=T$. Thus $x(t)\equiv y(t)$ $\forall t\in\mathbb{R}$

So we have now a Continuation of solution by Reflection along x_1x_3 plane (!)

We now trace t in the backward direction letting t to decrease down from 0 in $(-\infty, 0]$
This time $x_2(t)$ increases on some time interval $(-t', 0]$

And $x_1(t)$, $x_3(t)$ likewise decrease on $(-t', 0]$ while staying in the first orthant. (Look at $\dot{x}_1(t)$ and $\dot{x}_3(t)$)

Two cases arise:

(i) There is a finite time $-K < 0$ such that
$x_1(-K) = 0$ or $x_3(-K) = 0$ not both
(Both cannot be zero at the same finite $-K$ because that $\Rightarrow$ $(x_1(-K), x_2(-K), x_3(-K))$ is a rest point which cannot be since $x(t)$ is a non const. solution)

(ii) $\lim_{t \to -\infty} x_1(t) = 0$ or $\lim_{t \to -\infty} x_3(t) = 0$

while $x_1(t) > 0$: $x_3(t) > 0$ for $-\infty < t \leq 0$.
Note that $\lim_{t \to -\infty} x_3(t) = 0 \Rightarrow \lim_{t \to -\infty} \dot{x}_1(t) = 0$ (using ODE)

$\Rightarrow \lim_{t \to -\infty} x_1(t)$ exists

and $x_2(t)$ increases to a limit > 0 as $t \to -\infty$
The limit is then the rest point $(0, \sqrt{C_1/B}, 0)$ (on the ellipsoid (5.24)' p. 220)
We shall see that (ii) does arise and there are TWO heteroclinic cycles.

Let us take up the first case:
Assume $x_3(-K) = 0$ and $x_1(-K) > 0$
Now that we had $y(t) \equiv x(t)$ for all $t \in \mathbb{R}$ (Equation (5.27) on p 221), for $t = -K$ we get

$x_1(-K) = y_1(-K) = x_1(2\tau + K)$ (equation (5.27))

Likewise $x_2(-K) = -x_2(2\tau + K)$ (5.28)'

$x_3(-K) = x_3(2\tau + K) = 0$ (5.28)''

Let us now use $x_3(-k) = 0$ to perform a reflection about x_1, x_2 plane (as was done on p 221)
Thus, we define $w(t)$ as follows. For $t \leq -k$

$$\begin{aligned} w_1(t) &= x_1(-2k-t) \\ w_2(t) &= x_2(-2k-t) \qquad (5.29) \\ w_3(t) &= -x_3(-2k-t) \end{aligned}$$

Both sides satisfy the system of Euler's ODEs (Check this) and agree when $t = -k$ (here we are using the fact that $x_3(-k) = 0$)

So $x(t) \equiv w(t)$ for all $t \in \mathbb{R}$.

In particular for $t = 2\tau + k$ in (5.29)

$$\left.\begin{aligned} w_1(2\tau+k) &= x_1(-3k-2\tau) \\ w_2(2\tau+k) &= x_2(-3k-2\tau) \end{aligned}\right\} \quad (5.30)$$

$$w_3(2\tau+k) = -x_3(-3k-2\tau) \qquad (5.30)'$$

But since $x(t) \equiv w(t)$ for all $t \in \mathbb{R}$ ~~We set $t = 2\tau + k$ in (5.29)~~

$$0 = x_3(2\tau+k) = w_3(2\tau+k) \qquad (\text{Use } (5.28)'')$$

So that both sides of $(5.30)'$ are zero.

So $(5.30)'$ can be rewritten as

$$x_3(2\tau+k) = x_3(-3k-2\tau) \qquad (5.30)''$$

(5.30) and $(5.30)''$ now gives us that $x(t)$ is a periodic solution

Case (ii) Since both equations

$$\begin{aligned} Ax_1^2 + Bx_2^2 + Cx_3^2 &= c_1 \qquad (\text{See } (5.24)' - (5.24)'') \\ A^2x_1^2 + B^2x_2^2 + C^2x_3^2 &= c_2 \end{aligned}$$

hold along the traj. Letting $t \to -\infty$ gives

$$B\left(\lim_{t\to-\infty} x_2(t)\right)^2 = c_1$$

$$B^2\left(\lim_{t\to\infty} x_2(t)\right)^2 = c_2 \qquad \text{or} \qquad c_2 = Bc_1$$

So the two ellipsoids are

$$\begin{aligned} Ax_1^2 + Bx_2^2 + Cx_3^2 &= c_1 \\ A^2x_1^2 + B^2x_2^2 + C^2x_3^2 &= Bc_1. \qquad \text{Mult first} \end{aligned}$$

by B and subtract:

$$A(A-B)x_1^2 + C(C-B)x_3^2 = 0 \qquad (5.31)$$

This is a pair of planes which is also an invariant set- (5.31) Meets $Ax_1^2 + Bx_2^2 + Cx_3^2 = c_1$ along two ellipses E_1, E_2 both passing through the rest points $\left(0, \pm\sqrt{\frac{c_1}{B}}, 0\right)$

E_1, E_2 are both heteroclinic cycles.

Note that if we take Initial Cond. on E_1 (or E_2) close to the rest point above but away from it the trajectory moves away from the vicinity of this rest point and converges to the antipodal one as $t \to \pm\infty$.
We say this rest point is an unstable rest point.

Remark: An alternate way to discuss Case (i) is that- the two ellipsoids (5.24)′ - (5.24)″ intersect- transversely and so is a Compact one dimensional manifold. This is a finite Union of Copies of Circles (two in this case). Now theorem applies and we are done.
The lengthy demonstration on pp 221-222 is important since the technique of Continuation via reflection has applications in other parts of differential equations.

Exercise: Take IC as a small ϵ- perturbation of the rest point $(0, 0, \alpha)$ (similarly for $(\beta, 0, 0)$)
For $(0,0,\alpha)$ the ellipsoids (5.24)′ - (5.24)″ are tangential to each other touching at $(0, 0, \pm\alpha)$
A perturbation causes a change in these ellipsoids and they now intersect along two closed curves (copies of circles) circuiting $(0, 0, \pm\alpha)$. These rest pts are Stable

We shall return to the Euler's equations for a spinning top in the next chapter in the context of complete integrability.

Alpha limit sets and Omega limit sets : We consider the system of ODEs $\dot{x} = f(x)$

The α-limit set of a solution $x(t)$ captures the behaviour as $t \to -\infty$

and ω-limit set captures the behaviour as $t \to \infty$

Def

Let $x(t)$ be a solution of $\dot{x} = f(x)$: $f: \Omega \to \mathbb{R}^n$ smooth

A point $p \in \Omega$ is called an ω-limit point of the given solution if there exists a sequence $t_n \to \infty$ such that

$$p = \lim_{n \to \infty} x(t_n)$$

A point $q \in \Omega$ is called an α-limit point of the given solution if there exists a sequence $t_n \to -\infty$ such that

$$q = \lim_{n} x(t_n)$$

The set of all ω-limit points of the given solution is called the ω-limit set of the sol.

The set of all α-limit points of the given sol is called the α-limit set of the solution.

Ex: If the solution is the constant sol. at a rest point ξ then the α as well as ω-limit set of this sol. is $\{\xi\}$

Ex: The α and/or ω-limit set can be empty: Look at $\dot{x}_1 = 1$

$\dot{x}_2 = 0$

For any IC (c_1, c_2) the sol. $(c_1 + t, c_2)$ has no

α or ω limit set points.

Notation: If x is a solution, $\omega(x)$ and $\alpha(x)$ denote the omega and alpha limit sets of this solution.

Theorem 5.10: The α and ω limit sets are closed.

proof: If $\omega(x)$ is empty then there is nothing to prove. So let (p_n) be a sequence in $\omega(x)$ such that $p_n \to p \in \Omega$. TPT $p \in \omega(x)$

For each n, there is a sequence $(t_{n,k})_k$ such that

$$t_{n,k} \to \infty \text{ as } k \to \infty$$

$$x(t_{n,k}) \to p_n \text{ as } k \to \infty$$

We can of course assume $t_{n,1} < t_{n,2} < t_{n,3} < \cdots$ for each n.

Now pass to a subseq. We can assume

$$t_{n+1,1} > t_{n,1} \; ; \; t_{n+1,2} > t_{n,2} \text{ etc.}$$

For each $n \in \mathbb{N}$, there exists $k(n) \in \mathbb{N}$ such that

$$|x(t_{n,k}) - p_n| < \frac{1}{n} \quad \text{for } k > k(n)$$

Again we may assume $k(n+1) > k(n)$.

Consider the diagonal sequence $x(t_{n,k(n)})$:

Now, $t_{n,k(n)} \to +\infty$ as $n \to \infty$

$$|x(t_{n,k(n)}) - p| \leq |x(t_{n,k(n)}) - p_n| + |p_n - p|$$

$$< \frac{1}{n} + |p_n - p| \to 0 \text{ as } n \to \infty$$

$\therefore p \in \omega(x)$ and the proof is complete

The following is an equivalent ~~Condition~~ definition

Thm
$$\omega(x) = \bigcap_{s>0} \overline{\{x(t) / t \geq s\}} \qquad (5.32)$$

$$\alpha(x) = \bigcap_{s<0} \overline{\{x(t) / t \leq s\}}$$

(bar denotes the closure wrt usual topology)

proof is easy: Assume $p \in (5.32)$ so that

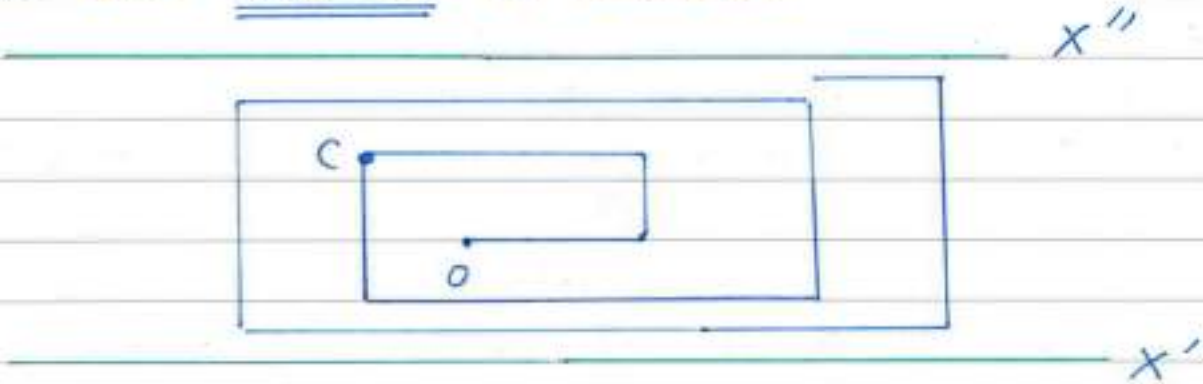
$$p \in \overline{\{x(t) / t \geq n\}} \text{ for each } n$$

with $\epsilon = 1/n$, there exists $p_n = x(t_n) : t_n \geq n$
s.t $|p_n - p| < 1/n$

We see that $p \in \omega(x)$. Conversely assume $p \in \omega(x)$
Choose $t_n \to +\infty$ s.t $x(t_n) \to p$.
Let $s > 0$ be arbitrary. We have to show
$p \in \overline{\{x(t) / t \geq s\}}$ for each $s > 0$
Let $\epsilon > 0$. There exists $n_0 \in \mathbb{N}$ s.t
$|x(t_n) - p| < \epsilon$ for $n \geq n_0$.
Select n so large that $t_n > s$ and we see that
$B_\epsilon(p)$ intersects $\{x(t) / t \geq s\}$; (ϵ arb.)
$\therefore p \in \overline{\{x(t) / t \geq s\}}$ proof is Complete

Note that each of the sets
$C_s = \overline{\{x(t) / t \geq s\}}$ is Connected
and the family $\{C_s\}$ is nested.

Ex: If $\{C_s\}$ is a nested family of Connected subsets of a metric space the intersection of all these sets need not be Connected



In the above figure Let us Consider the part of the Curve (polygonal curve) from C onwards emanating away from O. Take the set of all such with C varying
The intersection is simply the union of two disjoint lines $X' \cup X''$ and this is disconnected.
See for example the book: J.G. Hawking and G. Young, Topology. Addison Wesley, 1961 for more information on such things

We conclude that $x(t)$ is a periodic trajectory

As another example let us look at the graph of $\sin \frac{1}{x}$:

$$X_n = \{ (x, \sin 1/x) \;/\; 0 < x < \tfrac{1}{n+1} \} \; ; n = 0, 1, 2, 3, \ldots$$

Evidently X_n is connected.

So any Y such that $X_n \subset Y \subset \bar{X}_n$ is also connected

So let $Y_n = X_n \cup \{ (0, q) \;/\; q \in \mathbb{Q}, -1 \leq q \leq 1 \}$

Then Y_n is connected for $n \geq 0$

But observe that

$$\bigcap_{n=0}^{\infty} Y_n = \{ (0, q) \;/\; q \in \mathbb{Q} : -1 \leq q \leq 1 \}$$

Which is d is connected !

However, it is a general theorem in topology (see Hocking and Young p. 43) that if the sets C_s are compact and connected (nested* of course) then $\bigcap C_s$ is connected

In our case $C = \bigcap_{s>0} C_s = \bigcap_{n \in \mathbb{N}} C_n$ and we shall prove the more restricted version:

Theorem 5.11: Suppose (C_n) is a decreasing sequence of compact and connected subsets in a metric space X then the intersection $\bigcap C_n$ is connected

Proof: Let C be the intersection and assume that it is disconnected. Then, there exists a continuous surjective function $f : C \to \{0,1\}$

By Tietze's extn thm, this extends continuously to $f_1 : C_1 \to [0,1]$ and f surj $\Rightarrow f_1$ surj (intermediate value thm)

There exists $p_1 \in C_1$ such that $f_1(p_1) = 1/2$

Let $f_2 = f_1|_{C_2} : C_2 \to [0,1]$ which is again an extn of f and so f_2 takes both the values 0 and 1 $\Rightarrow$ f_2 surjective

So $\exists\ p_2 \in C_2$ s.t. $f_2(p_2) = \frac{1}{2}$

Proceeding thus, we construct a seq (p_n) of points such that $p_n \in C_n$

$f_n(p_n) = f_1|_{C_n}(p_n) = \frac{1}{2}$. Thus $f_1(p_n) = \frac{1}{2}$

Let (p_{n_k}) be a subseq converging to $p \in C_1$

First $f_1(p_{n_k}) \to f_1(p)$

$\therefore\ f_1(p) = \frac{1}{2}$

Next, $p_{n_2}, p_{n_3}, \ldots$ all lie in C_2. Conv. to $p \in C_2$

$f_{n_k}(p_{n_k}) = f_2(p_{n_k})$ $(k \geq 2)$

because each f_n $(n \geq 2)$ restricts is a restriction of f_2

---

* Can be replaced by slightly weaker Cond.

$\therefore f_2(P_{n_k}) = \frac{1}{2}$ for all k

$\therefore f_2(p) = \frac{1}{2}$

So we see that for each ℓ

$P_{n_\ell}, P_{n_{\ell+1}}, \ldots$ all belong to C_ℓ

and so $p \in C_\ell$: $f_\ell(p) = \frac{1}{2}$

Now we get that $p \in \bigcap_{n=1}^{\infty} C_n = C$

$f(p) = f_1(p) = \frac{1}{2}$ Contradiction.

Likewise the α-limits

Now we are in a position to show

Thm 5.12' Let S be a forward (backward) invariant set for the ODE $\dot{x} = f(x)$

Assume that S is Compact then the ω-limit set (resp α-limit set) of any solution with IC in S is Compact and Connected. It is non empty.

proof: $C_s = \overline{\{x(t) / t \geq s\}}$ is a closed subset of S (this is true even if S is merely closed and invariant)

C_s is Compact since S is Compact.

C_s is the closure of a Connected set $x([s, \infty))$ and so is Connected. Finally if $s_1 < s_2$ then

$C_{s_2} \subseteq C_{s_1}$ so the previous thm applies and $\bigcap C_s$ is Compact and Connected.

It is non empty since each C_s is non empty and Compactness with finite intersection property implies $\bigcap C_s \neq \phi$.

Let us now show

Thm (5.12)'' The α-limit set and ω-limit set of a Sol. are invariant sets.

pf: Let $p \in \omega(x)$ and q be a point on the solution curve with IC p

(Both of these could be clubbed as one theorem)

That is to say $q = \phi(t_0, p)$

$\phi(t, c)$ is the unique sol. of the IVP

$$\dot{x} = f(x); \quad x(0) = c$$

Now $p \in \omega(x)$ means there is a sequence $t_n \to \infty$ such that $\phi(t_n, x(0)) \to p$.

Let us now look at the points

$$q_n = \phi(t_0 + t_n, x(0))$$

which lie on the forward trajectory $\{x(t) / t \geq 0\}$

We use the basic flow property $\phi_{s+t} = \phi_s \circ \phi_t$: (See p. 209)

$$\phi(t_0 + t_n, x(0)) = \phi(t_0, \phi(t_n, x(0))) \to \phi(t_0, p) = q$$

(Continuity wrt IC)

So we get that $q_n \to q$ showing that $q \in \omega(x)$

Cor 5.13:

Suppose $f: \Omega \to \mathbb{R}^n$ smooth and $x(t)$ sol. of $\dot{x} = f(x)$ such that $\lim_{t \to \infty} x(t)$ exists and is in Ω

then

this limit is a rest point

pf: $p = \lim_{t \to \infty} x(t) \Rightarrow \omega(x) = \{p\}$

But this means the singleton $\{p\}$ is invariant (forwards) and this implies p is a rest point.

We have given another demonstration of the fact that theorem 5.1 (p. 185)

Now we prove the important result topological nature

Thm 5.14 Suppose K is an invariant set for a system of ODEs $\dot{x} = f(x)$ such that K is homeomorphic to the closed unit ball in $\mathbb{R}^n$ then K contains a rest point

Proof: As usual $t \mapsto \phi(t,p)$ is the sol. of the IVP

$\dot{x} = f(x)$; $x(0) = p$

For each $t_1 > 0$, the map $p \mapsto \phi(t_1, p)$

maps K to itself

and hence has a fixed point p_1

Ex: Show that if a space X has the fixed point property (namely, any continuous $f: X \to X$ has a fixed pt) then any space Y homeomorphic to X also has the fixed point property. Brouwer's thm asserts that the closed unit ball in $\mathbb{R}^n$ has the fixed point property*

Now p_1 is either a rest point in which case the proof is over or else

$\phi(t_1, p) = p_1 = \phi(0, p_1)$ and we have a periodic solution with period t_1

p_1 of course would depend on t_1. We proceed along this line

Now let $T_m \to 0$: $T_m > 0$: $\phi(T_m, p_m) = p_m$

For each m, we have a periodic trajectory with period T_m (in case p_m is a rest point the proof is complete)

(p_m) has a convergent subseq conv to say $q \in K$

Work with this subseq and at the outset we may assume WLOG $p_m \to q$.

Claim: q is a rest point

Begin with

$$|\phi(t,q) - q| \leq |\phi(t,q) - \phi(t, p_m)| + |\phi(t, p_m) - p_m| + |p_m - q|$$

First and third terms $\to 0$ as $m \to \infty$

*) For a proof see the book

C. C. Pugh: Real mathematical analysis, Springer Verlag NY 2002. Note that Pugh is a well known researcher in the field of DE and Dynamical Systems. So we choose to cite this work in this Course on ODEs.

To deal with the middle one we proceed as follows. For $t > 0$ select a natural # $k_m(t)$ such that

$$k_m(t)\,T_m \leq t < k_m(t)\,T_m + T_m$$

Then

$$\phi(t, p_m) = \phi(t - k_m(t)T_m, p_m)$$

since T_m is a period.

But then $0 \leq t - k_m(t)T_m < T_m \to 0$ as $m \to \infty$

so

$$\phi(t, p_m) \to \phi(0, q) \quad \text{as } m \to \infty$$

$$\phi(t, p_m) - p_m \to \phi(0, q) - q = 0 \quad \text{as desired.}$$

So we get

$$\phi(t, q) = q \quad \text{for all } t \in [0, \infty)$$

which means q is a rest point.

Ref: J. Hale: ODEs, pp 48-49, Dover Pub, 1980.

Cor 5.15

If a planar system of ODEs

$$\dot{x} = P(x, y)$$
$$\dot{y} = Q(x, y)$$

has a periodic solution on a convex domain $\mathcal{L}$ on which P, Q are smooth, then this periodic trajectory encloses a rest point

proof: The closed trajectory is a Jordan curve and together with its interior (in the sense of the JCT) forms a closed set $\mathcal{D}$ which is homeomorphic to the closed unit disc

This $\mathcal{D}$ is evidently an invariant set. The boundary is one complete trajectory and so if a point p in this $\phi(t, p)$ must remain in $\partial\mathcal{D}$

If $p \in \text{Int}\,\mathcal{D}$ and $\phi(t, p) \in \mathbb{R}^2 - \mathcal{D}$

then the points p and $\phi(t, p)$ must be in the same path component of the complement $\mathbb{R}^2 - \partial\mathcal{D}$ which is a contradiction.

So $\mathcal{D}$ invariant and must contain a rest point

Alternate proof of Cor 5.15:

Let γ be a periodic trajectory in Ω (Ω Convex)
Using Stokes' theorem one checks

$$\oint_\gamma \frac{P\,dQ - Q\,dP}{P^2+Q^2} = 0 \quad \text{if} \quad P^2+Q^2 \neq 0 \text{ on } D \qquad D = \gamma \cup (\text{Int}\,\gamma) \qquad (5.33)$$

Assume then that there are no rest points in D. We shall compute this integral in a different way and get a non zero value leading to a contradiction well!

$$\oint_\gamma \frac{P\,dQ - Q\,dP}{P^2+Q^2} = \int_0^T \frac{P\dot{Q} - Q\dot{P}}{P^2+Q^2}\,dt \qquad (5.34)$$

$$= \int_0^T \left(\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2+\dot{y}^2)}\right) dt \qquad (5.34)'$$

T = period of the solution.

Now the integrand in $(5.34)'$ is $k(s)\,ds$ where $k(s)$ is the curvature measured with respect to the arc length parametrization - Recall that $\frac{dt}{ds} = (\dot{x}^2+\dot{y}^2)^{-\frac{1}{2}}$
So the integral in question is

$$\int_0^L k(s)\,ds \qquad (5.34)''$$

Now the Umlaufsatz from elementary diff geom. gives that the integral $(5.34)''$ equals $\pm 2\pi$
(See for example p 40 of the book
C. Bär, Elementary diff. geom; Cambridge Univ. press 2010).

Next we prove the Dulac Bendixon negative Criterion:

Thm 5.16 (Dulac - Bendixon):
Consider the system of ODEs $\dot{x} = f(x)$
$f: \Omega \to \mathbb{R}^2$ smooth Ω Convex Open set in $\mathbb{R}^2$
If $\text{Div}\, f(x) \neq 0$ throughout Ω then
the system of ODEs cannot admit any periodic solutions.

proof is a direct application of Green's-Thm:
Suppose ∃ a periodic solution γ with period T
Along γ:

$$P\dot{y} - Q\dot{x} = \dot{x}\dot{y} - \dot{y}\dot{x} = 0$$

So $$\oint_{\gamma} (P\,dy - Q\,dx) = 0.$$

Green's-Thm (Ω is Convex!) now gives

$$\iint_{\text{Int}\,\gamma} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx\,dy = 0$$

but this is plainly a Contradiction

Since $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ maintains the same sign throughout Ω and the integral cannot be zero.

Exercise Prove the following more general Version of Dulac-Bendixon Criterion:
Suppose there exists a smooth $\beta : \Omega \to \mathbb{R}$: Ω Convex and $\text{Div}(\beta P, \beta Q) \neq 0$ on Ω
then $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$ does not admit a periodic solution.

As an application Consider the Lotka-Volterra model taking into account Social friction among members of the Same Species (See p 94 of J.D. Murray's Math. Biology, volume I, Springer 2002)

$$\dot{x} = ax - bxy + \epsilon x^2 \quad ; \quad \epsilon, \delta > 0$$
$$\dot{y} = -by + exy + \delta y^2 \quad a, b, c, e \text{ as before}$$

System is Considered in the first quadrant.
Here the divergence $\neq$ does vanish but take $\beta(x,y) = 1/xy$ and apply the Generalized Version of Dulac Bendixon!

$$\frac{\partial}{\partial x}(a - by + \epsilon x) + \frac{\partial}{\partial y}(-b + ex + \delta y) = \epsilon + \delta > 0 \;(!)$$

periodic solutions are ALL LOST (!)

VI - Non linear Systems - Integral invariants, Jacobi's thm and Complete integrability

In this Chapter we discuss another important aspect of autonomous nonlinear systems:

$$\dot{x} = f(x) \qquad (6.1)$$

Where $f: \Omega \to \mathbb{R}^n$ is smooth, Ω open in $\mathbb{R}^n$. This is the notion of Complete integrability

The system (6.1) is said to be Completely integrable if there exists $(n-1)$ functionally indep (global) first integrals (namely first integrals defined on Whole of Ω). In Contrast local first integrals exist in the nbd of any non rest point and there are always $(n-1)$ functionally indep local integrals. The precise definitions follow.

Often one obtains $(n-2)$ functionally indep first integrals through physical / geometrical Considerations. If a System exhibits a Continuous One parameter group of Symmetries there will be an associated first integral. We shall however not prove this thm (due to Noether) but give one example of what a one parameter group of Symmetry is.

Let $\alpha \in \mathbb{R}$ and $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$^{\alpha}T(x,y) = (X,Y)$$

$X = x\cos\alpha - y\sin\alpha$ and $Y = x\sin\alpha + y\cos\alpha$; rotation of $\mathbb{R}^2$ about angle α fixing origin.

$\{T_\alpha \mid \alpha \in \mathbb{R}\}$ forms a group.

Now look at the System of ODEs:

$$\dot{x} = y : \quad \dot{y} = -x \qquad (6.2)'$$

$$\dot{X} = \dot{x}\cos\alpha - \dot{y}\sin\alpha = y\cos\alpha + x\sin\alpha = Y$$

$$\dot{Y} = \dot{x}\sin\alpha + \dot{y}\cos\alpha = y\sin\alpha - x\cos\alpha = -X$$

$(6.2)'$ goes over to

$$\dot{X} = Y, \quad \dot{Y} = -X \qquad (6.2)''$$

and we see that $\{T_\alpha\}$ is a one parameter

group of symmetries. The associated first integral is x^2+y^2

Such first integrals will be said to arise out of geometrical considerations. Physical first integrals are those that spell out laws of Conservation Such as the law of Conservation of energy for the pendulum:

$$\dot{x} = y : \quad \dot{y} = -\frac{g}{\ell}\sin x \qquad (6.3)'$$

$$\phi(x,y) = \frac{y^2}{2} - \frac{g}{\ell}\cos x \qquad (6.3)''$$

We have seen the immense use of first integrals. To Summarize: Two principal sources of first int:

Geometry (via symmetries)
Physics (Conservation laws)

Often, one obtains $(n-2)$ functionally indep first-integrals ~~of~~ using the above and the missing one (last integral!) is obtainable via a theorem of C.G.J. Jacobi provided the system (6.1) satisfies the Condition $\operatorname{div}(\rho f) = 0 \qquad (6.4)$ for some smooth $\rho : \Omega \to \mathbb{R}$, $\forall \rho$ One must of course exclude trivial cases such as $\rho \equiv 0$ (the precise hypothesis will be stated later). Such a ρ is called a Jacobi Multiplier.

Jacobi Multipliers are special instances of what are known as integral invariants. Integral invariants is a notion introduced by Poincaré in his researches on Celestial Mechanics

A celebrated result due to Joseph Liouville, (See the introductory Chapter) is well known in fluid mechanics namely if $\operatorname{div}(f) = 0$ then the flow $\{\Phi_t\}_{t\in\mathbb{R}}$ of the system of

ODEs $\dot{x} = f(x)$ (recall that $\Phi_t : \Omega \to \Omega$ is the map described as follows: $\Phi_t(p)$ is the unique solution $\phi(t,p)$ of the ODE with IC p) that preserves volume. This means, for any open set $G \subset \Omega$,

$$\text{Vol}(\Phi_t(G)) = \text{Vol}\, G. \qquad (6.4)$$

The result easily generalizes (even to non-autonomous systems) and we shall discuss this in the light of the continuity equation in hydrodynamics. For a detailed commentary on Liouville's thm see J. Lützen's biography Joseph Liouville, master of pure and applied math. Springer verlag 1990 p657 ff

Although Liouville's thm is quite old rooted in fluid mechanics, its use in dynamical systems - specifically in regard to stability issues - is due to H. Poincaré. In his celebrated memoir* (1890) on the three body problem he proves the recurrence theorem that bears his name.

We shall discuss some important examples on complete integrability and also exhibit a case of non existence of global first integrals. The most important example is that of the top spinning under the influence of gravity (and the integrable cases thereof). We shall also indicate the use of Liouville's thm in geometrical optics. References will be provided as we go along. After this (somewhat lengthy) introduction we come to the basic definitions.

(*) An English translation due to B.D. Popp is available: The three-Body problem and the eq^ns of dynamics Tr. B.D. Popp, Springer 2017

We first formalize the notion of the flow generated by a system of ODEs $\dot{x} = f(x)$ ($f: \Omega \to \mathbb{R}^n$ open). If the IVP for this system with IC p has solution $\phi(t,p)$ with maximal interval $\neq \mathbb{R}$ all we get is a <u>local flow</u>

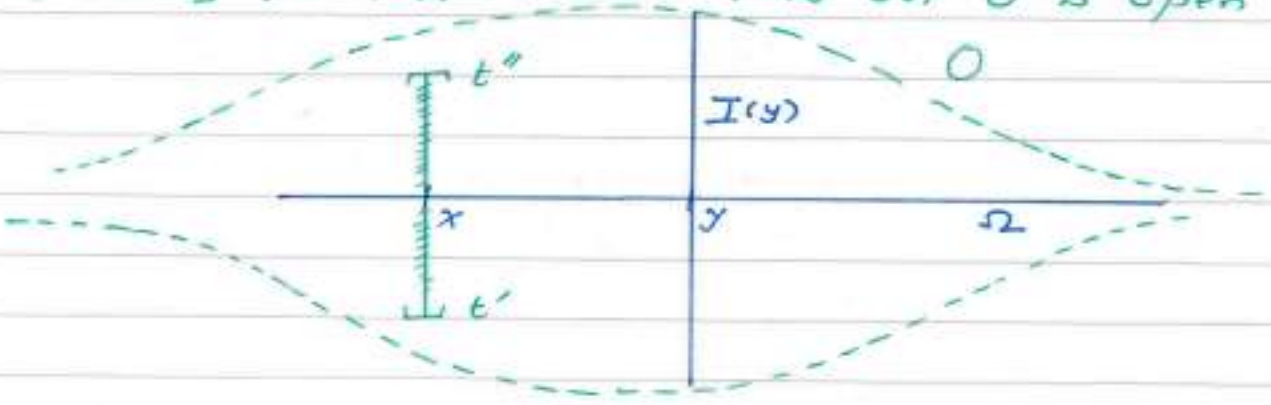
For each $p \in \Omega$, let $I(p)$ be the maximal interval of existence of the sol. of the IVP

$$\dot{x} = f(x); \quad x(0) = p$$

and $\phi(t,p)$ the corresp. sol. We construct the open set

$$O = \{ (t,y) / \; t \in I(y) \; ; \; (t,y) \in \mathbb{R} \times \Omega \}$$

We HAVE TO PROVE that this set O is open (!)



<u>Lemma 6.1</u>: Suppose for $x \in \Omega$, $t \in I(x)$ namely $\phi(t,x)$ is defined

and $\phi(s, \phi(t,x))$ is defined that is,

$$s \in I(\phi(t,x))$$

then $s+t \in I(x)$ i.e $\phi(t+s, x)$ is defined and further

$$\phi(s+t, x) = \phi(s, \phi(t,x))$$



Sol. starting at x ends at $\phi(t,x)$ in time t

Sol. starting at $\phi(t,x)$ ends at $\phi(s, \phi(t,x))$ in time s

Then, Sol. starting at x ends at $\phi(s, \phi(t,s))$ in time $t+s$.

proof is simple. We do it for $s, t > 0$. The other cases are similar. Idea is reflection that we employed in chapt 5 in conn. with Euler's eqn

$I(x) = (\alpha, \beta)$ say : $\alpha < t < \beta$.

We have to show

$s + t < \beta$ if $s \in I(\phi(t, x))$

($s + t > \alpha$ since $s > 0$). Well, define

$$\psi(\tau) = \begin{cases} \phi(\tau, x) & \text{if } \alpha < \tau \leq t \\ \phi(\tau - t, \phi(t, x)) & \text{if } t \leq \tau \leq t + s \end{cases}$$

Clearly ψ is Continuous on $(\alpha, t+s]$ and notice that $\phi(\tau - t, \phi(t, x))$ is defined even beyond $t+s$ (!) (check this)

The one sided derivatives of ψ at t agree and so ψ is even differentiable

ψ satisfies the ODE (Easy. Check this)

and $\psi(0) = \phi(0, x) = x$

$\therefore$ ψ is a sol. of the IVP with IC x

and this is defined in a nbd of $t+s$

This means $t + s \in I(x)$ (first part established)

Now both. $s \mapsto \phi(s+t, x)$ and

$s \mapsto \phi(s, \phi(t, x))$

satisfy the ODE and have the same value at $s = 0$

$\Rightarrow \phi(s+t, x) = \phi(s, \phi(t, x))$

Second part of the thm is now established

The next lemma is basic analysis

Lemma 6.2 : Suppose $f : \Omega \to \mathbb{R}^n$ is Lipschitz on a ball around each point $p \in \Omega$ then

$f|_K$ is Lipschitz for each Compact $K \subset \Omega$

The hypothesis says that each $p \in \Omega$ has a ball $B_\epsilon(p)$ around it s.t $f|_{B_\epsilon(p)}$ is Lipschitz (the Lipschitz Const may depend on p)

Suppose $f|_K$ is not Lipschitz - then for each n $\exists$ a pair of points $x_n, y_n \in K$ s.t.

$$|f(x_n) - f(y_n)| \geq n\,|x_n - y_n| > 0$$

Now $\exists\, M > 0$ s.t. $|f(x)| \leq M$ for all $x \in K$

Passing to subsequences we may assume that

$$x_n \to x_0 : \; y_n \to y_0$$

$$0 < n|x_n - y_n| \leq |f(x_n) - f(y_n)| \leq 2M$$

$$\Rightarrow x_0 = y_0.$$

Now let $\epsilon > 0$ s.t. $f|_{B_\epsilon(x_0)}$ is Lipschitz with Const. L.

Now $x_n, y_n \in B_\epsilon(x_0)$ for all $n \geq n_0$. (Some n_0)

$\therefore\; |f(x_n) - f(y_n)| \leq L\,|x_n - y_n| \quad \forall\, n \geq n_0.$

$\therefore\; 0 < n|x_n - y_n| \leq L\,|x_n - y_n| \quad \forall\, n \geq n_0$

$\therefore\; n \leq L \quad \forall\, n \geq n_0$ Contradiction.

A smooth $f: \Omega \to \mathbb{R}^n$ is locally Lipschitz (trivial) and we shall employ the lemma to smooth f. The next lemma is key to proving O is open

Lemma 6.3: Suppose $f: \Omega \to \mathbb{R}^n$ is smooth and $x(t)$ is a sol. of the IVP

$$\dot{x} = f(x); \quad x(0) = p \in \Omega$$

defined on $[t', t''] \subset I(p)$

There exists a nbd U of p such that for all $q \in U$ the solution of the IVP ODE with Initial Cond q is defined on $[t', t'']$

Moreover, $\exists$ a Const $L > 0$ such that

$$|\phi(t, q) - \phi(t, p)| < |p - q| \exp L(t - t_0)$$

$$t' \leq t_0 \leq t''$$

Proof: Well $x([t', t''])$ is a compact subset of Ω so $\exists\, \epsilon > 0$ such that the "ϵ-tube" around $x([t', t''])$ is contained in Ω namely,

$$K = \{ y \,/\, \text{dist}(y, x([t', t'']) \leq \epsilon \} \subset \Omega$$

We shall assume $t' < 0 < t''$.

$f|_K$ is Lipschitz with const L

Take $\delta = \epsilon/2 \exp -L|t'-t''|$
If $q \in B_\delta(p)$ then $q \in \Omega$ and

$$|p-q| < \epsilon/2 \exp -L|t'-t''|$$

Now we shall prove that the solution $\phi(t,q)$ (of the ODE with IC q) is defined on $[t',t'']$
That is $I(q) \supset [t',t'']$

Suppose $t \in I(q)$ and in $[t',t'']$
Gronwall's lemma shows (with notation $x(t) = \phi(t,p)$)

$$|\phi(t,q) - \phi(t,p)| \leq |p-q| \exp L|t'-t''| < \epsilon/2 \qquad (6.5)$$

Thus,
$\phi(t,q)$ is trapped in a compact subset of Ω for $q \in B_\delta(p)$ and $t \in I(q) \cap [t',t'']$

By the general principles of Chapt I, the max. interval of existence must thus be at least as large as $[t',t'']$

Summarizing we have shown:

$$\{p\} \times [t',t''] \Rightarrow \subset O$$
$$\Rightarrow B_\delta(p) \times [t',t''] \subset O$$

Thm 6.4: O is an open set in $\mathbb{R} \times \Omega$
The map $(t,x) \mapsto \phi(t,x)$
$O \longrightarrow \mathbb{R}^n$
is Continuous.

We have proved O is open. Next let $(t_0,p) \in O$

$$|\phi(t,q) - \phi(t_0,p)| \leq |\phi(t,q) - \phi(t,p)| + |\phi(t,p) - \phi(t_0,p)| \qquad (6.6)$$

Now Choose $c>0$ s.t

$$[t_0-c, t_0+c] \times \{p\} \subset \mathcal{O}$$

and then a tube

$$[t_0-c, t_0+c] \times B_\gamma(p) \subset \mathcal{O}$$

$|\phi(t,p) - \phi(t,q)|$ can be made small using the estimate (6.5) while

$|\phi(t,p) - \phi(t_0,p)|$ can be made small using Continuity of $s \mapsto \phi(s,p)$

Details are left for the reader.

Def (local flow) The map $\mathcal{O} \longrightarrow \mathbb{R}^n$ (6.7)

$$(t,p) \longmapsto \phi(t,p)$$

described above is called the local flow generated by $\dot{x} = f(x)$

The flow is said to be global or Complete if $\mathcal{O} = \mathbb{R} \times \Omega$

We sometimes say that the ODE $\dot{x} = f(x)$ defines a Complete flow or that the System of ODEs $\dot{x} = f(x)$ is Complete if $\mathcal{O} = \mathbb{R} \times \Omega$

So for example the following systems are Complete: (i) Lotka Volterra

(ii) Pendulum

(iii) Any linear system $\dot{x} = f(x)$

(iv) Euler's equations:

$A\dot{x}_1 = (B-C)x_2x_3$; $B\dot{x}_2 = (C-A)x_1x_3$ and

$C\dot{x}_3 = (A-B)x_1x_2$

Where $0 < C < B < A$.

Exercises: (i) $\dot{x} = x^2+1$ (in $\mathbb{R}$) is not Complete but $\dot{x} = \exp(-x^2)$ is Complete

(ii) Use Gronwall's lemma to prove that if $\|f(x)\| \leq C\|x\| \quad \forall x \in \Omega$

then $\dot{x} = f(x)$ is Complete

Let $f: \mathbb{R} \to \mathbb{R}$ be smooth positive and

$$\int_\alpha^\infty \frac{ds}{f(s)} < \infty \quad \text{for some } \alpha > 0$$

Show that $\dot{x} = f(x)$ cannot be complete

Local first integrals: Given a system of ODEs $\dot{x} = f(x)$: $f: \Omega \to \mathbb{R}^n$ smooth
A local first integral in a nbd of $p \in \Omega$ is a smooth $\varphi: N \to \mathbb{R}$ (N nbd of p) such that $\nabla\varphi \cdot f \equiv 0$ and $\nabla\varphi \neq 0$ a.e in N.

Effect of a coordinate transf.

Let us consider a system of ODEs $\dot{x} = f(x)$ (6.8)'
$f: \Omega \to \mathbb{R}$ a smooth function
and $F: \tilde{\Omega} \to \Omega$ be a coord transf.
(i.e F is a diffeomorphism of $\tilde{\Omega}$ onto Ω)
Say $x = F(y)$; $y \in \tilde{\Omega}$
Then

$$\dot{x} = (DF(y))\dot{y}$$
$$\therefore f(F(y)) = (DF(y))\dot{y}$$
$$\text{So} \quad \dot{y} = (DF(y))^{-1} f(F(y)) \qquad (6.8)''$$

In classical language the effect of F is to map $f(x)$ to $(DF(y))^{-1} f(F(y))$ and this is called Contravariant transformation law. You will see this being stated as: A vector field is a Contravariant-tensor of rank one. We shall not use this language in this course and so shall say no more on this.

Ex: Check that if ϕ is a first integral of (6.8)' then $\tilde{\phi} = \phi \circ F$ is a first int of (6.8)''
Write $\nabla\phi \cdot f$ as $(\nabla\phi) f$ thinking of

$\nabla\phi$ as a $1 \times n$ matrix and f as $n \times 1$ matrix
Direct application of Chain rule !

We are now ready to prove the important-
Thm 6.5 (Rectification-thm)

Consider the system $\dot{x} = f(x)$ (6.9)
$f: \Omega \to \mathbb{R}^n$ smooth and fix $p \in \Omega$ such that-
$f(p) \neq 0$. Then, there exists a neighborhood
N of p and a diffeomorphism
$F: \tilde{N} \to N$: $\tilde{N}$ nbd of 0
such that (6.9) transforms into
$\dot{y} = [1, 0, \ldots, 0]^T$ in $\tilde{N}$ (6.9)'

Before taking up the proof of thm 6.5 let us discuss a simple Corollary
Cor 6.6: For the system (6.9) there exists on N a set of $(n-1)$ functionally indep. & local first integrals

We shall define precisely the notion of functional indep but all we need to know is that functional indep of a set of real valued functions
$\phi_1, \ldots, \phi_k : N \to \mathbb{R}$ is equiv guaranteed if
$\nabla\phi_1, \ldots, \nabla\phi_k$ lin. indep almost everywhere on N
In fact this can be taken as a provisional def (!) of this concept.
proof of the Cor is simple:
For (6.9)' we clearly have $(n-1)$ first integrals
$y_2, y_3, \ldots, y_n$ all defined on $\tilde{N}$
(coord functions)
Then $y_2 \circ F^{-1}, \ldots, y_n \circ F^{-1}$ are first integrals of the original system (6.9)

That is to say if $F^{-1} = (G_1, \ldots, G_n)$ in components then $G_2, G_3, \ldots, G_n$ are first integrals of (6-8) Evidently these are functionally indep since

$$\text{Rank } DG = n$$

(whereby $\nabla G_2, \ldots, \nabla G_n$ everywhere lin indep on N)

Summarizing, we have proved that in a nbd of a non-rest point the system of ODEs admits $(n-1)$ functionally indep local first integrals.

Proof of the rectification theorem:

Without loss of generality we may assume that the ~~rest~~ non rest point p is Origin Performing a linear change of variables (Const Coeff) we may assume that

$$f(0) = \hat{e}_1 = [1, 0, \ldots, 0]^T$$

For $(p_2, \ldots, p_n)$ (with suff. small norm) let $\phi(t, p_2, \ldots, p_n)$ be the unique sol. of the ODE $\dot{x} = f(x)$ with IC $(0, p_2, \ldots, p_n)$ So,

$$\frac{d\phi}{dt} = f(\phi(t, p_2, \ldots p_n)) \; ; \; \left.\frac{\partial \phi}{\partial t}\right|_{t=0} = f(0, p_2, \ldots, p_n)$$

Next, using the principles of Chapt I (differentiability wrt IC and the variational equation)

$$\frac{\partial \phi}{\partial p_j} \; (j = 2, 3, \ldots, n) \text{ exist and } \left.\frac{\partial \phi}{\partial p_j}\right|_{t=0} = \frac{\partial}{\partial p_j}\phi(0, p)$$

$$= \hat{e}_j \quad (j = 2, \ldots, n)$$

Now $\dfrac{\partial \phi}{\partial t}$ at the origin is $f(0, \ldots, 0) = \hat{e}_1$

So $\det(D\phi) = 1$ at the origin

$D\phi$ is the derivative of the map

$(t, p_2, \ldots, p_n) \longmapsto \phi(t, p_2, \ldots, p_n)$ (6·10)

This map is a local diffeo of a nbd N of the origin onto a nbd $\tilde{N}$ of the origin by the inverse function thm (Rudin: Principles ... Chapter 9)

Let $G(t, p_2, \ldots, p_n)$ be the map given by (6·10) Let us compute

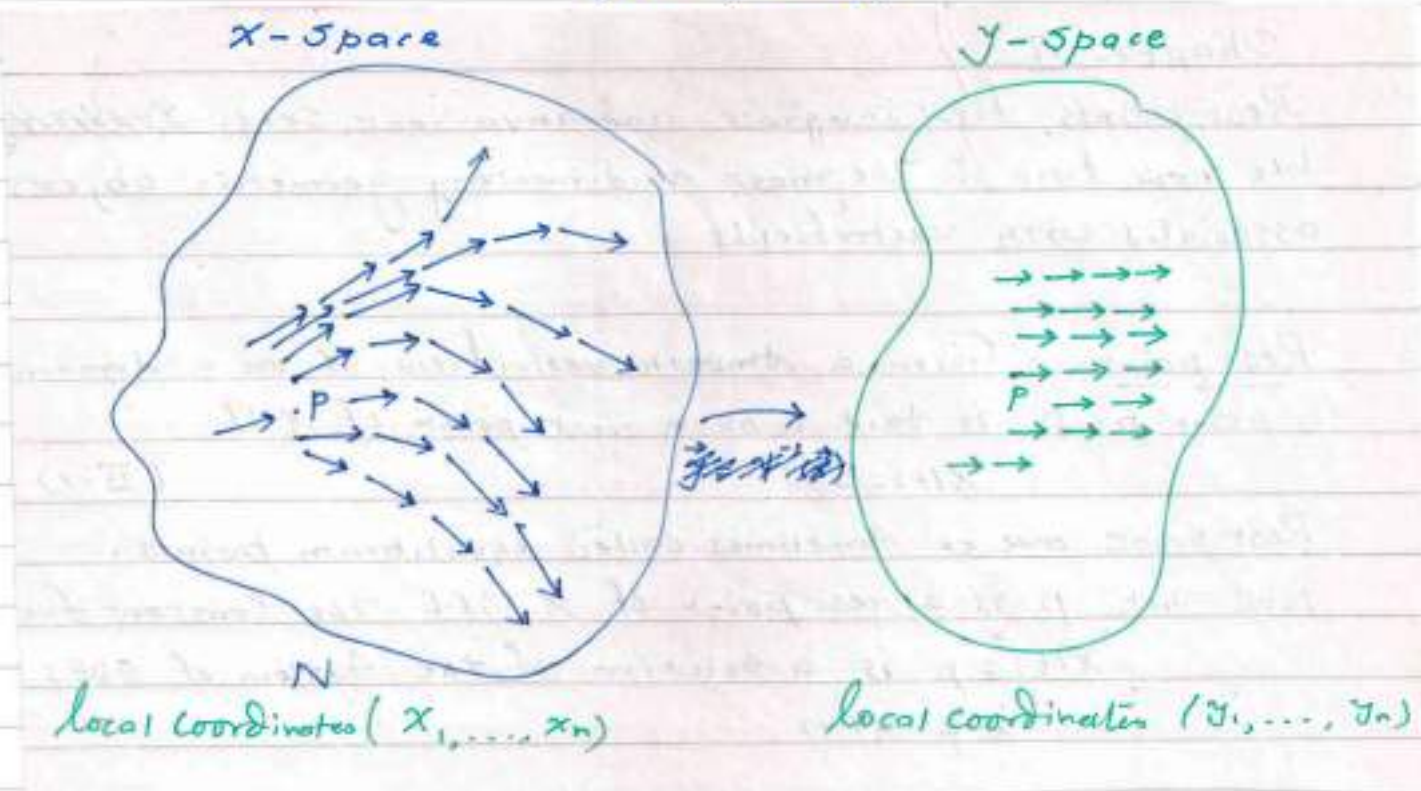
$$(DG(t, p_2, \ldots, p_n))\hat{e}_1 = \frac{\partial G}{\partial t} = \frac{\partial \phi}{\partial t}$$

$$= f(\phi(t, p_2, \ldots, p_n))$$

$$= f(G(t, p_2, \ldots, p_n)$$

$$\therefore \hat{e}_1 = (DG(t, p_2, \ldots, p_n))^{-1}(f \circ G)(t, p_2, \ldots, p_n)$$

Now referring to the transformation rule for ODEs (see p 244)



We see that the transformed ODE in the new coordinates $(y_1, \ldots, y_n) = \phi(t, p_2, \ldots, p_n)$ is precisely $\dot{y} = \hat{e}_1$. Proof is Complete.

Example: Non existence of any global first integrals

Let us return to the example on p 190 (eqn (5.12))

$$\left.\begin{aligned} \dot{x} &= y + x(1-x^2-y^2) \\ \dot{y} &= -x + y(1-x^2-y^2) \end{aligned}\right\} \quad (6.11)$$

We have seen that if $(x(t), y(t))$ is a non constant solution and

$$S(t) = (x(t))^2 + (y(t))^2$$

Then $S(t)$ is monotone increasing and $S(t) \to 1$ as $t \to +\infty$. That is the distance of $(x(t), y(t))$ from the unit circle tends to 0 as $t \to \infty$

Writing in terms of polar coordinates we see that $\frac{d\theta}{dt} = 1$

So that the trajectory $t \longmapsto (x(t), y(t))$ spirals outwards and the spiral approaches the unit circle

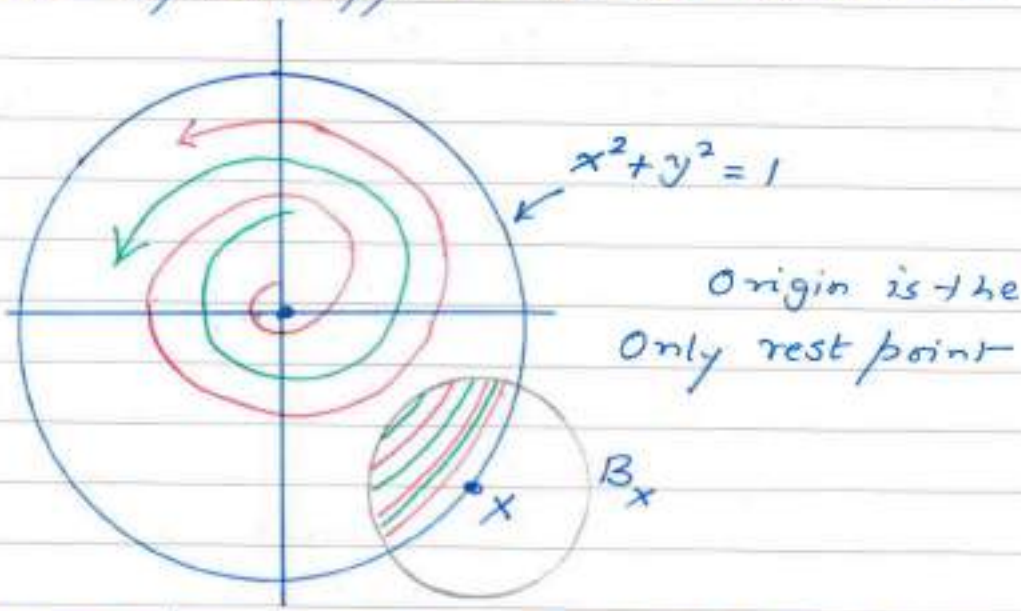


Figure depicts two distinct solutions spiraling outwards. The arms of the spirals approach the unit circle arbitrarily closely.

Let us take a point X on the unit circle and a disc B_x around it. The spiraling arms slice this disc B_x and the cuts are depicted as red and Green arcs interlacing

Suppose ϕ is a first integral, since $\nabla\phi \neq 0$ a.e ϕ must be non constant (a.e) in B_x

We show that this cannot be so

Let $p, q \in B_x$ s.t $\phi(p) \neq \phi(q)$

Then clearly p, q must lie on distinct trajectories say p, q lie on red and green spirals resp.

Then along each of the green arcs in B_x ϕ takes the value $\phi(p)$

Along each of the red arcs in B_x, ϕ takes the value $\phi(q)$

But since X is a limit point of all these green arcs as well as all the red arcs,

$$\phi(p) = \phi(x) \quad \text{and} \quad \phi(q) = \phi(x)$$

Contradiction since $\phi(p) \neq \phi(q)$.

So ϕ is constant throughout B_x again contradiction since $\nabla\phi \neq 0$ almost everywhere.

Functional Independence

This important notion was more commonly treated in Calculus texts in the first half of the 20th cent probably inspired by the book of E. Goursat and transl into English by E. R. Hedrick: A Course in Math. Analysis, Vol I, Ginn and Co. 1904 See for example G. A. Gibson's Advanced Calc, Macmillan Co; 1966; p143 ff (there is also a ref to a work of Bateman on p143). This topic seems to have gone out of vogue in modern treatments despite it being important.

A good though terse treatment is given in R. Narasimhan's Analysis on real and complex manifolds pp 25-28 as an application of Sard's thm North-Holland, Masson & CIE Paris 1973.

The interested reader must consult the important article by W. F. Newns, Functional dependence American Math Monthly, Vol 74 (1967) 911-920. Newns's treatment is, set against a general abstract framework and comprehensive with other important references.

Let us begin with the def as given in R. Narasimhan's book (p 25)

Def 6.2: Suppose $\phi_1, \ldots, \phi_k : G \to \mathbb{R}$ are smooth functions defined on an open set G in $\mathbb{R}^n$ We say $\phi_1, \ldots, \phi_k$ are functionally dependent on a subset $S \subset G$ if

$\exists$ a smooth $\psi : \Omega \to \mathbb{R}$ where

(i) Ω is an open set containing the image $\phi(S)$; $\phi = (\phi_1, \ldots, \phi_k)$

(Ω open in $\mathbb{R}^k$ of course)

(ii) $\psi(\phi_1(x), \ldots, \phi_k(x)) \equiv 0$ on S

(iii) ψ is not identically zero on any open ball contained in Ω. That is to say $\psi^{-1}(0)$ is a closed nowhere dense subset of Ω.

Condition (i) of course is natural since the Composite function $\psi(\phi_1(x), \ldots, \phi_k(x))$ must make sense.

(ii) expresses the fact that there is a functional relation* among the $\phi_1, \ldots, \phi_k$

(iii) needs a word of explanation.

The functions $(x_1, x_2) \mapsto x_1$, $(x_1, x_2) \mapsto x_2$ are obviously functionally indep on $\mathbb{R}^2$ (in fact on every open subset of $\mathbb{R}^2$)

---

*) In classical literature of 19th cent. ψ would be called a Syzygy (!) An interesting term. See D. Hilbert: Lectures on Th. of Alg. Inv. Camb. Univ Press 1993

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function s.t.-

$$\psi(t) = \begin{cases} 1 & \text{if } |t| \leq \frac{1}{2} \\ 0 & \text{if } |t| \geq 1 \end{cases}$$

Look at $\phi_1(x_1, x_2) = x_1 \psi(x_1)$

$\phi_2(x_2, x_2) = x_2 \psi(x_2)$

ϕ_1, ϕ_2 are functionally indep on any open set G intersecting the open ball $\{x_1^2 + x_2^2 < 1/4\}$ (Why ?)

Now let $g(y_1, y_2)$ be $\equiv 0$ on $y_1^2 + y_2^2 < 2$

and $g(y_1, y_2) = y_1^2 + y_2^2$ on $y_1^2 + y_2^2 > 3$ say

Then the Composite

$g(\phi_1, \phi_2) \equiv 0$ despite the fact that-

ϕ_1, ϕ_2 functionally indep on $\mathbb{R}^2$. Problem arises due to the fact that $g \equiv 0$ on that open set where ϕ_1, ϕ_2 are functionally indep

Condition (iii) in the def, which demands that the Zero set $g^{-1}(0)$ has no interior points is violated here This condition (iii) is quite useful to eliminate such peculiarities and is sufficiently mild.

Let us Compare Condition (iii) with the following Condition (iii)': $\nabla \psi \neq 0$ almost everywhere

Obviously Condition (iii)' $\Rightarrow$ Condition (iii)

Also functional dependence on $S \Rightarrow$ functional dep. on $\tilde{S}$ for every $\tilde{S} \subset S$

Or, functional indep on a subset set $\Rightarrow$ functional indep on every superset-

It is useful to keep an analogy in mind from LA. Concerning linear independence.

$$\phi_1(t) = \begin{cases} t^3 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \quad ; \quad \phi_2(t) = \begin{cases} 0 & \text{if } t \geq 0 \\ t^3 & \text{if } t < 0. \end{cases}$$

$\phi_1 |_{(0,\infty)}$ and $\phi_2 |_{(0,\infty)}$ are lin dependent-

but ϕ_1, ϕ_2 are lin. indep as functions on whole of $\mathbb{R}$

We leave it to the reader to ponder over this matter.

Some simple observations first:

Suppose $\phi_1, \ldots, \phi_k$ are smooth functions on Ω

$F: \Omega \to \tilde{\Omega}$ a diffeomorphism

$\phi_1, \phi_2, \ldots, \phi_k$ functionally dependent / indep. on Ω iff $\phi_1 \circ F^{-1}, \ldots, \phi_k \circ F^{-1}$ are functionally dependent / independent on $\tilde{\Omega}$.

In other words, these notions are coord independent-

We shall need some thm on Mult. Var. Calc.

(i) Constant rank thm

(ii) Sard's theorem

The first is a direct consequence of Implicit function thm - See for example R. Narasimhan, Analysis on real and complex manifolds cited earlier (p 18)

OR

C. C. Pugh, Real Math. Analysis, Springer 2002 (p190ff) (cited earlier)

The other result is

Thm 6.7: (Sard's thm) Suppose Ω is an open set in $\mathbb{R}^n$ and $F: \Omega \to \mathbb{R}^m$ is a smooth map The set of critical values of F has zero Lebesgue measure.

In particular regular values exist. If $q \in \mathbb{R}^m$ is a regular value and $F^{-1}(q)$ non empty - then $F^{-1}(q)$ is a smooth manifold of dim $n-m$

Recall that q is a regular value iff $F^{-1}(q)$ does not contain any critical pts.

A point $p \in \Omega$ is critical if $DF(p): \mathbb{R}^n \to \mathbb{R}^m$ fails to be surjective. That is Rank $DF(p) \not\geq m$

For a system of ODEs we shall be looking at an m-tuple $(\phi_1, \ldots, \phi_m)$ of first integrals and apply these results.

We cannot prove Sard's thm (see Raghavan Narasimhan's book cited above).

Smooth Urysohn lemma: Recall that if X is a metric space and A is a nonempty closed subset of X, there is a continuous $f: X \to \mathbb{R}$ s.t. $f^{-1}(0) = A$.

The smooth version is that:

If A is a nonempty closed subset of $\mathbb{R}^n$ there exists a C^∞ function $\phi: \mathbb{R}^n \to \mathbb{R}$ such that $\phi^{-1}(0) = A$. (R. Narasimhan p. 25)

Thm 6.8' Suppose $\phi_1, \ldots, \phi_k : G \to \mathbb{R}$ are smooth functions such that at some point $p \in G$ $\nabla\phi_1(p), \ldots, \nabla\phi_k(p)$ are linearly independent then $\phi_1, \ldots, \phi_k$ are functionally indep in a nbd of p (and hence functionally indep on G)

proof $F = (\phi_1, \ldots, \phi_k): G \to \mathbb{R}^k$

Rank $DF(p) = k$

$\Rightarrow$ Rank $DF(x) = k$ for all x in a nbd N of p

Rank thm now implies $F(N)$ is an open set in $\mathbb{R}^k$

Suppose $\exists\ \psi: \Omega \to \mathbb{R}$: $\Omega \supset F(N)$

s.t $\psi(\phi_1, \ldots, \phi_k) \equiv 0$ on N and $\psi^{-1}(0)$ closed nowhere dense in Ω

$\nabla\psi(F(x))\, DF(x) \equiv 0 \quad \forall x \in N$

$\Rightarrow \nabla\psi(F(x)) \in \ker (DF(x))^T = (0)$ (why?)

$\Rightarrow \nabla\psi \equiv 0$ on $F(N)$

$\Rightarrow \psi$ is Const on the Open set $F(N)$

The equation $\psi(\phi_1, \ldots, \phi_k) \equiv 0$ shows that

$\psi \equiv 0$ on the Open set $F(N)$

Which Contradicts the fact that $\psi^{-1}(0)$ nowhere dense.

proof is Complete.

The Other Case is When Rank $DF(x) < k$ for all $x \in G$.

Thm 6.8'': Notation as in Thm (6.8)': $F = (\phi_1, \ldots, \phi_k)$

If Rank $DF(x) < k$ for all $x \in G$

then $\phi_1, \ldots, \phi_k$ are functionally dependent on every Compact Subset of G.

proof: $F(G)$ Contains only Critical values and so has Zero Lebesgue Measure.

If K is a Compact Subset of G then $F(K)$ is a Compact Subset G with Zero Lebesgue Meas.

So in particular, $F(K)$ is nowhere dense

Now select $\psi \in C^\infty(\mathbb{R}^k)$ such that

$\psi^{-1}(0) = F(K)$

then $\psi(\phi_1, \ldots, \phi_k) \equiv 0$ on K and the proof is Complete.

Remark: There Could be several non trivial relations among $\phi_1, \ldots, \phi_k$

for if ψ is a relation so is $e^\psi - 1$

but these two would not Count as indep relations!

So how many independent relations are there?

Or how many indep. Syzygies exist between $\phi_1, \ldots, \phi_k$?

This kind of question is discussed in the classical works (Goursat, Gibson etc.) and the Condition is that Rank $DF(x) \equiv \ell < k$ for all x in a nbd N of p.

The following is a supplement to Thm 6.8′ - 6.8″

Thm 6.9: Let $\phi_1, \ldots, \phi_k : G \to \mathbb{R}$ be smooth functions such that

$$\text{Rank}(DF(x)) \equiv \ell < k \quad \text{for all } x \in G$$

$$F = (\phi_1, \ldots, \phi_k) : G \to \mathbb{R}^k$$

(i) For each point $p \in G$ there is a nbd N of p such that $\phi_1|_N, \ldots, \phi_k|_N$ are functionally dependent

(ii) There are $k - \ell$ indep relations among $\phi_1|_N, \ldots, \phi_k|_N$.

Proof: WLOG assume $p = 0$ and $F(p) = 0$

By rank thm, $\exists$ a nbd N of 0 in $\mathbb{R}^n$ and a nbd U of 0 in $\mathbb{R}^k$ and diffeomorphisms $Q_1 : \tilde{N} \to Q_1(\tilde{N}) = N$

$$Q_2 : U \to Q_2(U)$$

such that

$$(Q_2 \circ F \circ Q_1)(y_1, \ldots, y_n) = (y_1, \ldots, y_\ell, 0, \ldots, 0) \quad (6.12)$$

$$Q_2((\phi_1 \circ Q_1)(y), \ldots, (\phi_k \circ Q_1)(y)) = (y_1, \ldots, y_\ell, 0, \ldots, 0) \quad (6.12)'$$

Write $$Q_2(z_1, \ldots, z_k) = (\psi_1(z), \ldots, \psi_k(z)) \quad (6.12)''$$
$$(z_1, \ldots, z_k) \in U$$

$\nabla\psi_1, \ldots, \nabla\psi_k$ are lin indep throughout U since Q_2 is a diffeomorphism

Equation (6.12)′ now reads:

$$\begin{matrix} \psi_{\ell+1}((\phi_1 \circ Q_1)(y), \ldots, (\phi_k \circ Q_1)(y)) = 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{for all } y \in \tilde{N}. \\ \psi_k((\phi_1 \circ Q_1)(y), \ldots, (\phi_k \circ Q_1)(y)) = 0 \end{matrix}$$

Writing $Q_1(y) = x$ we see that

$$\nabla\psi_j(\phi_1(x), \ldots, \phi_k(x)) = 0 \quad \text{for all } x \in \Omega_1 \cap \tilde{N} \tag{6.13}$$

$j = \ell+1, \ldots, k$.

The fact that $\nabla\psi_j$ $(j = \ell+1, \ldots, k)$ are lin indep implies that each ψ_j $(j = \ell+1, \ldots, k)$ is a nontrivial relation among the functions $\phi_1|_N, \ldots, \phi_k|_N$ and there are $k-\ell$ indep. relations (!)

Def 6-3 (Complete integrability) A system of ODEs

$$\dot{x} = f(x): \quad f: \Omega \to \mathbb{R}^n \text{ smooth},$$

(Ω open in $\mathbb{R}^n$) is said to be Completely integrable if there exist $(n-1)$ first integrals

$$\phi_1, \ldots, \phi_{n-1}: \Omega \to \mathbb{R}$$

that are functionally independent on Ω

Ex: Euler's equations: $0 < C < B < A$

$$A\dot{x}_1 = (B-C)x_2x_3 \; ; \; B\dot{x}_2 = (C-A)x_1x_3$$
$$C\dot{x}_3 = (A-B)x_1x_2$$

is a Completely integrable system since we have two functionally indep first integrals on $\mathbb{R}^3$

$$\phi(x) = Ax_1^2 + Bx_2^2 + Cx_3^2$$
$$\psi(x) = A^2x_1^2 + B^2x_2^2 + C^2x_3^2$$

The following are Completely integrable

The Lotka Volterra system

The pendulum equation

Any linear autonomous system $\dot{x} = Ax$ with A diagonalizable (Proof?)*

The system $\dot{x} = y + x(1-x^2-y^2)$

$$\dot{y} = -x + y(1-x^2-y^2)$$

is not Completely integrable since there are no global first integrals! (*) Hint: Use left eigen vectors

The phrase Complete integrability ought to mean that the solutions of the system of DEs must be obtainable in some explicit "explicit-form" or perhaps computable form and as explicit as possible

Let us begin with a simple example:

$$\dot{x} = y \;;\; \dot{y} = -x$$

Assume IC s.t $x^2+y^2 = 1$ (x^2+y^2 is the first Int)

Take a provisional parametrization of this curve $(\cos\xi(t),\ \sin\xi(t))$

$$x(t) = \cos\xi(t) \;;\; y(t) = \sin\xi(t)$$
$$\dot{x}(t) = -\sin\xi(t)\,\dot{\xi} \;;\; \dot{y} = (\cos\xi)\,\dot{\xi}$$
$$\therefore \sin\xi(t) = (-\sin\xi(t))\,\dot{\xi} \;;\; -\cos\xi = (\cos\xi)\,\dot{\xi}$$

Note that either of these gives the Variable Separable ODE $\frac{d\xi}{dt} = -1$

$$\therefore \xi(t) = -t + c \quad \text{from which we get-}$$

the sol. $x(t) = \cos(c-t),\ y(t) = \sin(c-t)$

Let us now try this out with Euler's equations:

$$A\dot{x}_1 = (B-C)x_2x_3 \;;\; B\dot{x}_2 = (C-A)x_1x_3 \;;\; C\dot{x}_3 = (A-B)x_1x_2$$

with its two first integrals

$$\phi = Ax_1^2 + Bx_2^2 + Cx_3^2 = c_1 \qquad (6\cdot13)$$
$$\psi = A^2x_1^2 + B^2x_2^2 + C^2x_3^2 = c_2 \qquad (6\cdot13)'$$

To this we adjoined (see p 220)

$$\xi(t)^2 = (x_1(t))^2 + (x_2(t))^2 + (x_3(t))^2 \qquad (6\cdot13)''$$

and then (See p 221)

$$x_1(t)^2 = \frac{c_1(B+C) - c_2 - \xi BC}{(A-B)(C-A)} \quad \text{etc;}$$

(cyclic rotation of letters)

$$x_2(t)^2 = \ldots \;;\; (x_3(t))^2 = \ldots$$

$\xi(t)$ can be thought of as a provisional parametrization of the intersection of the two ellipsoids (6·13) - (6·13)'

$$x_1\dot{x}_1 + x_2\dot{x}_2 + x_3\dot{x}_3 = \left(\sum \left(\frac{B-C}{A}\right)\right) x_1 x_2 x_3$$

$$\therefore \quad \frac{d}{dt}(\xi^2) = 2\sum\left(\frac{B-C}{A}\right) x_1 x_2 x_3 \qquad (6.14)$$

$$\frac{dt}{d\xi} = \frac{\sum\left(\frac{B-C}{A}\right)(A-B)(B-C)(C-A)}{\xi\left(\prod(* c_1(B+C) - c_2 - \xi BC)\right)^{\frac{1}{2}}}$$

(cyclic perm. of symbols)

(Recall that x_1^2, x_2^2, x_3^2 can be determined in terms of ξ^2 via (6.13) - (6.13)''. Assume that we are in the first orthant to begin with)

Again we get an variable separable first-order ODE for the provisional parameter $\xi(t)$ along the curve of intersection of (6.13) - (6.13)'

Denoting RHS of (6.14) as $F(\xi)$

$$t = \int_{\xi(0)}^{\xi} F(\lambda)\,d\lambda = G(\xi) \qquad (6.15)$$

Integral on RHS of (6.15) is an elliptic integral

After performing this integration to get $G(\xi)$ there remains the problem of

Inverting the equation $t = G(\xi)$ (6.16)

to obtain ξ as a function of t

and this would ultimately provide us

$x_1(t)$, $x_2(t)$ and $x_3(t)$ explicitly as functions of t via equations

$$x_1(t)^* = \left(\frac{c_1(B+C) - c_2 - \xi(t)BC}{(A-B)(C-A)}\right)^{\frac{1}{2}} \qquad (6.17)$$

$x_2(t) = \ldots$ and $x_3(t) = \ldots$

This clearly illustrates that the availability of $(n-1)$ functionally indep first integrals

$\phi_1, \phi_2, \ldots, \phi_{n-1}$ with linearly indep gradients $\nabla\phi_1, \ldots, \nabla\phi_{n-1}$ reduces the problem of integrating (finding solutions of) the system of ODEs to

(i) A quadrature: $t = \int_{\xi(0)}^{\xi} F(\lambda)\,d\lambda = G(\xi)$

namely Computing (in principle) this integral

And

(ii) Inverting the (transcendental) equation $t = G(\xi)$

This is precisely the meaning of the claim found in classical works that under $(n-1)$ func. indep first Int- reduces the problem to a quadrature and an inversion.

The Case of the two body problem:

In the case of the two body problem the above scheme goes through (after using a result due to Jacobi that we shall describe later) leading to the Inversion of the celebrated Kepler equation:

$$\frac{2\pi t}{T} = \phi - e\sin\phi \qquad (6.18)$$

This is quite Non trivial (!)

For details we refer the reader to the following important works:

(i) D. Boccaletti and G. Pucacco: Theory of Orbits, vol I, Springer 2004

(ii) L. A. Pars Treatise on analytical dynamics Heinemann, London 1965. See p398 ff particularly p400 (equation 21.2.5)

We shall return to the two body problem later but the reader may wonder if the reduction

to quadrature and inversion - demonstrated for Euler's equation for a top was merely a fortuitous accident or is it possible to see this as a special case of a general theorem.
The citation in L.A. Pars certainly hints at the latter.

Thm 6.10: Suppose the system of ODEs
$\dot{x} = f(x)$; $f: \Omega \to \mathbb{R}^n$ smooth
(Ω open in $\mathbb{R}^n$) $\phi_1, \ldots, \phi_{n-1}$
admits $(n-1)$ global first integrals such that
$\nabla\phi_1, \ldots, \nabla\phi_{n-1}$ are almost everywhere
linearly indep
We can reduce the problem of solving the
IVP $\dot{x} = f(x)$, $x(0) = c$
to a quadrature and inversion for almost all
IC c

proof: By Implicit function thm we solve the
system $\phi_1(x) = p_1, \ldots, \phi_{n-1}(x) = p_{n-1}$ (6.19)
($p_j = \phi_j(c)$)
Obtaining say $x_1, \ldots, x_{n-1}$ as functions of x_n
That is to say, in a nbd of c the curve of
intersection (6.19) has a provisional parametrization
$\xi \mapsto (x_1(\xi), \ldots, x_{n-1}(\xi), \xi)$
Or perhaps some other parametrization
$\xi \mapsto (x_1(\xi), \ldots, x_n(\xi))$ giving all coord
equal status.
We may assume that $x(0) = c$ (the Initial Cond)
$\left.\frac{dx}{d\xi}\right|_{\xi=0}$ = tangent to the curve (6.19)
of intersection of loci $\phi_i = p_i$ $(i = 1, 2, \ldots n-1)$

Since this is a regular curve,

$$\left.\frac{dx}{d\xi}\right|_{\xi=0} \neq 0.$$

and this vector is a multiple of $f(c)$
($f(c) \neq 0$ otherwise c is a rest point and the entire problem of solving the IVP is trivial)

Note: $\left.\frac{dx}{d\xi}\right|_{\xi=0}$ and $f(c) = \dot{x}(0)$ (dot refers to time derivative)
are both tangent vectors to the sol. curve
(the parametrizations are only different)

Reversing the parametrization if necessary
we may assume that

$$\left\langle \left.\frac{dx}{d\xi}\right|_{\xi=0}, f(c) \right\rangle > 0 \qquad (6.20)$$

Now $\nabla \phi_j$ $\perp$ to the level sets $\phi_j = p_j$.

While $\xi \mapsto x(\xi)$ lies on these

$$\nabla \phi_j \cdot \frac{dx}{d\xi} = 0 \qquad (j = 1, 2, \ldots, n-1)$$

Further $\nabla \phi_j \cdot f(x) = 0$ (∵ Each ϕ_j is a first-integral)

This implies

$$\frac{dx}{d\xi} = \lambda(\xi) f(x(\xi)) \qquad (6.20)'$$

$\lambda(\xi)$ is a smooth function of ξ along the curve (6.19)

$\lambda(0) > 0$ thanks to (6.20)

Now <u>define</u>

$$t = \int_0^{\xi} \lambda(s)\,ds = g(\xi) \qquad (6.21)$$

$g'(\xi) > 0$ so (6.21) is a reparametrization of the curve of intersection (6.19)

We can write ξ as a function of t

(Problem of quadrature and Inversion).

Now we show that the composite function

$t \mapsto x(\varphi^{-1}(t))$

satisfies the IVP.

$$\frac{d}{dt} x(\varphi^{-1}(t)) = \frac{dx}{d\xi}\frac{d\xi}{dt}$$

$$= \frac{dx}{d\xi}\frac{1}{\lambda(\xi)} = f(x(\xi))$$

$$= f((x \circ \varphi^{-1})(t))$$

(by (6.20')) So the ODE is satisfied

$\varphi^{-1}(0) = 0$ and $x \circ \varphi^{-1}(0) = x(0) = c$

Showing that the scheme outlined works in general.

Jacobi Multipliers and Liouville's thm.

In many cases for a system of n DEs

$\dot{x} = f(x)$

$f: \Omega \to \mathbb{R}^n$ smooth, fewer than $(n-1)$ functionally indep first integrals are available

A remarkable thm of C.G.J. Jacobi * asserts that $(n-2)$ functionally indep first integrals suffice provided a multiplier (integral invariant of order n) is available. We now turn to the discussion of this and look at the problem of spinning top moving under the influence of gravity.

Def 6.4: (Jacobi Multiplier) A smooth function

$\rho: \Omega \to \mathbb{R}$

is said to be a Jacobi multiplier for the system of ODEs $\dot{x} = f(x)$ if

$$\mathrm{Div}(\rho f) = 0 \qquad (6.22)$$

and $\rho \neq 0$ almost everywhere

(*) Vorlesungen über Dynamik, Vierzehnte Vorlesung, S. 111, Drück und Verlag von G. Reimer, Berlin 1884.

Exercise: Suppose ρ_1, ρ_2 are both Jacobi Multipliers and $\rho_2 \neq 0$ then $\frac{\rho_1}{\rho_2}$ is a first-integral.

Sol: $\nabla\rho_1 \cdot f + \rho_1 \operatorname{Div} f = 0$

$\nabla\rho_2 \cdot f + \rho_2 \operatorname{Div} f = 0$

Mult first by ρ_2, second by ρ_1, and subtract we get $\left(\frac{\rho_2 \nabla\rho_1 - \rho_1 \nabla\rho_2}{\rho_2^2}\right) \cdot f = 0$.

The Spinning top under Gravity:

Jacobi's thm asserts that if $(n-2)$ indep first-integrals are available and a (Jacobi) multiplier then an additional (indep) first integral can be determined rendering the system completely integrable.

Historically the most notable situation to which this thm was employed, is the study of a spinning top moving under the influence of gravity

We first write down the Governing Equations:

$$\begin{aligned} A\dot{p} + (C-B)qr &= Mg(y_0\gamma'' - z_0\gamma') \\ B\dot{q} + (A-C)pr &= Mg(z_0\gamma - x_0\gamma'') \\ C\dot{r} + (B-A)pq &= Mg(x_0\gamma' - y_0\gamma) \\ \dot{\gamma} &= r\gamma' - q\gamma'' \\ \dot{\gamma}' &= p\gamma'' - r\gamma \\ \dot{\gamma}'' &= q\gamma - p\gamma' \end{aligned} \qquad (6.23)$$

(p, q, r) instantaneous angular velocity of the top

$(\gamma, \gamma', \gamma'')$ Unit vector, expressed in terms of the moving coordinate system fixed to the top, of the vertical direction

(x_0, y_0, z_0) Coord. of the center of gravity of the top

See p295 of

P. Kochina: Love and Mathematics

Mir Publishers, 1985

See also the imp work of
R. Cooke, Mathematics of Sonya Kovalevskaya, Springer Verlag 1984, p 151 ff

It is easy to see that the system of ODEs admits the following three functionally indep first-integrals:

$$\gamma^2 + \gamma'^2 + \gamma''^2 (= 1) \qquad (6.24)$$

$$A p\gamma + B q\gamma' + C r\gamma'' \qquad (6.24)'$$

$$Ap^2 + Bq^2 + Cr^2 - 2Mg(x_0\gamma + y_0\gamma' + z_0\gamma'') \qquad (6.24)''$$

The last is the energy of the top.
Since there is no torque in the vertical direction, the z-Component of angular momentum is Conserved and this is precisely the Constancy of (6.24)'
The first of course is the geometrical Condition that $(\gamma, \gamma', \gamma'')$ is a vector of length 1.

Ex: Writing $f(p, q, r, \gamma, \gamma', \gamma'')$ for the RHS of (6.23),
Show that $\operatorname{Div} f = 0$
So the constant 1 is a Jacobi Multiplier
According to the theorem of Jacobi,

If one can find one More (indep) first integral then the system of ODEs would be Completely integrable
This missing first integral was referred to as The Mathematical Mermaid (!)
Die Mathematische Nixe (!!)
by the German mathematicians.
The Search for the Mathematical Mermaid turns out to be the theme of a Large and Important

Chapter in the historical development of Mechanics, DEs and Complex function theory (Abelian functions etc;)

See the Comprehensive treatment of the subject by V. V. Goluber, Lectures on the integration of the equations of motion of a Rigid body about a fixed point,
Israel program for Sci transl. Jerusalem 1960
(Transl. from Russian by J. Shorr-Kon)

Three Classical Cases when the missing first integral is available

(I) $x_0 = y_0 = z_0 = 0$ (Euler): $A^2p^2 + B^2q^2 + C^2r^2$

(II) $A = B = C$ (Poinsot)
$Apx_0 + Bqy_0 + Crz_0$ additional integral

(III) Lagrange: $A = B$; $x_0 = y_0 = 0$
Cr is an additional integral

The Search for the Mathematical Mermaid was an outstanding problem.
In 1888 S. Kowalevskya found a fourth case
$A = B = 2C$ (known today as Kowalevskya's top)
integration involves hyperelliptic / abelian functions. This work won her the prestigious Bordin prize in 1888
(Paper appeared in 1889 in Acta Mathematica Vol 12, pp 77-232)

We now turn to the proof of Liouville's thm and its generalization, Jacobi's thm and the discussion on Integral Invariants in general
(Again, a good ref is L.A. Pars: ke. Treatise on Analytical dynamics cited earlier)

For a brief while we shall return to Nonautonomous Nonautonomous systems of Chapt I.

$$\frac{dx}{dt} = v(x,t) \qquad (6.25)$$

$v : \Omega \to \mathbb{R}^n$ but this time Ω is an open set in $\mathbb{R}^{n+1}$

We know from the general principles of Chapt 1 that for each $(t_0, c) \in \Omega$ the sol. of (6.25) with IC $x(t_0) = c$ exists call it $\phi(t, c)$
(Again we shall if Convenient assume $t_0 = 0$)
and this makes sense on an interval $|t - t_0| \le h_0$

If L is a fixed Compact subset of $\mathbb{R}^n$ such that $\{t_0\} \times L \subset \Omega$
then there is a common h s.t $\phi(t,c)$ exists for all $c \in L$ and $|t - t_0| \le h$

For such t, we have a map

$$\phi_t : L \to \mathbb{R}^n \quad \text{(defined on an open set containing } L\text{)}$$
$$c \mapsto \phi(t,c)$$

The Only thing is that the flow property

$$\phi_t \circ \phi_s = \phi_{s+t}$$ DOES NOT MAKE SENSE any more but this is of no Consequence for us.

However,

the maps $\{\phi_t\}$ are all injective

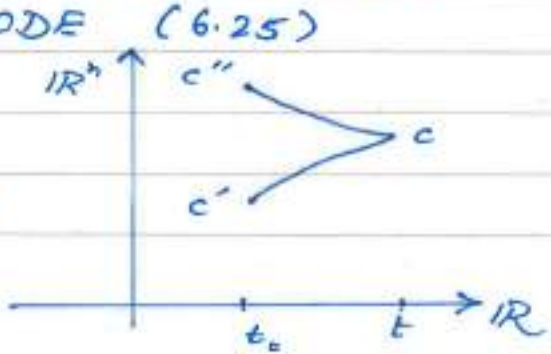
Suppose $\phi_{t_1}(c') = \phi_{t_1}(c'')$ that means

$t \mapsto \phi_t(c') = \phi(t, c')$ and $t \mapsto \phi(t, c'')$ are two solutions of the ODE (6.25) with IC c', c'' resp.

But agreeing at time $t = t_1$

This Contradicts Uniqueness of IVP

IVP $\dot{x} = v(x,t)$
$x(t_1) = c$



The maps ϕ_t are all diff. wrt c
(Smoothness wrt IC of Solutions of IVP)
Now note that for $j = 1, 2, \ldots, n$
$\frac{\partial}{\partial c_j} \phi_t(c)$ all satisfy the same linear

ODE: $\frac{d}{dt}\left(\frac{\partial \phi_t}{\partial c_j}\right) = D_x v(x,t)\left(\frac{\partial \phi_t}{\partial c_j}\right) : x = \phi_t(c)$

$\left.\frac{\partial \phi_t}{\partial c_j}\right|_{t=t_0} = \hat{e}_j$ (Variational Eqn)

So by Abel-Liouville formula

$$\det\left(D\phi_t(c)\right) = \exp \int_{t_0}^{t} \text{Trace}\left(A(s)\right) ds$$

$$A(s) = D_x v(x,s)\Big|_{x = \phi_s(c)}$$

So each ϕ_t is a local diffeo of L onto its image
Being injective, it is a global diffeo of L onto its image.

Much of this is a repeatition of the basic principles of Chapt I.
For $K \subset L$ (take K to be an open set with Cpt closure if that's pref)

$K_t = \phi_t(K)$; K is quite arb.*

Que: Under what Conditions is it true that-

$$\text{Vol}(K_t) = \text{Vol}(K) \; ?$$

The question is answered in the following thm and the answer is that

$$\text{Div } v \equiv 0$$

We shall infact prove the following more general version.

See A.J. Chorin and J.E. Marsden, Math. Intro to fluid Mech. p 11 (Webnotes, Springer 1992)

(*) Except that it is a Borel set in L

Let us introduce a smooth measure

$$\rho(x,t)\,dx_1 \dots dx_n$$

and seek ρ s.t

$$\frac{d}{dt}\left(\int_{K_t} \rho(x,t)\,dx_1 \dots dx_n\right) = 0 \qquad (6.26)$$

for all initial sets $K_{t_0} \subset\subset \Omega$

That is, using ch. of var. formula:

$$\frac{d}{dt}\int_{K_0} \rho(\phi_t(c),t)\,|D\phi_t(c)|\,dc_1 \dots dc_n = 0 \quad \forall\ K_{t_0} \subset\subset \Omega. \qquad (6.26)'$$

Performing the time differentiation of the integrand

$$\left(\frac{\partial \rho}{\partial t} + \nabla\rho(\phi_t(c),t)\cdot v\right)|D\phi_t(c)| + \rho(\phi_t(c),t)\,\frac{\partial}{\partial t}|D\phi_t(c)|$$

But by Abel-Liouville formula

$$\frac{\partial}{\partial t}|D\phi_t(c)| = (\text{Trace } D_x v(\phi_t(c),t))\,|D\phi_t(c)| = (\text{Div } v)\,|D\phi_t(c)|$$

So the integral condition (6.26)' translates to the condition:

$$\int_{K_{t_0}} (\rho_t + \nabla\rho\cdot v + \rho\,\text{Div}\,v)\,|D\phi_t(c)|\,dc_1 \dots dc_n$$

Since K_{t_0} was arbitrary,

Necessary and sufficient condition for the constancy of $\int_{K_t} \rho(x,t)\,dx_1 \dots dx_n$ is that the following

Continuity Equation must hold:

$$\rho_t + \nabla\rho\cdot v + \rho\,\text{Div}\,v = 0 \qquad (6.27)$$

If ρ is the density of a fluid moving with velocity v then (6.27) expresses the law of Conservation of mass.

Thm 6.11: With the set up above
$t \mapsto \phi(t,c)$ sol. of the non autonomous system $\dot{x} = v(x,t)$; $x(t_0) = c$
$\rho(x,t)$ is a smooth real valued function on Ω
K_0 is an arb. Borel subset of a fixed $L \subset\subset \Omega$ (L compact)
$K_t = \phi_t(K_0)$: ϕ_t is the map $c \mapsto \phi_t(c)$ (defined for $c \in L$; $|t - t_0| \leq h$)
Each ϕ_t is a diffeomorphism
$I(t) = \int_{K_t} \rho(x,t)\, dx_1 \ldots dx_n$;
is constant in time iff equation (6.27) holds.

Returning back to autonomous systems:

System of ODEs: $\dot{x} = f(x)$ (6.28)
$f: \Omega \to \mathbb{R}^n$ smooth; Ω open in $\mathbb{R}^n$
In the following $\rho: \Omega \to \mathbb{R}$ smooth (and this means ρ is independent of time)
Equation (6.27) simplifies to $\mathrm{Div}(\rho f) = 0$. (6.27)'

Thm 6.12 $\rho(x)$ is a Jacobi Mult. for (6.28) iff the measure $\rho(x)\, dx_1 \ldots dx_n$ is invariant under the flow $\{\phi_t\}$
namely for $K_0 \subset \Omega$ (Borel set)
$$\frac{d}{dt} \int_{K_t} \rho(\phi_t(c))\, dx_1 \ldots dx_n \equiv 0$$
Proof is clear from the foregoing discussion. Most imp is the case $\rho \equiv 1$ flow preserves Lebesgue Measure.

Example: The following question arises naturally.
Suppose $\mathrm{Div} f = 0$
and $\phi: \Omega \to \mathbb{R}$ is a first integral

Then the hypersurface $\phi^{-1}(c)$ is invariant-
If A_0 is a piece of this level set
Is it true that Area A_0 = Area A_t
$A_t = \phi_t(A_0)$.
Ans is No. It is not the area Meas that is preserved but the measure $\frac{ds}{|\nabla\phi|}$. We shall prove this later.

Consider
$\dot{x}_1 = x_2x_3$; $\dot{x}_2 = x_1x_3$; $\dot{x}_3 = x_1x_2$
in the first orthant-
$x_1^2 - x_2^2$ is a first integral.
Consider the level surface $x_1^2 - x_2^2 = 1$
parametrize it as: $(\cosh\theta, \sinh\theta, x_3)$ (6.29)
Subst into the ODEs. Check $\dot{\theta} = x_3$, $\dot{x}_3 = \cosh\theta\sinh\theta$
Show that the area of a piece of the surface
is $\int_{A_0} (\cosh^2\theta_0 + \sinh^2\theta_0)^{1/2}\, d\theta_0\, dx_3^0$
and $\int_{A_t} (\cosh^2\theta + \sinh^2\theta)^{1/2}\, d\theta\, dx_3$; Area A_t = $\phi_t(A_0)$
Now Area A_0 = Area A_t would imply
$(\cosh^2\theta + \sinh^2\theta)^{1/2}$ is a Jacobi Mult for the system (6.29) and this is false (!)

Exercises:

(I) Let $\phi(t,c)$ be the unique sol. of
$\dot{x} = A(t)x$; $x(0) = c$
(a) Show that for each t, the map $c \longmapsto \phi_t(c) = \phi(t,c)$ is linear and that if B is any set
$$\frac{\text{Vol}(\phi_t(B))}{\text{Vol}(B)} = \exp \int_0^t \text{Trace}(A(s))\, ds$$
(b) deduce that if $\int_0^\infty |\text{Trace } A(s)| < \infty$ then
$\phi_t(B)$ cannot shrink to zero as $t \to \infty$

If $\mathrm{Vol}(B) > 0$.

(C) Use Gronwall's lemma to prove that if $\int_0^\infty \|A(s)\| ds < \infty$ then $\lim_{t\to\infty} \phi(t,c)$ exists for each $c \in \mathbb{R}^n$ and the map $c \mapsto \lim_{t\to\infty} \phi(t,c)$ is injective

II) E. A. Jackson, Perspectives in of Nonlinear dynamics p46 Vol I, Camb. Univ. Press 2008.

$\dot{x}_1 = 5x_1 - 6x_1^2 x_2$

$\dot{x}_2 = -3x_2 + 4x_1 x_2^2$

Let $g(x) = x_1^2 x_2^4$

Show that g is a Jacobi multiplier

Since g grows with $\|x\|$, B is a ball say $\mathrm{Vol}(B_t)$ must shrink if its centre goes to infinity. Similar situation occurs in collapse of ray tube in nonlinear geometrical Optics !*

III) The Lorenz System. This is the system of ODEs

$$\dot{x} = \sigma(y-x)$$
$$\dot{y} = rx - y - xz \qquad (6.30)$$
$$\dot{z} = -bz + xy \qquad : \sigma, r, b \text{ pos. } (r \geq 0)$$

E. A. Jackson: Perspectives in NL dynamics Vol II, p138ff Contains a very detailed account of the Origin and significance of this important system that has served as an important paradigm in a lot of research on NL dynamics and Chaos. (Camb. Univ Press, 2008). See also the IMP. REF

T. Matsumoto, M. Komuro, H. Kokubu, R. Tokunaga (next page)

Compute $\frac{d}{dt}(x^2+y^2+(z-r-\sigma)^2)/2$

$$= -\sigma x^2 - y^2 - b\left(z - \frac{r}{2} - \frac{\sigma}{2}\right)^2 + \frac{1}{4}b(r+\sigma)^2$$

RHS is negative outside an ellipsoid in $\mathbb{R}^3$

(*) When the ray tube collapses the intensity σ^2 must become infinite. G K. Srinivasan and V. D. Sharma: Jour Math. Analysis App (2003) 636-637

Deduce that all solutions have $\mathbb{R}$ as the maximal interval of existence.

Calling the RHS of (6.30) as $f(x)$, note that

$$\text{Div } f = -\sigma - 1 - b < 0$$

Imitating the proof of Liouville's theorem

$$\text{Vol}(\Phi_t(B)) \to 0 \quad \text{as} \quad t \to \infty$$

for any Borel set B with finite volume.
The convergence to zero is exponentially fast

(c) When $\sigma = \frac{1}{2}$, $r = 0$, $b = 1$
The system admits a first integral $\dfrac{y^2+z^2}{(x^2-z)^2} = I$ $(x^2 - z \neq 0)$

Note: $x^2 - z = 0$ is an invariant set
so if $x(0)^2 \neq z(0)$ then $x(t)^2 \neq z(t)$ for all t

Directly checking $\dot{I} = 0$ along solutions can be quite troublesome!

Imp. Ref: T. Matsumoto et. al. Bifurcations: Sights, Sounds & Mathematics, Springer (1993. p. 302ff

$$\frac{d}{dt}(y^2+z^2) = 2y\dot{y} + 2z\dot{z} = -2(y^2+z^2)$$

So $y(t)^2 + z(t)^2 = (y(0)^2 + z(0)^2)\, e^{-2t}$

$$\frac{d}{dt}(x(t)^2 - z(t)) = 2x\dot{x} - \dot{z} = x(y-x) - (xy - z) = -x^2 + z$$

$$\therefore x(t)^2 - z(t) = (x(0)^2 - z(0))\, e^{-t}$$

We see at once that

$\dfrac{y^2+z^2}{(x^2-z)^2}$ is constant along solutions.

Check that $x^4 - 2x^2 z - y^2 = 0$ is an invariant-locus. $(x^2-z)^2 = y^2+z^2$ initially $\Rightarrow$ this eqn holds always. Consider two cases $x(0)^2 = z(0)$:

$x(0)^2 \neq z(0)$

(d) $J(t) = (x(t))^2 + (y(t))^2 + (z(t))^2 - z(t)$

Satisfies the ODE $\dot{J} + J = -(y(0)^2 + z(0)^2)\, e^{-2t}$

(See E.A. Jackson, Vol II p 145 ff)

(e) Show that if $0 < r < 1$, $b > 0$, $\sigma > 0$ then the Lorenz system has a unique rest point namely the origin

Is there an invariant ellipsoid containing the origin ?

Find $Df(0)$ and determine the eigenvalues of $Df(0)$. This will be needed in the chapter on stability.

(f) Examine whether the z-axis is invariant- Any solution starting in the z axis must approach the origin as $t \to \infty$. Prove or disprove.

The Kepler problem : We shall only discuss those aspects that are germane to this chapter namely first integrals, Jacobi mult and Jacobi's thm Subject is vast and the book of Boccaletti & Pucaccao cited earlier or Pars treatise must be consulted for in depth studies ...

See the paper of W. Kaplan (cited in Boccalletti ...)

The problem involves the mathematical study of two point masses moving under the mutual gravitational influence of each other As is well known, the problem can be reduced to a system of 4 ODEs

(See Boccaletti et.al. p171 ff) given by

$$\dot{x}_1 = p_1 \;:\; \dot{x}_2 = p_2 \qquad (6.3)$$

$$\dot{p}_1 = -\frac{x_1}{r^3} \;;\; \dot{p}_2 = -\frac{x_2}{r^3} \;:\; r = (x_1^2 + x_2^2)^{1/2}$$

The system admits the following two first integrals : There is supposed to a constant made up of masses of the point mass objects, G the universal gravitational Const. But this Const can be set $= 1$ by

rescaling the time variable.

$$p_1\dot{p}_1 + p_2\dot{p}_2 = \frac{-x_1p_1 - x_2p_2}{r^3} = -\frac{(x_1\dot{x}_1 + x_2\dot{x}_2)}{r^3}$$

$$\therefore \frac{d}{dt}(p_1^2+p_2^2) = \frac{-1}{r^3}(r^2)^{\cdot} = \frac{-2}{r^2}\dot{r} = \left(\frac{2}{r}\right)^{\cdot}$$

$\therefore$ $E = p_1^2 + p_2^2 - \frac{2}{r}$ is a first integral (6.32)
(total Energy)

And.

$$p_2\dot{x}_1 - p_1\dot{x}_2 = 0$$

$$\frac{d}{dt}(p_2x_1 - p_1x_2) - (\dot{p}_2x_1 - \dot{p}_1x_2) = 0$$

but directly $\dot{p}_2x_1 - x_2\dot{p}_1 = 0$

So we get the second integral

$$p_2x_1 - p_1x_2 \qquad (6.32)'$$

Constancy of angular momentum (note that in the reduction of the original problem to (6.31) the direction of angular momentum has already been accounted and so (6.32)' is the Constancy of the magnitude of angular momentum)

Denoting RHS of (6.31) by f we directly see

$$\mathrm{Div} f = 0$$

So there is an integral invariant (Jacobi mult)
So by Jacobi's thm there must be one more indep first integral lurking in the background. This is not easy to guess and is the inclination of the Laplace-Runge Lenz vector to the line joining the Origin and perihelion. The expression is Ugly! Appears on p317 of W. Kaplan's paper: Topology of the two body problem, American Math. Monthly vol 49 (1942) 316-323.

See p172 of Boccaletti - Puccacao.

$$\cos^{-1}\left(\frac{x_1}{r}\right) - \cos^{-1}\left[\frac{(x_1p_2 - x_2p_1)^2 - r}{r\,|x_1p_2 - x_2p_1|\sqrt{E+1}}\right] \qquad (6.32)''$$

E is (6.32)'

How do Jacobi multipliers transform when the system of ODEs is subjected to a diffeomorphism?

We address this issue since it will be needed in the proof of Jacobi's-thm.

Let us now consider a system of ODEs

$$\dot{x} = f(x) \qquad (6.33)$$

$f: \Omega \to \mathbb{R}^n$ is a smooth map: Ω open in $\mathbb{R}^n$

Subject (6.33) to a coordinate transformation

$$F: \tilde{\Omega} \to \Omega \quad : \quad x = F(y)$$

$y = (y_1, \ldots, y_n)$ coordinates on $\tilde{\Omega}$

$x = (x_1, \ldots, x_n)$ coordinates on Ω

The transformed ODE is

$$\dot{y} = g(y) \qquad (6.33)'$$

Where $g(y) = (DF(y))^{-1}(f \circ F)(y) \qquad (6.33)''$

Thm 6.13: Let $\{\phi_t\}$ be the flow generated by (6.33) and

$$\psi_t = F^{-1} \circ \phi_t \circ F$$

then $\{\psi_t\}$ is the flow generated by (6.33)'

Proof: Evidently $\{\psi_t\}$ is a group of diffeo. of Ω

Let $c' \in \tilde{\Omega}$ and $c = F(c')$

$$\frac{d}{dt}\left(F^{-1} \circ \phi_t (F(c')) \right)$$

$$= \frac{d}{dt}\left(F^{-1}\left(\phi_t(c) \right) \right)$$

$$= (DF^{-1})(\phi_t(c))\ \dot{\phi}_t(c)$$

$$= (DF)^{-1}(F^{-1}(\phi_t(c))\ f(\phi_t(c))$$

(using Inv. function thm),

$$= ((DF)^{*}(\psi_t(c'))\, f(F(\psi_t(c'))$$

$$\frac{d}{dt}\psi_t(c') = (DF(\psi_t(c')))^{-1} f(\psi_t(c')) = g(\psi_t(c'))$$

This shows that $\psi_t(c')$ satisfies

the ODE (6.33)'. At $t=0$ we see that-

$$\psi_t(c')\Big|_{t=0} = c'$$

proving the result.

Thm 6.14: Assume that under the diffeomorphism

$$F: \tilde{\Omega} \to \Omega$$

the system (6.33) transf to (6.33)'

If $\rho(x)$ is a Jacobi multiplier for (6.33)

then $(\rho \circ F)\,|DF|$ is a Jacobi multiplier for the transf. system (6.33)'.

proof. Let K_0 be a Borel set in $\tilde{\Omega}$

$$K = F(K_0) \subset \Omega$$

$$\psi_t(K_0) = K_0^t \subset \tilde{\Omega}$$

$$\phi_t(K) = K^t \subset \Omega$$

K^t and K_0^t Corresp. under F (trivial. Check)

$$\int_K \rho(c)\, dc_1 \dots dc_n = \int_{K_t} \rho(x)\, dx_1 \dots dx_n \qquad (6.34)$$

Where $x = \phi_t(c)$

Writi $c = F(c')$ and $x = F(y)$

Then $y = (F^{-1} \circ \phi_t \circ F)(c') = \psi_t(c')$ (6.35)

But (6.34) implies

$$\int_{K_0} \rho(c)\, dc_1 \dots dc_n = \int_{K_0^t} \rho(y)\, dy_1 \dots dy_n$$

$$\int_{K_0} (\rho \circ F)(c')\, |DF(c')|\, dc_1' \dots dc_n'$$

$$= \int_{K_0^t} (\rho \circ F(y))\, |DF(y)|\, dy_1 \dots dy_n$$

Thanks to (6.35) the last eqn says

$(\rho \circ F)\,|DF|$ is the Jacobi Mult for the transformed system.

Before taking up proof of Jacobi's thm we insert here one more exercise

Ex (Lorenz System. More analysis)

a) For the system (6.30) (p 271) assume $0<r<1$
Find a smooth function ϕ s.t

$$\frac{d}{dt}\phi(x(t), y(t), z(t)) < 0 \quad \text{for all } t>0$$

and try to examine the behaviour of sol. as $t\to\infty$
ϕ is a "measure" of dist from origin
Do solutions $\to 0$ as $t\to\infty$ (using ϕ can such Conclusions be drawn ?)

(b) Rescaling variables - Strained coordinates
put $r = (\sigma\epsilon^2)^{-1}$: $x = \tilde{x}/\epsilon$, $y = \tilde{y}/\sigma\epsilon^2$
$z = \tilde{z}/\sigma\epsilon^2$; $t^{\sim} = t/\epsilon$
The Lorenz system transforms to (dropping $\sim$)

$$\begin{aligned} \dot{x} &= y - \sigma\epsilon x \\ \dot{y} &= x - xz - \epsilon y \qquad (6.36) \\ \dot{z} &= xy - \epsilon b z \end{aligned}$$

See p 180 of E. A. Jackson's, Perspectives in nonlinear dynamics vol II (Camb. Univ. press 2008)
When $\epsilon=0$ the system has zero divergence
has two first integrals $\frac{1}{2}x^2 - z$; $\frac{1}{2}y^2 - z + \frac{1}{2}z^2$ (*)
Show that integration of the system can be completed using elliptic functions.
See p 187 of Jackson's book cited above for further details and references
Curve of intersection of (*) Consists of
(i) One closed curve if $k_1+1 > \sqrt{2k_2+1}$
(ii) Two closed curves if $k_1+1 < \sqrt{2k_2+1}$
linked ?
(iii) Pair of homoclinic curves if

$$k_1+1 = \sqrt{2k_2+1}$$

$k_1 = \frac{1}{2}x^2 - z$; $k_2 = \frac{1}{2}y^2 - z + \frac{1}{2}z^2$. Values assumed by first integrals ...

 (Provisional Version)

Thm 6.15 (Jacobi's last integral - theorem)

Assume given $(n-2)$ functionally indep first integrals (global) for the system of ODEs

$$\dot{x} = f(x) \qquad (6.36)$$

$f: \Omega \to \mathbb{R}^n$ smooth, Ω open in $\mathbb{R}^n$

If the system admits a Jacobi multiplier then one can obtain an additional first integral completing a family of $(n-1)$ ~~first~~ functionally indep first integrals.

This is only a provisional version ** (See p 288)

Proof: Let $\phi_1, \dots, \phi_{n-2}$ be the given family of functionally indep first integrals

Assume $\det \left(\frac{\partial \phi_i}{\partial x_j} \right)_{1 \le i,j \le n-2} \neq 0$ a.e (*)

Perform the change of variables

$$y_1 = \phi_1, \dots, y_{n-2} = \phi_{n-2}$$
$$y_{n-1} = x_{n-1}, \quad y_n = x_n$$

Check that $\left(\frac{\partial y_i}{\partial x_j} \right) = \begin{bmatrix} \left(\frac{\partial \phi_i}{\partial x_j} \right) & ? \\ 0 & I_2 \end{bmatrix}$

So that the IFT* is applicable in a nbd of any point where (*) hold. Let then $x = F(y)$ be the inv. transf.

Assume that (6.36) transforms into

$$\dot{y} = g(y) \qquad (6.36)'$$

Under this Coord transf. By the basic (Contravariant-) transf. rule $g(y) = (DF(y))^{-1} (f \circ F)(y)$

$$g_j(y) = \sum_{k=1}^{n} \frac{\partial y_j}{\partial x_k} f_k(x) \quad ; j = 1, 2, \dots, n \qquad (6.37)$$

For $j = 1, 2, \dots, n-2$ this reads

$$g_j(y) = \nabla \phi_j \cdot f = 0$$

So that the transformed system (6.36)' assumes the rather simple form:

(*) Much of this has local validity.
(**) Globalizing this will involve more topological ideas —

$$\dot{y}_1 = 0, \ldots, \quad \dot{y}_{n-2} = 0 \qquad (6.38)$$

$$\frac{dy_{n-1}}{dt} = g_{n-1} \; ; \; \frac{dy_n}{dt} = g_n \qquad (6.38)'$$

Which implies $g_n\, dy_{n-1} - g_{n-1}\, dy_n = 0$ (6.39)

Now, the transf. system (6.38) - (6.38)' has Jacobi Multiplier

$$\tilde{\rho}(y) = (\rho \circ F)\,|DF| \qquad (6.40)$$

This means

$$\mathrm{Div}(\tilde{\rho} g) = 0 \qquad (6.41)$$

But this means

$$\frac{\partial}{\partial y_{n-1}}(\tilde{\rho} g_{n-1}) + \frac{\partial}{\partial y_n}(\tilde{\rho} g_n) = 0$$

meaning $\tilde{\rho}$ is an integrating factor for the first order ODE (6.39). Through one quadrature we get one more relation

$$\psi_{n-1}(y_{n-1}, y_n) = c_{n-1} \qquad (6.42)$$

In addition to the first integrals of (6.38) - (6.38)':

$$\psi_1 = y_1, \ldots, \psi_{n-2} = y_{n-2} \qquad (6.42)'$$

The list (6.42) - (6.42)' is then a complete list of $(n-1)$ functionally indep integrals of the transformed system whereby we get a list of $(n-1)$ functionally indep integrals of the original system completing the proof.

Note that the process involves only a quadrature (and possibly an inversion)

Validity

In short the procedure outlined for the specific systems - of starting with $(n-2)$ integrals (in the presence of a Multiplier) involving quadratures and inversion, has been established in general ...

We now round up this Chapter with some discussion on Integral Invariants.

However, before doing so we would like to mention an extremely important example of a Completely integrable System arising in geometry

Right at the outset of the Course, in the Chapter on Genesis we mentioned geodesics on a surface in $\mathbb{R}^3$ - Curves of unit speed with zero acceleration (See references cited on p5)

On a Sphere such Curves are great circles.

The problem of finding geodesics on a surface in $\mathbb{R}^3$ leads to a system of 4 ODEs. These 4 ODEs assume the form

$$\dot{z}_k = -\frac{1}{2} \sum_{i,j=1}^{2} \frac{\partial g^{ij}}{\partial x_k} z_i z_j \qquad (6.43)$$

$$\dot{x}_k = \sum_{i=1}^{2} g^{ki} z_i$$

(g_{ij}) are the components of the Metric tensor and $(g^{ij}) = (g_{ij})^{-1}$.

(6.43) is a Hamiltonian System of ODE namely a System in the form (2n ODEs)

$$\dot{x}_k = \frac{\partial H}{\partial p_k} \; ; \; \dot{p}_k = -\frac{\partial H}{\partial x_k} \qquad (6.44)$$

for $H = H(x_1, \ldots, x_n; p_1, \ldots, p_n)$ Smooth.

Note that RHS of (6.44) has zero divergence.

For (6.43) $n=2$ and the Hamiltonian H is

$$H = \Sigma \frac{1}{2} g^{ij} z_i z_j$$

Also, note that H is always a first integral for (6.44)

For the geodesic equations (6.43) of 4 ODEs we always have one first integral

And $g=1$ is the Jacobi Multiplier.

One more first integral suffices to render the Geodesic Equations Completely Integrable.

This happens always for Surfaces of revolution

e.g a paraboloid $z = x^2+y^2$

A remarkable instance is that of a triaxial ellipsoid which admits another (indep) first integral due to Joachimsthal

Unfortunately we cannot get into the details here for doing so would be too long a digression we offer a couple of references:

Courant R and Hilbert D. Methods of Math phy, Vol II, pp 111-113, Wiley India 2008

D. Genin, B. Khesin and S. Tabachnikov, Geodesics on an ellipsoid in Minkowski Space, Web preprint May 1, 2007

It is natural to employ ellipsoidal coord that we mentioned in Chapt 5 (pp 194-195)

For general information see my Course notes MA847 (Topics in Geom. Autumn 2024) available in the website - particularly pp. 112-115

Integral Invariants:

When the system $\dot{x} = f(x)$ admits a Jacobi multiplier we have seen that

$$I(t) = \int_G \rho(x)\, dx_1 \ldots dx_n \qquad (6.45)$$

is constant in time. In otherwords the smooth measure $\rho(x)\, dx_1 \ldots dx_n$ is invariant under the flow $\{\phi_t\}$ generated by the ODEs. The integrand in (6.45) can be thought of either as an n-form in Ω or as a density of order n

These objects are integrated on an n-dim manifold.

Suppose $F: \Omega \to \mathbb{R}$ is a first integral for the system of ODEs, we know that

Each level surface $F^{-1}(c) = \Sigma$ is an invariant set; Σ is a smooth $(n-1)$ dim manifold if c is a regular value (almost all values are regular - Sard's Thm)

Are there $(n-1)$ forms or densities on Σ that are invariant under the restriction

$$\phi_t : \Sigma \to \Sigma \quad (?)$$

Such entities would be called integral invariants of order $(n-1)$.

We have seen that if G is a piece of Σ then Area $G \neq$ Area $(\phi_t(G))$ in general (See the example on p.270)

However, we shall prove that the density

$$\frac{dS}{|\nabla F|}$$ is invariant along Σ

dS = area element on Σ. This will be the prime result in this last part of this Chapter.

H. Poincaré systematically developed the theory of integral invariants of arbitrary order k $(1 \leq k \leq n)$ in his researches on Celestial Mechanics.

We have already cited his 1890 memoir transl in English by B. D. Popp.

See also Vol 3 of his New Methods in Celestial Mechanics Amer. Inst. Phy; Ed. Daniel. L. Goroff for a more extensive treatment.

~~The standard~~ L. A. Pars' Treatise on analytical Mech. is among the few texts to treat this topic at some length.

While integral invariants of order n are by far most important a special case of int. inv. of order 1 features in Hamiltonian Mechanics

This is the ~~Cartan's~~ one form $\sum p_k dq_k$ for the Hamiltonian system

$$\dot{p}_v = \frac{\partial H}{\partial q_v} \; ; \quad \dot{q}_v = -\frac{\partial H}{\partial p_v} \qquad (v = 1, 2, \ldots, n) \qquad (6\text{-}46)$$

This is an example of relative integral invariant (See L. A. Pars p 433)

To understand Poincaré's line of reasoning, let us turn to his 1890 memoir, specifically p 42 of B. D. Popp's translation:

Poincaré first generalizes the notion of first-integrals as follows:

Consider a function $\Psi : \Omega \times \Omega \to \mathbb{R}$ such that Ψ is constant along pairs of solutions namely

$$\Psi(y_1(t), \ldots, y_n(t); z_1(t), \ldots, z_n(t)) = \Psi(C'; C'') \; ; \quad C', C'' \in \Omega$$

$y(t)$, $z(t)$ are solutions of the ODE with respective IC C' and C''.

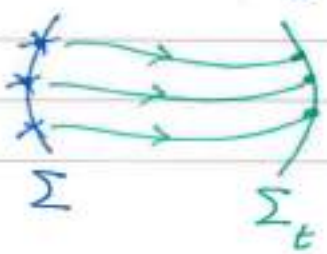
Likewise one can look at $\Psi : \Omega \times \Omega \times \Omega \to \mathbb{R}$ that are constant along triples of solution and Finally one can think of a ~~Continuous~~ function Ψ of a continuum of arguments namely

Ψ is constant along a continuum of solutions...

Now, a continuum of solutions is a family parametrized by its initial conditions (which is also a continuum)

These, initial conditions could form say a piece of a curve, surface or volume of space

Call any of these by Σ



We have a family of solution curves emanating out of each point of Σ (as IC)

$$\Sigma_t = \phi_t(\Sigma)$$

$\phi_t(\Sigma)$ is simply the aggregate (terminal) of states $\{\phi_t(c) / c \in \Sigma\}$

A function of $\phi_t(\Sigma)$ is then an integral

$$\int_{\phi_t(\Sigma)} \omega \qquad (6.47)$$

where ω is a diff one / two or three form according as Σ is a piece of a curve / surface or volume

(note that $\phi_t(\Sigma)$ is as smooth as Σ since ϕ_t is a diffeomorphism)

Instead of a diff k-form ω one could work with a density on the k-manifold $\phi_t(\Sigma)$ ($k = 1, 2$ or 3)

We say ω is a invariant k-form if the integral (6.47) is constant in time in which case we have an integral invariant of order k. Poincaré provides examples of integral invariants of orders 1 and 2

We cannot elaborate further since that requires an understanding of differential forms

The above discussion is meant to whet the readers appetite for these circle of ideas.

Summarising

(i) Poincaré generalizes the notion of first integrals leading to his idea of integral invariants

(ii) Applies it to proof of the famous recurrence thm that bears his name

This is the notion of Poisson stability.

We shall discuss stability in Chapt 7.

Exercise: We have seen that

$$\dot{x} = -y + x(1-x^2-y^2)$$
$$\dot{y} = x + y(1-x^2-y^2)$$

does not admit a first integral on $\mathbb{R}^2$

Does it admit a Jacobi multiplier g? Say on the domain $0 < x^2+y^2 < 1$??

Try looking for g that depends only on x^2+y^2.

Case of smooth invariant hypersurface (Integral inv of order $n-1$)

Thm 6.16: Consider the system $\dot{x} = f(x)$

$\operatorname{Div} f = 0$; $f: \Omega \to \mathbb{R}^n$ smooth

Ω open in $\mathbb{R}^n$

Assume that $F: \Omega \to \mathbb{R}$ is a first integral and $\nabla F \neq 0$ along the locus $\{x \in \Omega / F(x) = \lambda\}$ and generally assume $F^{-1}(a,b)$ does not contain any critical points.

Let $\lambda_0 \in (a,b)$ and A be a bounded (Borel) set on $F^{-1}(\lambda_0)$

$$I_t(A) = \int_{\phi_t(A)} \frac{dS_{\lambda_0}}{|\nabla F|} \qquad (6.48)$$

is constant in time; dS_{λ_0} = Area element along $F^{-1}(\lambda_0)$

Thus $\frac{dS_{\lambda_0}}{|\nabla F|}$ is an integral invariant of order $(n-1)$.

Proof. Since integrals are additive we may assume WLOG that A is tiny (contained in a sufficiently small ball)

WLOG let us assume $\frac{\partial F}{\partial x_n} \neq 0$ so that locally the surface $F = \lambda$

Can be parametrized as

$$(y_1, \ldots, y_{n-1}) \longmapsto (y_1, \ldots, y_{n-1}, G(y_1, \ldots, y_{n-1}, \lambda)) \quad (6.49)$$

Recall from Calculus that the area element dS_λ is given by

$$dS_\lambda = (1 + |\nabla G|^2)\, dy_1 \ldots dy_{n-1} \quad (6.50)$$

$$= \frac{|\nabla F|}{(\partial_n F)}\, dy_1 \ldots dy_{n-1}$$

$$\therefore \frac{dS_\lambda}{|\nabla F|} = \frac{dy_1 \ldots dy_{n-1}}{\partial_n F} \quad : \quad \nabla F,\ \partial_n F \text{ evaluated at } (y_1, \ldots, y_{n-1}, G(y_1, \ldots, y_{n-1}, \lambda))$$

Now perform the change of variables

$$T: (y_1, \ldots, y_{n-1}, \lambda) \longmapsto (x_1, \ldots, x_n) = (y_1, \ldots, y_{n-1}, G(y_1, \ldots, y_{n-1}, \lambda)) \quad (6.51)$$

The slice $\lambda = \text{const}$ in y-space goes over to the locus $F = \lambda$ (which has parametrization (6.49))

U is a box type nbd in y-space on which T is defined.

Note that Jacobian of T is $\frac{\partial G}{\partial \lambda}$

Now $F(y_1, \ldots, y_{n-1}, G(y_1, \ldots, y_{n-1}, \lambda)) = \lambda$

$\therefore \partial_n F \cdot \frac{\partial G}{\partial \lambda} = 1 \quad \therefore \frac{\partial G}{\partial \lambda} \neq 0$ and the change of variables (6.51) is licit

Also

$$\int_{T(U)} dx_1 \ldots dx_n = \int_U \frac{\partial(x_1, \ldots, x_n)}{\partial(y_1, \ldots, y_{n-1}, \lambda)}\, dy_1 \ldots dy_{n-1}\, d\lambda$$

$$= \int_U \frac{\partial G}{\partial \lambda}\, dy_1 \ldots dy_{n-1}\, d\lambda \quad : \quad G_\lambda = \frac{1}{\partial_n F}$$

$$= \int_U \frac{dy_1 \ldots dy_{n-1}}{\partial_n F}\, d\lambda$$

$$= \int_U \frac{dS_\lambda}{|\nabla F|}\, d\lambda = \int_{\lambda_0}^{\lambda_0 + h} d\lambda \int_A \frac{dS_\lambda}{|\nabla F|}$$

Taking the box U to be thin in the λ direction...

Now, under the Change of Var. T
The System $\dot{x} = f(x)$ goes over to

$$\dot{y}_j = g_j(y, \lambda) \; ; \quad \dot{\lambda} = g_n(y, \lambda)$$

we shall show that $g_n(y, \lambda) = 0$ so that λ is Constant

$$\int_{T(U)} dx_1 \ldots dx_n = \int_{\lambda_0}^{\lambda_0 + h} d\lambda \int_A \frac{dS_\lambda}{|\nabla F|} \qquad (6\text{-}52)$$

A is a piece on the slice $F = \lambda$
(Image of the slice λ Const under T)
Now if $U = \tilde{\phi}_t(U_0)$: $A_t = \phi_t(A_0)$
$T(U) = \phi_t(T(U_0)) = \phi_t(U)$

(Flows of the system $\dot{x} = f(x)$ and transf. system Conjugate via T)

Diff (6·52) wrt t. Use the Constancy of $\int_{\phi_t(U)} dx_1 \ldots dx_n$ in time

$$0 = \int_{\lambda_0}^{\lambda_0 + h} d\lambda \frac{d}{dt} \int_{A_t} \frac{dS_\lambda}{|\nabla F|} \qquad (\because \dot{\lambda} = 0)$$

Divide by h and let $h \to 0$ we get that-

$\int_{A_t} \frac{dS_\lambda}{|\nabla F|}$ is Constant in time.

Claim: $g_n(y, \lambda) = 0$ so that $\dot{\lambda} = 0$.

This is easy: $x_n = \varphi(y_1, \ldots, y_{n-1}, \lambda)$

$$\dot{x}_n = \sum_{j=1}^{n-1} \frac{\partial \varphi}{\partial y_j} \dot{y}_j + \frac{\partial \varphi}{\partial \lambda} \dot{\lambda}$$

But $\frac{\partial \varphi}{\partial y_j} = -\frac{\partial F}{\partial x_j} (\partial_n F)^{-1}$: $\frac{\partial \varphi}{\partial \lambda} = (\partial_n F)^{-1}$

$$\therefore (\partial_n F)\dot{x}_n = -\sum \frac{\partial F}{\partial x_j} \dot{x}_j + \dot{\lambda}$$

since $x_j = y_j$ for $j = 1, 2, \ldots, n-1$.

$\therefore \nabla F \cdot \dot{x} = \dot{\lambda}$

But F is a first int. $\therefore \dot{\lambda} = 0$.

proof is Complete.

Coda to Chapt 6: Material here is folklore but it is difficult to suggest a text that would adequately cover all these topics. The general idea connecting first integrals with Complete integrability dates back at least to Jacobi (1841) and appears in Volume 4 of his Collected works pp 149-255 Druck und Verlag 1886 Berlin. The write up is in Latin! Turn to p 188 ff

Thm 6.16 is also quite standard but a Complete and clear proof is not easy to locate in books (See L.A. Pars p446)

Note on Thm 6.15: To make this precise and water tight would require lot more topology! Let us illustrate the difficulty through a simple example

$\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1$; $\dot{y}_1 = y_2$; $\dot{y}_2 = -\omega y_1$

pair of harmonic oscillators with freq 1, ω

The system has zero divergence, admits two first integrals

$\phi_1 = x_1^2 + x_2^2$; $y_2^2 + \omega^2 y_1^2 = \phi_2$

now $\nabla\phi_1$, $\nabla\phi_2$ are lin. indep

on $\mathbb{R}^4 - (V_1 \cup V_2) = G$.

$V_1 = \{(x_1, x_2, 0, 0)\}$; $V_2 = \{(0, 0, y_1, y_2)\}$.

G is Connected but not Simply connected (Ex)

Topology of G is non trivial.

If ω is irrational,

the Solution $(x(t), y(t))$ develops on the torus T: $\phi_1 = c_1$, $\phi_2 = c_2$ and winds around T densely.

If there is a 3rd first integral ϕ_3, then $\phi_3 = c_3$ would slice

The torus T on a compact invariant set K
This cannot be ! The entire curve
$(x(t), y(t))$ would be in K and at the
same time be dense in T
$\Rightarrow$ K is dense in T and at the same time
K would have to be compact.

We cannot address these topological issues
here which is why Thm 6.15 is termed as
provisional. Tightening the nuts and bolts would
be beyond the scope of this course.

VII - Nonlinear Systems - Stability

This chapter, unlike its two preceeding ones, is fairly standard and treated substantially in many books. We shall heavily use the book of P. Glendinning that has been cited earlier on a couple of occasions: Stability, Instability and Chaos, Cambridge Univ. Press, 1994. There are themes treated in Glendinning's book such as Poincaré's linearization theorem and its ramifications due to C.L. Siegel. These themes are less commonly found in books and so we shall dwell upon them in some detail. (ODE books)

The notion of Stability is undoubtedly one of the most important one in the theory of diff. equations and dynamical systems. The notion captures the idea that if a physical system is correctly modelled then, a small change in data ought to produce only a small change in the solution. This would be a natural requirement inasmuch as scientific data is bound to contain tiny experimental errors. Solutions being too sensitive to such (observational) errors would limit the scope of the theory.

This philosophy is captured in J. Hadamard's notion of well-posedness in PDEs (See p 38 of J. Hadamard: Lectures on Cauchy's problem in linear PDEs, Yale Univ. Press 1923. See also his Comments on p 33 and p 35)

The main issue of course is clarifying, mathematically, the meaning of "Closeness of solutions" The notion of closeness of initial conditions in the context of ODEs is easily settled since all norms in $\mathbb{R}^n$ are equivalent.

The space of solutions is a function space on which there could be several distinct ideas of closeness. It should not thus be surprising that there are several distinct notions of stability in the literature

(i) Poisson Stability
(ii) Lyapunov Stability
(iii) Orbital Stability to name a few.

It is useful to mention here that a particular type of unstable behaviour occurs in many physical systems notably the Lorenz system encountered in the last Chapter. This subclass of systems are called Chaotic Systems. For a detailed account of this the reader is referred to Chapt 11 of P. Glendinning's book. Here Chaos is explained loosely (provisionally) as systems where in solutions are non-periodic and trapped in bounded regions but solutions with close enough IC separate rapidly in time (p. 290). Precise definitions are provided later in this book with illuminating examples.

As for separation of orbits recall that Gronwall's lemma gives us that separation of (nearby) orbits is atmost exponential in time:

$$|x'(t) - x''(t)| \leq |x'(0) - x''(0)| \exp Lt$$

$x'(t)$, $x''(t)$ are solutions with IC $x'(0)$, $x''(0)$ resp. In particular on Compact time intervals one can ensure that solutions remain close if IC are suff. close. Thus, Stability must take into account the behaviour of solutions (with nearby IC) over unbounded intervals of time.

Let us begin with def of Lyapunov Stability - Poisson stability

Will be taken up later

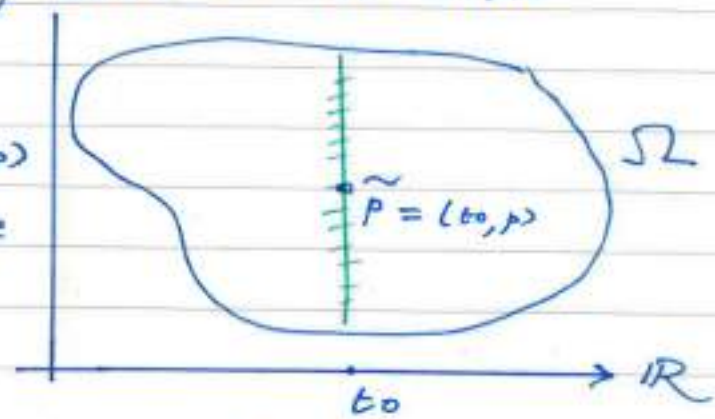
Def 7.1 (Lyaponov Stability) (cf Glendinning p 27)

Let us Consider the System of non autonomous ODEs

$$\frac{dx}{dt} = f(t,x) \qquad (7.1)$$

$f: \Omega \to \mathbb{R}^n$ smooth; Ω is open in $\mathbb{R}^{n+1}$.
Fix a time t_0 and we work with IC p s.t $(t_0, p) \in \Omega$ namely look at the slice

$\Omega_{t_0} = \Omega \cap (\{t_0\} \times \mathbb{R}^n)$



A point $\tilde{p} \in \Omega_{t_0}$: $\tilde{p} = (t_0, p)$ is said to be Lyaponov stable if given $\epsilon > 0$, $\exists\ \delta > 0$ such that

(i) for all $\tilde{q} \in B_\delta(\tilde{p}) \cap \Omega_{t_0}$: $\tilde{q} = (t_0, q)$ the solution with IC $x(t_0) = q$ denoted by $\phi(t, t_0, q)$ exists on $[t_0, \infty)$

(ii) $|\phi(t, t_0, q) - \phi(t, t_0, p)| < \epsilon \quad \forall\ t \geq t_0$.

Note: In the non autonomous case there is no point taking $t_0 = 0$ (except for notational simplicity) Most of the examples we discuss will be autonomous in that case $t_0 = 0$ and $\Omega = \mathbb{R} \times \Omega_0$; Ω_0 open in $\mathbb{R}^n$. Glendinning makes these assumptions at the outset (retaining non autonomous case but $\Omega = \mathbb{R}^{n+1}$ and $t_0 = 0$)

We also say the solution $\phi(t, t_0, p)$ is Lyaponov Stable.

Before stating the next two def. of quasi-asymptotic and asymptotic stability we make a remark on the scope of this notion

Remark: It must be kept in mind that def 7.1 although natural, is quite restrictive. The examples below also indicate why there are several definitions of stability in the literature

Ex 7.1: Consider the pendulum equation

$$\dot{x} = y \; ; \quad \dot{y} = -\frac{g}{\ell} \sin x \qquad (7.2)$$

Let us prescribe initial conditions

$$x(0) = \theta_i \; ; \; y(0) = 0 \quad \text{for } i = 1, 2 \qquad (7.2')$$

and $\phi_1(t)$, $\phi_2(t)$ be the Corresp. Solutions.

We know that the period of the pendulum depends upon the (initial) amplitude θ

Ex: Find an expression of the quarter period in terms of θ (= Max amplitude)

Use the first integral $\frac{y^2}{2} - \frac{g}{\ell} \cos\theta$

Recall the idea of quadrature and inversion from the last Chapter !

$\frac{y^2}{2} - \frac{g}{\ell} \cos x = E$: When $x = \theta$ (Max Amp), $y = 0$

So $E = -\frac{g}{\ell} \cos\theta$ and we get-

$\frac{y^2}{2} = \frac{g}{\ell} (\cos x - \cos\theta)$. Now Consider the swing of the pendulum from Mean pos. to the extreme right in time $T/4$

$$\frac{dx}{dt} = y = \sqrt{\frac{2g}{\ell}} \, (\cos x - \cos\theta)^{1/2} \; : \; 0 \le t \le T$$

gives

$$\frac{T}{4} = \sqrt{\frac{\ell}{2g}} \int_0^{\theta} (\cos x - \cos\theta)^{-1/2} \, dx \qquad (7.3)$$

Equation clearly reveals that T is dependent on

the Max ampl. θ = initial amp. (pendulum was released from extreme pos. from rest)

Let us now follow the two solutions ϕ_1, ϕ_2 with IC (7.2)'. Assume $T_2 > T_1$
After one epoch of time T_1, ϕ_1 returns back to its initial pos. but ϕ_2 lags behind (!) by say Δ
After n iterations the pendulums would differ in position by amount $n\Delta \pmod{2\pi}$
So there will be times when the two pendulams would be far apart. So $\phi_1(t)$ is not stable in the sense of Lyaponov.

Ex 7.2: Consider the Kepler problem consisting of the Sun S and a Comet C along an elliptical orbit with semi major axis a and period T.
Kepler's 3rd law gives $T^2 = (\text{Const})\, a^3$
Now if the IC are perturbed so that a changes to a' and T to T' (say $T' > T$)
then after completion of time T on the unperturbed orbit, C would return back to its starting pos. but along the perturbed orbit there would be a lag by amount Δ say. After n iterations through successive epochs the position along the perturbed orbit would lag by $n\Delta$ and there would be times when the corresp. solutions $\phi(t)$ and $\phi'(t)$ would be Far Apart.
Again, Lyaponov Stability fails in this case.

Ex: Discuss these in greater detail.
In both these examples the solutions are periodic trajectories and there is another definition that would be more relevant!

Orbital Stability : Suppose $\phi(t)$ is a periodic orbit of an autonomous system with IC p
We say $\phi(t)$ is orbitally stable if given $\epsilon > 0$ there is $\delta > 0$ s.t for $q \in B_\delta(p)$ the solution $\phi(t, q)$ (with IC q) stays within distance ϵ from the set (compact)

$$\{ \phi(t, p) / \ 0 \le t < \infty \}$$

Another formulation of this notion is in terms of the Poincaré Section (see M. Hirsch and S. Smale; Diff Eqns, dynamical sys and Lin Alg Academic Press 1974 p. 285)
We shall provide this formal def when we need this.
(see p47 of P. Glendinning)
We move on to asymptotic stability; again in the general set up

Def 7.2 (Asymptotic Lyapunov stability):
We again assume that the set up is as in Def 7.1.
(a) We say that a sol. $\phi(t, p)$ with IC $x(t_0) = p$ is quasi asymptotically stable if $\exists\ \delta > 0$ (there is no epsilon here!) such that for all $\tilde{q} \in B_\delta(\tilde{p}) \cap \Omega_{t_0}$: $\tilde{q} = (t_0, q)$

the solution $\phi(t, t_0, q)$ with IC $x(t_0) = q$ satisfies $|\phi(t, t_0, q) - \phi(t, t_0, p)| \to 0$ as $t \to \infty$
(this of course necessitates that for all such q, the sol. $\phi(t, t_0, q)$ exists on $[t_0, \infty)$)

(b) Sol $\phi(t, t_0, p)$ is Asymptotic Lyapunov Stable if it is both Lyapunov and quasi asymp. stable

Does this ~~means~~ imply the convergence

$$\lim_{t \to \infty} |\phi(t, t_0, q) - \phi(t, t_0, p)| = 0$$

is unif wrt q. ?

This point needs to be clarified. Prove or Disprove.

(or lack thereof)

The simplest case is of course stability of equilibrium points.

(i) For the pendulum $\dot{x} = y$; $\dot{y} = -\frac{g}{\ell}\sin x$ the origin $(0,0)$ is Lyapunov stable but not quasi asymptotically stable

(ii) Same is the case with Lotka Volterra System The equilibrium sol is Lyapunov stable but not quasi asymp. stable

(iii) $\dot{x} = y - x(1-x^2-y^2)$
$\dot{y} = -x - y(1-x^2-y^2)$

$$x\dot{x} + y\dot{y} = \frac{1}{2}\dot{\rho} = -\rho(1-\rho^2) \quad ; (\rho = x^2+y^2)$$

Any solution with $0 < x(0)^2 + y(0)^2 < 1$ spirals inwards, and the trajectory has a limit as $t \to \infty$ this limit being the unique equilibrium point $(0,0)$.

The origin is asymp. Lyapunov stable

(iv) We have already seen that non-constant solutions of Pendulum equation Cannot be Lyapunov stable.
Same is the case of elliptic orbits in the two body problem

An example of a quasi asymp. stable but not Lyapunov stable is given on p. 31 of Glendinning:

$$\dot{x} = x - y - x(x^2+y^2) + xy(x^2+y^2)^{-\frac{1}{2}}$$
$$\dot{y} = x + y - y(x^2+y^2) - x^2(x^2+y^2)^{-\frac{1}{2}}$$

There are two rest points $(0,0)$ and $(1,0)$
$(1,0)$ is a quasi asymp. stable but not stable. Details are slightly involved

but Completely elementary and the reader must Consult the cited reference.

Def 7.8:

Basin of attraction: Again, Consider an autonomous system $\dot{x} = f(x)$ and p be a rest point which is asymptotically Lyapunov stable. In particular $\exists\ \delta > 0$ s.t for all $q \in B(p)$ the solution $\phi(t,q)$ with IC q, tends to the equilibrium point p as $t \to +\infty$

The basin of attraction of p is the largest (maximal) nbd of p on which the above happens viz for all q in this nbd, $\phi(t,q) \to p$ as $t \to \infty$

In Ex (iii) on p 296 the basin of attraction of the rest point $(0,0)$ is the {open unit-disc $\{(x,y) / x^2+y^2<1\}$.

Determining the basin of attractim for rest pt of a physical system is usually a nontrivial task.

(v) Look at the Lorenz system (p. 273) with $0<r<1$, $b>0$, $\sigma>0$

We know that $(0,0,0)$ is the unique equilibrium point of

$$\dot{x} = \sigma(y-x);\quad \dot{y} = rx - y - xz,\quad \dot{z} = -bz + xy$$

We shall see that the origin is asymptotically Lyapunov stable (See p52 of Glendinning)

The basin of attraction in this case is whole of $\mathbb{R}^n$

Now we come to the discussion of Lyapunov functions which are generalizations of first integrals in the sense that they are in some sense, like the "energy" of the system but it can "dissipate" since

the system need not be conservative
(See p 195 of M. W. Hirsch, S. Smale and R. L. Deveney, Diff Eqn, dynamical sys. and intro to Chaos, 2nd Ed., 2004)

Without further ado we come to the definitions.
Recall that $V: \Omega \to \mathbb{R}$ is a first integral if

$$\nabla V \cdot f = 0 \quad \text{on } \Omega \qquad (7.4)$$

Now, we are interested in the behavior of solutions whose IC lie in a vicinity of an equilibrium point p so our V need be defined only in such a vicinity.
Second, dissipative terms in the DE (examples will be given below) would preclude the identity (7.4) even locally. (7.4) spells out law of Conservation of "energy" which must fail if there is dissipation
We must now weaken (7.4) to a condition

$$\nabla V \cdot f \leq 0 \qquad (7.4)'$$

in a nbd of the equilibrium point.
Then, if $x(t)$ is any solution of $\dot{x} = f(x)$ near p then

$$\frac{d}{dt} V(x(t)) = \nabla V \cdot f \Big|_{x(t)} \leq 0$$

So that V decreases along trajectories.
Thus we are led to conclude that the sublevel sets $\{V \leq c\}$ would turn out to be trapping regions and solutions within them would converge to the rest-point or atleast remain close to it.

The Much of the above heuristic discussion can be formalized and elevated to the status of theorems - Thms of Lyapunov and the LaSalle invariance principle.

Def 7-4: (Lyaponov function)
(See p. Glendinning p32)

Consider the autonomous system $\dot{x} = f(x)$ with a rest point $p \in \Omega$

Let G be an open nbd of p such that $\bar{G} \subset \Omega$ and $V: \bar{G} \to \mathbb{R}$ a smooth function such that

(i) $V(p) = 0$
(ii) $V(x) > 0$ if $x \in \bar{G} - \{p\}$
(iii) $\nabla V(x) \cdot f(x) \leq 0$ for all $x \in G$

V is called a Lyaponov function on G for the given system.

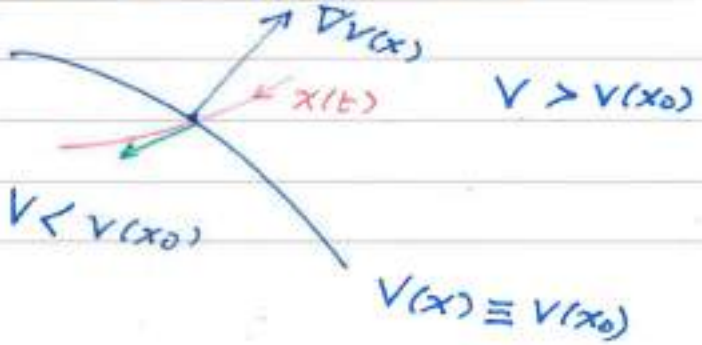
If $\nabla V(x) \cdot f(x) < 0$ for all $x \in \bar{G} - \{p\}$ we say V is a strict Lyaponov function on G (this will be needed in Lyaponov's second thm)
(See p38 of Glendinning's book)

Lyaponov functions are useful to determine (compact) trapping regions and get some hold on understanding the geometry of the basin of attraction of p.

(i)-(iii) State that p is a strict Min. of V on G and so if V is regarded as some form of energy then we are at the lowest energy level at p.
The system may be dissipative and energy need not be conserved (as indicated on p 298)
Level sets of V are not invariant since V is not nec. a first integral. How do solution curves intersect the level sets?
Look at the level set

$$V(x) \equiv V(x_0)$$

$\nabla V(x)$ is the outer pointing normal



emanating out of $V(x) \leq V(x_0)$ and pointing into $\nabla(x) > V(x_0)$

The figure above shows a solution curve $x(t)$ passing through a point x on the level set $\{x \in \mathcal{G} / V(x) = V(x_0)\}$

The tangent vector $\dot{x}(t)$ makes an obtuse angle with $\nabla V(x(t))$ is what the condition (iii) says and this translates to the statement that $x(t)$ enters into $\{V(x) < V(x_0)\}$ from $\{V(x) > V(x_0)\}$ at the point where it intersects the level set $V(x) \equiv V(x_0)$.

Now, if the sublevel sets $S = \{x \in \bar{\mathcal{G}} / V(x) \leq V(x_0)\}$ are compact then S would be forward invariant as per the above (geometric) observation and in particular if $q \in S$ the solution $t \mapsto \phi(t,q)$ exists for all $t \geq 0$

Let us now formalize these with a simple analytical proof.

Thm 7.1 (Bounding lemma):

Let $\mathcal{G}$ be a bounded open set in Ω (recall that $f: \Omega \to \mathbb{R}^n$ smooth, the ODE in question is $\dot{x} = f(x)$) and $V: \bar{\mathcal{G}} \to \mathbb{R}$ be a Lyapunov function* for the system $\dot{x} = f(x)$ (with equilibrium point $x_0 \in \mathcal{G}$ say)

Let $x_0 \in \mathcal{G}$:

Suppose $V(x) > V(x_0)$ for all $x \in \partial\mathcal{G}$

then $S = \{x \in \bar{\mathcal{G}} / V(x) \leq V(x_0)\}$

is a compact forward (i.e positively) invariant set in $\mathcal{G}$.

That is, $q \in S \Rightarrow \phi(t,q) \in S \quad \forall t \geq 0$

((*) See Note II on p301)

Proof: First observe that the hypothesis $V(x) > V(x_0)$ along ∂G shows that $S \subset G$ (and in particular $\partial S \subset G$)

Let $q \in S$, $\phi(t,q)$ sol. of IVP with IC q. Assume this sol. has $[0,T)$ as its max interval of existence.

If $\phi(t,q)$ exits S for some $t' \in [0,T)$ since $S \subset G$, we can assume that

$$\phi(t,q) \in G \text{ for } 0 \le t \le t'$$

and $\phi(t',q) \in G - S$

Thus $V(\phi(t,q))$ is defined for $0 \le t \le t'$ — so we apply the MVT

$$V(\phi(t',q)) - V(\phi(0,q)) = t' \nabla V(\phi(c,q)) \cdot f(\phi(c,q))$$

for some $c \in (0,t')$

RHS ≤ 0. Now look at the LHS:

$$V(\phi(t',q)) - V(q) \ge V(\phi(t',q)) - V(x_0) \quad (\because q \in S)$$

But $\phi(t',q) \in G - S \Rightarrow V(\phi(t',q)) > V(x_0)$

Contradiction.

Note: Without the hypothesis $V(x) > V(x_0)$ along ∂G we cannot ensure $S \subset G$ which is needed for us to select a t' with $\phi(t',q) \in G - S$.

The fact that $T = +\infty$ is now clear since S is compact, from the general principles laid down in Chapter I.

Note II: Also note that the equilibrium point p did not play any role in the bounding lemma.

It is true for any $V: \bar{G} \to \mathbb{R}$ not nec. a Lyapunov function satisfying the two conditions: For some $x_0 \in G$

$$V(x) > V(x_0) \quad \forall x \in \partial G$$

$$\nabla V \cdot f(x) \le 0 \text{ for all } x \in \bar{G}$$

Since we shall be using this mainly to prove results on stability of equilibrium points we have stated in the bounding lemma that V is a Lyaponov function.

It must be remembered that the bounding lemma has wider scope. The conditions $V>0$ and $V(p)=0$ in the def of Lyaponov function has not been used in the proof of Thm 7.1

Only $\nabla V \cdot f \leq 0$ was used.

We are now ready to prove

Thm 7.2 (Lyaponov's first Stability Thm):

Suppose that the system $\dot{x}=f(x)$ has a rest-point $p \in \Omega$ ($f:\Omega \to \mathbb{R}^n$ smooth ; Ω open in $\mathbb{R}^n$) and $V:\bar{G} \to \mathbb{R}$ is a Lyaponov function with $p \in G$

Then, p is a Lyaponov stable point namely, the constant function $x(t) \equiv p$ is stable in the sense of Lyaponov.

proof: Begin with $\epsilon>0$. Assume $B_\epsilon(p) \subset G$.

$V(p)=0$, $V(x)>0$ for $x \neq p$

$\therefore \mu = \inf\{V(x) \;/\; \|x-p\| = \epsilon\} > 0$

and this infimum is attained at a point ξ

$V(\xi) = \mu > 0 \;:\; \|\xi - p\| = \epsilon$

Now the open set

$B_\epsilon(p) \cap \{x \;/\; V(x) < \mu\}$

Contains p and Δ_0 is non empty! $\exists\, \delta>0$ s.t

$\overline{B}_\delta(p) \subset B_\epsilon(p) \cap \{x \;/\; V(x) < \mu\}$

Take $x_0 \in B_\delta(p)$ such that

$$V(x_0) = \sup \{ V(x) / x \in B_\delta(p) \}$$

so that $V(x_0) < \mu$

Now, let us apply the bounding lemma with $V : \bar{B}_\epsilon(p) \to \mathbb{R}$ taking the above x_0.

Observe that along $\partial B_\epsilon(p)$:

$$V(x) \geq \mu > V(x_0)$$

(the condition $\nabla V \cdot f \leq 0$ of course holds)

Second $V(x) \leq V(x_0)$ if $x \in B_\delta(p)$

The set $B_\delta(p) \subset S = \{ x \in \bar{B}_\epsilon(p) / V(x) \leq V(x_0) \}$

and S as we know from bounding lemma is forward invariant

Thus, $q \in B_\delta(p) \Rightarrow \phi(t, q) \in B_\epsilon(p) \quad \forall t \geq 0$

proving stability.

With additional hypothesis we get asymptotic stability given by Lyapunov's second theorem.

Before that it would be useful to recall the notion of omega limit points and omega limit sets from Chapter 5:

Lyapunov's first thm offers us stability of rest points and in particular if $q \in B_\delta(p)$ the ω-limit set of q is a nonempty compact connected set.

If we can prove that $\omega(q)$ reduces to a singleton for each $q \in B_\delta(p)$ then it would follow from the principles laid down in Chapter 5 that $\phi(t, q) \to p$ for each $q \in B_\delta(p)$ $(t \to \infty)$

P. Glendenning deduces this from a general result La Salle's principle but we shall establish this directly

Thm 7.3 (Lyaponov's Second Thm on Stability)

Suppose p is a rest point of the system $\dot{x} = f(x)$ and assume $V: \bar{\mathcal{C}} \to \mathbb{R}$ is a strict Lyaponov function for this system in the vicinity of p.

Then, p is an asymptotically stable equilibrium point. (The additional feature is that $\nabla V \cdot f < 0$ on $\mathcal{C} - \{p\}$)

Proof: We already already know that p is stable and so given $\epsilon > 0$. $\exists\, \delta > 0$ s.t.

$$q \in B_\delta(p) \Rightarrow \phi(t,q) \in B_\epsilon(p) \;:\; t \geq 0.$$

Well, $V(\phi(t,q))$ is strictly motone dec. since

$$\frac{d}{dt} V(\phi(t,q)) = \nabla V \cdot f \Big|_{\phi(t,q)} < 0$$

and being pos. $\lim_{t\to\infty} V(\phi(t,q))$ exists $= c$ say

Claim: If $r \in \omega(q)$ then $V(r) = c$

Easy: Take $t_n \to \infty$ s.t $\phi(t_n, q) \to r$

Cont. of V at once gives $V(r) = c$

Next, $\omega(q)$ is invariant (pos.) and nonempty (since it is contained in a Compact forward inv)

$$r \in \omega(q) \Rightarrow \phi(t,r) \in \omega(q) \text{ for all } t \geq 0$$
$$\Rightarrow V(\phi(t,r)) = c \quad \forall\, t \geq 0 \text{ by the claim.}$$
$$\therefore \nabla V(\phi(t,r)) \cdot f(\phi(t,r)) = 0$$

But V being strict Lyaponov function,

$$\phi(t,r) = p \text{ for all } t \geq 0$$
$$\therefore r = p$$
$$\therefore \omega(q) = \{p\}$$

Now using the general principles of Chapt 5 we conclude $\lim_{t\to\infty} \phi(t,p) = p$. (Ex: Check this)

Hint:
Well, if $t_n \to \infty$, $(\phi(t_n, p))$ cannot have two subsequences conv. to distinct limits!

Examples:

We begin with the most important case of a Conservative mechanical system where the force is derived from a potential:

$$\ddot{x} = -\nabla U(x) \quad ; \quad U : \Omega \to \mathbb{R} \text{ smooth}$$

Ω open in $\mathbb{R}^n$

Put $\dot{x} = q$ and we get the system

$$\dot{x} = q \; ; \quad \dot{q} = -\nabla U(x) \qquad (7.1)$$

$\dot{x} \cdot \nabla U + q\dot{q} = 0$ So that

$V(x,q) = U(x) + \frac{1}{2}|q|^2$ is a first integral (total energy of the system)

Assume U is bounded below and $-c = \inf U$

Working with $V + c$ in place of V we may assume $c = 0$. Further assume $U(x_0) = 0$

i.e. x_0 is a point of minimum for U

$\nabla U(x_0) = 0$ and $(x_0, 0)$ is a rest point of the system of ODEs (7.1)

$V(x_0, 0) = 0$; $V(x,q) > 0$ away from the rest point. $\nabla V(x,q) \cdot f = 0$

So V is a Lyapunov function but not a strict Lyapunov function

The rest point $(x_0, 0)$ is stable though not asymptotically stable (Discuss this for the pendulum. Identify U in this case)

Note: It is NOT necessary that x_0 be a global Min of U. The above analysis goes through even if x_0 is a local Min.

Then take V in a vicinity of $(x_0, 0)$

A large number of systems get subsumed in the above example. For example Discuss the Lotka-Volterra and Duffing's equations

Example (Glendinning, p. 36):

Consider the system

$$\left.\begin{aligned}\dot{x} &= -x + x^2 - 2xy \\ \dot{y} &= -2y - 5xy + y^2\end{aligned}\right\} \qquad (7.2)$$

Call the RHS $f(x,y)$

After a little algebra

$$x\dot{x} + y\dot{y} = -x^2(1 - x + 2y) - y^2(2 + 5x - y)$$

$$\therefore \nabla V \cdot f = -x^2(1 - x + 2y) - y^2(2 + 5x - y)$$

$$\leq 0 \quad \left(\text{with } V(x,y) = \tfrac{1}{2}(x^2 + y^2)\right)$$

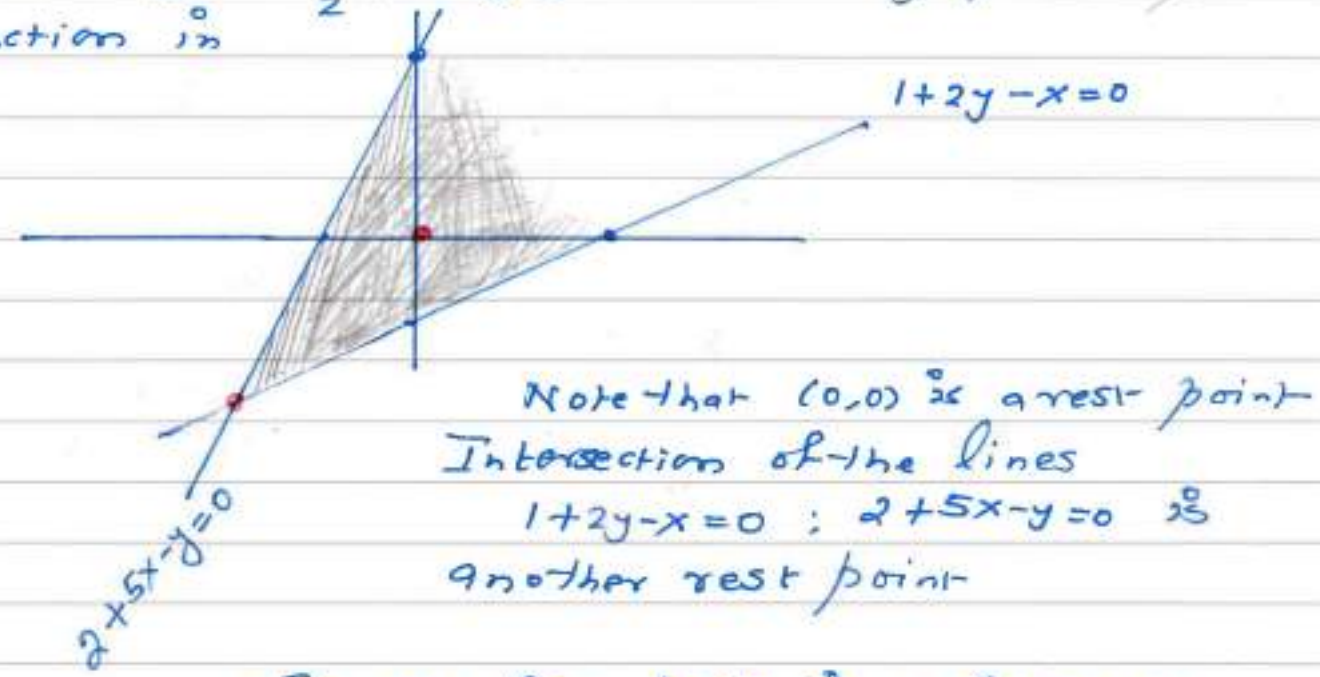
as soon as we ensure (non neg)

$$1 - x + 2y \geq 0 \text{ and}$$

$$2 + 5x - y \geq 0$$

$V(x,y) > 0$ if $(x,y) \neq (0,0)$ and $V(0,0) = 0$.

So $V(x,y) = \frac{1}{2}(x^2 + y^2)$ will be a Lyapunov function in



Note that $(0,0)$ is a rest point

Intersection of the lines

$1 + 2y - x = 0$: $2 + 5x - y = 0$ is another rest point

So one can work in a compact disc $\overline{B_\epsilon(0,0)}$ in the shaded region

$V : \overline{B_\epsilon(0,0)} \to \mathbb{R}$ is a

Strict Lyaponov function and so the origin is asymptotically stable.

Ex. Discuss the stability of (0,0) of the system of ODEs $\dot{x} = -x+y$, $\dot{y} = -x-y^3$
(Barreira. L and Valls. C: Ordinary Diff. Equations - qualitive-theory, AMS 2012 p 117)

Discuss the stability of (0,0) for the system
$\dot{x} = y - xy^2$; $\dot{y} = -x^3$
Barreira - Valls, p. 119
Try $V(x,y) = x^4 + 2y^2$

The book of L. Barreira and C. Valls contains numerous examples and on p 121 we have the following Instability-thm we now state and prove

Thm 7.4 (Instability thm):

Consider the autonomous system $\dot{y} = f(y)$ (7.3) with rest point p.
Assume that in a nbd N of p we have a C^1 function $V: N \to \mathbb{R}$ such that
(i) $V(p) = 0$ and V takes pos. values* in every nbd. of p. (Note: V is not pos. on a nbd)
(ii) $\nabla V \cdot f > 0$ on the punctured nbd $N - \{p\}$

Then p is unstable.

proof. Suppose for every $\delta > 0$, $\exists\ q \in B_\delta(p)$ such that the max. interval of existence of $\phi(t,q)$ is $[0, T_q)$ with $T_q < \infty$

Then there is nothing to show since it violates the first Cond. for stability
(*) This is different from saying $V \geq 0$ on a nbd ...

So let us assume there is a nbd W of p such that for $q \in W$, $\phi(t,q)$ exists for all $t \geq 0$

Further if the rest point is stable there would be a nbd W s.t for all $q \in B_\delta(p)$ ($\exists\ \delta > 0$ (there exists such a δ)

$\phi(t,q) \in W$ for all $t \geq t_0$

We may clearly assume $\bar{W}$ is compact.

Now,

for each $\delta > 0$, we can select a $q \in B_\delta(p)$ s.t $V(q) > 0$ and in view of hypothesis (ii)

$$\frac{d}{dt} V(\phi(t,q)) > 0 \text{ for } t > 0.$$

So $t \longmapsto V(\phi(t,q))$ is strictly increasing

$V(\phi(t,q)) \geq c > 0$ for all $t \geq t_0$

Now $V(p) = 0$

We can choose the nbd N of p so small that $\phi(t,q) \notin N$ for all $t \geq t_0$

and $\phi(t,q) \in \bar{W}$ (as above)

In otherwords

$\phi(t,q) \in \bar{W} - N$ (Compact) $\forall\ t \geq t_0$.

$M = \sup_{\bar{W}} V$ and $m = \inf \{ \nabla v \cdot f \text{ on } \bar{W} - N \}$ $(m > 0)$

$$\int_{t_0}^{t} \frac{d}{dt} V(\phi(t,q))\, dt \geq m(t - t_0)$$

$\therefore M \geq V(\phi(t,q)) \geq V(\phi(t_0,q)) + m(t - t_0)$

for all $t \geq t_0$

plainly a Contradiction

So the point p cannot be a stable rest point

To illustrate we take the following example from Barreira-Valls (p122)

$$\dot{x} = 3x + y^3 \ ; \ \dot{y} = -2y + x^2 \qquad (7.4)$$

$(0,0)$ is a rest point of the system

Consider $V(x,y) = x^2 - y^2$

$$\nabla V \cdot f = 6x^2 + 4y^2 + 2xy^3 - 2x^2y$$

Now check $\lim_{(x,y)\to(0,0)} \frac{\nabla V \cdot f}{6x^2+4y^2} = 1$

So that

$$\nabla V \cdot f \geq \frac{1}{2} \quad \text{in a nbd } B_\delta(0,0)$$

V takes positive values in every nbd of $(0,0)$ so that $(0,0)$ is unstable.

Note: In this specific example one could have used the principle of linearized stability namely the eigen values of $Df(0,0)$ are of opposite signs ...

Pendulum with Friction: Effect of dissipation

$$\ddot{x} + \frac{g}{\ell}\sin x + \alpha \dot{x} = 0 \quad ; \alpha \text{ const} \qquad (7.5)$$

$x(t)$ is the ~~amplitude~~ angular displacement of the pendulum from mean position.

As a system $\dot{x}_1 = x_2 \qquad (7.5)'$

$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1 - \alpha x_2$$

Let $V(x_1, x_2)$ be the energy $= \frac{x_2^2}{2} - \frac{g}{\ell}\cos x_1 + \frac{g}{\ell} \qquad (7.5)''$

$$\nabla V \cdot f = -\alpha x_2^2 \leq 0 \quad \text{if} \quad \alpha > 0.$$

So we see that energy dissipates as time evolves:

$$\frac{d}{dt} V(\phi(t,z)) \leq 0 \quad ; \quad \phi(t,z) \text{ is the sol. with IC } z.$$

Observe that $V(0,0) = 0$

$V \neq 0$ infact $V > 0$ in a punctured nbd of the origin

By Lyaponov's first thm, origin is a stable equilibrium.

Notice that $\nabla V \cdot f = -\alpha x_2^2 = 0$ whenever $x_2 = 0$ namely, V is not a strict Lyaponov function! So the 2nd thm of Lyaponov cannot directly be employed to conclude that the origin is asymptotically stable.

Physical Considerations does suggest that- as the pendulum loses energy with increasing t, its amplitude must diminish ultimately approaching the equilibrium sol. as $t \to +\infty$
How to derive this from mathematical analysis?

One way is to ~~critically~~ minutely reexamine the proof of Lyaponov's Second theorem on page 304.

First by stability, given $\epsilon > 0$, $\exists \delta > 0$ s.t.-
$q \in B_\delta(0) \Rightarrow \phi(t, q) \in B_\epsilon(0)$

Assume $q \neq 0$ so that $\phi(t, q)$ is non constant-

$$\frac{d}{dt} V(\phi(t, q)) = -\alpha (x_2(t))^2 \qquad (7.6)$$

$(\phi(t, q) = (x_1(t), x_2(t)))$

Now assume that at time $t = T$, $x_2(T) = 0$
(so that $\dot{x}_2(T) \neq 0$ - we have a non const sol.)

$$\frac{d^2}{dt^2} V(\phi(t, q))\Big|_{t=T} = -2\alpha x_2(T) \dot{x}_2(T) = 0 \qquad (7.6)'$$

and

$$\frac{d^3}{dt^3} V(\phi(t, q))\Big|_{t=T} = -2\alpha \dot{x}_2(T)^2 < 0$$

So we see (using Taylor expansion around $t = T$) that the function $t \mapsto V(\phi(t, q))$ is strictly decreasing even in a nbd of $t = T$.

So referring back to the proof on p304

$\lim_{t\to\infty} V(\phi(t,z))$ exists $= c$ say

Again, we show ω-limit set of z is a singleton namely the origin (See Chapt 5)

$r \in \omega(z) \Rightarrow \phi(t_n, z) \to r$ for a certain $t_n \to \infty$

$\Rightarrow V(\phi(t_n, z)) \to V(r)$

$\Rightarrow V(r) = c$

So $V(r) = c$ for all $r \in \omega(z)$

Again $\omega(z)$ being forward invariant.

$r \in \omega(z) \Rightarrow \phi(t, r) \in \omega(z) \quad \forall \ t \geq 0$

forward $\Rightarrow V(\phi(t,r)) = c$ for all $t \geq 0$

Entire traj $t \mapsto \phi(t,r)$ lies on the level set $V^{-1}(c)$

$\therefore \frac{d}{dt} V(\phi(t,r)) \equiv 0$

$\therefore (\nabla V \cdot f)(\phi(t,r)) = 0 \quad \forall \ t \geq 0 \quad (7.7)$

Write $\phi(t,r) = (y_1(t), y_2(t))$

and (7.7) implies $y_2(t) \equiv 0$ and $\dot{y}_1(t) \equiv 0$

$\Rightarrow (y_1(t), y_2(t)) \equiv (0,0)$ for all $t \geq 0$

$\therefore V(\phi(t,r)) = V(0,0) = 0$. i.e. $c = 0$

Thus

$\lim_{t\to\infty} V(\phi(t,z)) = 0$ as $t \to \infty$

$\therefore \lim_{t\to\infty} \left(\frac{(x_2(t))^2}{2} + \frac{g}{\ell}(1 - \cos x_1(t)) \right) = 0$

forcing $x_2(t) \to 0$ as $t \to \infty$

and $x_1(t) \to 0$ as $t \to \infty$

So $(0,0)$ is asymptotically stable.

Note. The key point here is that we were able to prove $t \mapsto V(\phi(t,z))$ is strictly decreasing

even across the time $t = \tau$

In this specific example none of the level sets $V^{-1}(c)$ for $c \neq 0$ contains an entire trajectory ($V^{-1}(0) =$ rest point $(0,0)$)

For a general system $\dot{x} = f(x)$ admitting a Lyapunov function in the nbd of a rest point p all we get is that $\frac{d}{dt} V(\phi(t,q))$ is ≤ 0 so it is non increasing and $\lim_{t \to \infty} V(\phi(t,q)) = c$ exists.

From this we can conclude $\omega(q) \subset V^{-1}(c)$
Next $\omega(q)$ is both forward as well as backwards invariant *

So if $r \in \omega(q)$ the entire trajectory $\phi(t,r)$ lies in $\omega(q)$ so

$$V(\phi(t,r)) = c \quad \forall\ t \in \mathbb{R}$$

Diff. wrt t we see that

$$(\nabla V \cdot f)(\phi(t,r)) \equiv 0$$

So the entire trajectory $\phi(t,r)$ lies inside the zero locus

$$E = \{ x \,/\, (\nabla V \cdot f)(x) = 0 \} \cap \Omega \qquad (7.8)$$

In particular, $\omega(q) \subset E$ and in particular the Omega limit points of $\phi(t,r)$ must lie in E.

In Conclusion,
$\omega(q)$ which is an invariant (both sided) set lies inside E given by (7.8)
So if M is the largest (two sided) invariant subset of E it follows that $\operatorname{dist}(\phi(t,q), M) \to 0$ as $t \to \infty$

that is $\phi(t,q) \to M$ as $t \to \infty$
that is the trajectory $\phi(t,q)$ must approach the set M.

(*) proof on p 315. 313

This is the content of the La Salle invariant-principle that we shall state below.

The interesting case will be when the largest invariant subset of E is the singleton $\{p\}$ (which happens for the damped pendulum) and we are able to conclude

$$\phi(t,q) \to p \quad \text{as} \quad t \to \infty$$

We now formalize

Thm 7.5 (La Salle invariance principle)

Let p be a rest point of the autonomous system $\dot{x} = f(x)$ and $V: \bar{G} \to \mathbb{R}$ be a Lyaponov function in a nbd G of p.

If $q \in G$ is such that $\{\phi(t,q) / t \geq 0\}$ is bounded and its limit points lie in G. (See p. 300. Use bounding lemma)

Then $\phi(q,t) \to M$ as $t \to +\infty$

M is the maximal (two sided) invariant subset of

$$E = \{x \in G / \nabla V \cdot f = 0\}$$

proof: See the discussion on pp 312-318.

→ namely, $\text{dist}(\phi(q,t), M) \to 0$ as $t \to \infty$

Lienard's Equation: This is an important class of second order ODEs that model electrical circuits and specifically those that exhibit limit cycle phenomenon such as the famous Van der Pol Oscillators. See the detailed account in M. W. Hirsch and S. Smale: Differential Equations, dynamical systems and linear algebra Acad. Press 1974, Chapt 10 (p 215 in particular)

(*) From p 312: $\omega(q)$ is both forward & backward invariant. Well, let $r \in \omega(q)$ so $r = \lim \phi(t_k, q): t_k \to \infty$

Let $s \in \mathbb{R}$ $\quad \phi(s,r) = \lim \phi(s, \phi(t_k, q))$

$= \lim \phi(s + t_k, q)$ and this lies in $\omega(q)$.

This is the ODE (See Glendinning p39)

$$\ddot{x} + h(x)\dot{x} + k(x) = 0 \qquad (7\cdot9)$$

Write $H(x) = \int_c^x h(\lambda)d\lambda$ (primitive of h)

So that the middle term in (7·9) is $\frac{d}{dt} H(x(t))$

So the ODE (7-9) reads

$$\frac{d}{dt}\left(\dot{x} + H(x)\right) + k(x) = 0 \qquad (7\cdot9)'$$

So it is convenient to write (7-9)' as a planar system:

$$\dot{x} = y + H(x) \qquad (7\cdot9)''$$
$$\dot{y} = -k(x)$$

The special case $k(x) = x$, $H(x) = x^3 - x$ is the Van der Pol oscillator equation. This is a basic model for relaxation oscillators.*

Again, with $V(x,y) = \frac{1}{2}(x^2+y^2)$

$$\nabla V \cdot f = x(y - H(x)) + y(-k(x))$$
$$= -x^2(1-x^2)$$

so $\nabla V \cdot f \leq 0$ if $-1 < x < 1$

So V is a Lyapnov function on the vertical strip $G_0 = \{(x,y) / -1 < x < 1\}$

Origin is stable but it is of interest to know if it is asymptotically stable

First we shall use the bounding lemma to obtain a compact forward invariant subset containing the origin in its interior.

Well, take any $x_0 \in \mathbb{R}^2$ with $\|x_0\| < 1$ say

$$\|x_0\| = 2/3.$$

$$V(x_0) = \frac{1}{3}$$

(*) An extremely important comprehensive work that's well worth pursuing is A. A. Andronov, A. A. Vitt and S. E. Khaiken, Theory of Oscillators, Pergamon Press 1966.

Take $G = \{ (x,y) \in \mathbb{R}^2 / \; x^2+y^2 \leq 2 \}$

Then $V(x,y) \geq V(x_0) = 1/3$ along ∂G

So the set

$$S = \{ (x,y) \in \mathbb{R}^2 \; / \; x^2+y^2 \leq 2/3 \}$$

is a compact forward invariant subset of G

So if $q \in S$ we can try to apply the La Salle principle.

We need to understand the largest (two sided) invariant subset of

$$E = \{ (x,y) \in \mathbb{R}^2 / \; \nabla V \cdot f = 0 \} \cap G$$
$$= \{ (0,y) / \; |y| \leq 1 \}$$

Of course $(0,0) \in E$ and is certainly (two sides inv)

Assume that a solution $\phi(t,q)$ meets E at a point $(0, y_0)$ with $y_0 \neq 0$; at time $t = T$

writing $\phi(t,q) = (x(t), y(t))$

we have $x(T) = 0, \; y(T) = y_0 \neq 0$

$$\therefore \dot{x}(T) = y(T) + H(x(T)) = y_0 + H(0) = y_0 \neq 0$$

So $x(t) \neq 0$ for $t \neq T$ in a nbd of T

So the trajectory $\phi(t,q)$ must then cross the y axis

So the Maximal (two sided) invariant subset M of E is just the origin (rest point)

So by La Salle invariance principle,

$\phi(t,q) \to 0$ as $t \to \infty$ and the origin is asymptotically stable.

Remark: In both the pendulum with dissipation and Lienard's eqn (van der pol osc) the situation was relatively simple inasmuch as we found the maximal, two sided, invariant subset of E is simply the origin. In general invariant sets can be quite complicated.

We shall discuss one example

Example (See p309 of the book: T. Matsumoto et.al; Bifurcations: Sights and Sounds, Springer 1993)

We revisit the Lorenz system:

$$\dot{x} = \sigma(y-x) \qquad (7.10)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = -bz + xy$$

Consider the function $V(x,y,z)$ given by

$$V(x,y,z) = rx^2 + \sigma y^2 + \sigma(z-2r)^2 \qquad (7.10)'$$

Call the RHS of (7.10) as f and we proceed in the usual way examining whether V is a "Lyapunov type function" (it does not satisfy the condition $V(p)=0$ at $p=(0,0,0)$ the rest point)

but we can think of applying the bounding lemma which has no ref. to rest points. Simple alg. gives:

$$(\nabla V \cdot f)(x,y,z) \qquad (7.11)$$
$$= -2\sigma\left(rx^2 + y^2 + b(z-r)^2 - br^2\right)$$

So $\nabla V \cdot f \geq 0$ precisely on the Compact set

$$C = \{(x,y,z) / \; rx^2 + y^2 + b(z-r)^2 \leq br^2\}$$

This is an ellipsoid. Center $(0,0,r)$ and $(0,0,0) \in C$

If $\alpha = \sup_C V$ in particular $\alpha \geq V(0,0,0) = 4r^2\sigma$

Then given $\epsilon > 0$, the entire level set $V = \alpha + \epsilon$ has to lie outside C.

Notice that $V = \alpha + \epsilon$ is also an ellipsoid. C'

The point $(0,0,r)$ lies inside C' so C lies entirely inside C'.

Let U = Set of points on C' and inside it

$$= \{(x,y,z) / \; V(x,y,z) \leq \alpha + \epsilon\}. \qquad (7.12)$$

$\nabla v \cdot f \geq 0$ precisely in C

$\Rightarrow \nabla v \cdot f < 0$ along $\partial U = C'$

So the any trajectory $\phi(t, q)$ hitting C' must enter inside namely into U

Thus U is a forward invariant Compact set

Claim: If A is any set. $\bigcap_{t \geq 0} \phi_t(A) = A_0$ is backwards invariant (Trivial)

Next if A is any forward invariant set then $\bigcap_{t \geq 0} \phi_t(A) = A_0$ is both forward and backward inv. (Easy).

So applying this to U we see that

$$\wedge = \bigcap_{t \geq 0} \phi_t(U) \qquad (7.13)$$

is both forward and backwards invariant

Ex: Why is $\wedge$ non empty? Well, $(0,0,r) \in U$

Is the z-axis an invariant set for the Lorenz system? Look at the Sol. with IC $(0,0,r)$

Recall that $(0,0,0) \in C \Rightarrow \alpha \geq 4r^2\sigma$

Sol. with IC $(0,0,r) \to (0,0,0)$ as $t \to \infty$...

Note: (origin is an interior point of U)

This $\wedge$ is going to be extremely Complicated

It has Zero Lebesgue Measure

Since $\operatorname{Div} f = -\sigma - 1 - b < 0$

So imitating the proof of Liouville's thm

$$\operatorname{Vol}(\phi_t(U)) = (\exp -ct)\operatorname{Vol}(U)$$

$$c = -\sigma - 1 - b.$$

So $\operatorname{Vol}(\wedge) = 0$.

Note that $q \in U \Rightarrow \omega(q) \in U$

If $r > 1$ then $Df(0)$ has a positive real eigenvalue and two neg eigen values.

So the origin is a saddle point (we shall define this) and origin is not a stable equilibrium point. According to a general thm in ODE there passes a curve γ through the origin which is an invariant curve whose points $\neq$ origin have omega limit sets contained in Λ and they are distinct from origin.

The above discussion is mainly to show that Λ does not reduce to a singleton (origin!) Infact Λ is an extremely complex object See the picture on p 303 of T. Matsumoto, M. Komuro, H. Kokubu and R. Tokunaga, Bifurcations: Sights and Sounds Springer 1993 for the values $r = 28$, $\sigma = 10$, $b = 8/3$.

Λ is called the Lorenz attractor.

If $r < 1$ then we shall show later that all solutions tend to the origin as $t \to \infty$. To see this, Let $V(x,y,z) = x^2 + \sigma y^2 + \sigma z^2$

$$\nabla V \cdot f = -2\sigma(x^2 + y^2 - (1+r)xy) - 2\sigma b z^2$$

(See M. W. Hirsch, S. Smale and R. Devane, Diff Eq^ns, dyn. sys and intro to chaos; 2nd Ed; Elsevier 2004; p 307)

Now, $x^2 + y^2 - (1+r)xy > (x^2+y^2) - 2xy \geq 0$ if $xy \neq 0$

and $x^2 + y^2 - (1+r)xy \geq x^2 + y^2$ if $xy \leq 0$

In any case $\nabla V \cdot f < 0$ if $(x,y) \neq (0,0)$ and $\nabla V \cdot f < 0$ away from the origin

V is a Strict Lyaponov function.

See Chapt 14 of Hirsch-Smale-Devaney's book cited above for more details.

Non autonomous Systems: We have been dealing primarily with autonomous systems in the second half of this course. The reader who is curious to learn about Stability of non autonomous systems could consult the (brief) account in P. Glendinning's book p 50 ff.

There is a corresponding notion of Lyapunov functions in this context as well.

$\nabla V \cdot f$ gets replaced by the operator:

$$\left(\frac{\partial}{\partial t} + f \cdot \nabla \right) V$$

Exercises on pp 52-53 of Glendinning's book are highly recommended and we shall not list those here.

Linearization and Stability via Linearization:

Def 7.5 (Sources, Sinks and Saddles):

Let p be a rest point of $\dot{x} = f(x)$; $f: \Omega \to \mathbb{R}^n$ smooth and $A = Df(p)$. We say

(i) The rest point p is a source if all eigen values of A have positive Real parts

(ii) p is called a sink if all eigen values of A have negative real parts.

(iii) p is called a saddle if it has eigen values λ, μ with $\mathrm{Re}\,\lambda > 0$ and $\mathrm{Re}\,\mu < 0$

See p 190 of M. W. Hirsch and S. Smale: Diff Eq^n's, dynamical Sys. and lin. alg; Acad Press 1974 page 190

Example: Show that the origin is a sink for the Lorenz system if $r<1$ and a saddle if $r>1$.

We now prove the basic result (Cor. of IFT)

Thm 7.6: Suppose p is a non isolated rest point of $\dot{x}=f(x)$ i.e $f(p_n)=0$ for a seq of distinct points $p_n \to p$ then $f(p)=0$ and $Df(p)$ is singular.

Follows at once from Inverse function thm.

Thm 7.7: Let p be a rest point of $\dot{x}=f(x)$ such that p is a sink that is, all eigen values of $Df(p)$ have (strictly) negative real parts then

p is asymptotically stable

One can give a proof using a Taylor expansion

$$f(x) = Df(p)(x-p) + G(x); \qquad (7.14)$$

$$\frac{G(x)}{\|x-p\|} \to 0 \text{ as } x \to p.$$

and using the Method of variation of parameters

See for example pp 312-313 of T. Matsumoto, et.al; Bifurcations: Sights, Sounds & Math, Springer 1993.

We shall however use the ideas based on Lyapunov theory.

We shall construct a strict Lyapunov function in a nbd of p; we then appeal to Thm 7.3 (Lyapunov's Second Theorem)

In what follows we shall set $p=0$ (WLOG)

We need a simple result from linear alg.

Lemma 7-8: Suppose A is a matrix with all its eigen-values having neg. real parts then, $\exists$ a real symmetric

matrix P such that P is positive definite and

$$A^TP + PA = -I \qquad (7.15)$$

Let us assume the truth of the lemma and move forward:

proof of Thm 7.7: Write the system of ODEs as

$$\dot{x} = Ax + G(x) \qquad (7.16)$$

$\frac{G(x)}{\|x\|} \to 0$ as $x \to 0$.

Now by $V(x) = x^TPx$

$$(\nabla V \cdot f)(x) = (2x^TP)(Ax + G(x))$$

$$= x^TP(Ax + G(x)) + (Ax + G(x))^TPx$$

$$(P^T = P)$$

$$= x^T(PA + A^TP)x + x^TPG(x) + G(x)^TPx$$

$$= -\|x\|^2 + \left(\frac{x^T}{\|x\|} P \frac{G(x)}{\|x\|} + \frac{G(x)^T}{\|x\|} P \frac{x}{\|x\|} \right) \|x\|^2$$

$$= -\|x\|^2 \left(1 - \frac{x^T}{\|x\|} P \frac{G(x)}{\|x\|} - \frac{G^T}{\|x\|} P \frac{x}{\|x\|} \right)$$

Note: We shall have to prove that P is positive definite. This will be done on p. 326.

But $\frac{G(x)}{\|x\|} \to 0$ as $x \to 0$ $\Rightarrow$ the stuff in the parenthesis $\geq \frac{3}{4}$ in an open ball $B_\epsilon(0)$

$\therefore (\nabla V \cdot f)(x) < 0$ on $B_\epsilon'(0)$

$V(x) > 0$ on $B_\epsilon'(0)$; $V(0) = 0$.

We see that V is a strict Lyapunov function and the proof is complete.

proof of the lemma: First assume A has distinct eigen values

Let $v_1, \ldots, v_n \in \mathbb{C}^n$ be an eigen basis.

Let Π = Set of all real symm. matrices and $L_A : \Pi \to \Pi$ be given by

$L_A(X) = A^TX + XA.$

We have to show $L_A(X) = -I$ for some X

We shall show that L_A is surjective

Suppose $X \in \ker L_A$.

$A^TX + XA = 0$ as matrices.

$\therefore A^TXv + XAv = 0$: v is an eigen vect of A with eigen value λ. $\operatorname{Re}\lambda < 0$

$\therefore A^T(Xv) = -\lambda Xv$

So $Xv \neq 0 \Rightarrow Xv$ is an eigen vector for A^T with eigen value $-\lambda$

$\Rightarrow A^T$ and hence A has an eigen value with pos. real part. Contradiction

Thus $Xv_j = 0$ for $j = 1, 2, \ldots, n \Rightarrow X = 0$.

$\therefore L_A$ is surjective

Proof in the general case is also similar.

Again we avoid Jordan Canonical form instead relying instead on the more elementary result:

If B is an $(n \times n)$ matrix, there is a basis* $v_1, \ldots, v_n \in \mathbb{C}^n$ s.t.

$$Bv_j = \lambda_j v_j + \sum_{k=1}^{j-1} c_{k,j} v_k$$

In familiar terms the matrix B is conjugate to an upper triangular matrix. Note that v_1 is an eigen-vect.

Exercise: Prove this

Returning to the proof of Lemma 7.8,

As in the previous case $Xv_1 = 0$

$$0 = (A^TX + XA)v_2 = A^T(Xv_2) + X(\lambda_2 v_2 + c v_1)$$
$$= A^T(Xv_2) + \lambda_2 Xv_2 \quad (\because Xv_1 = 0)$$

So again $Xv_2 \neq 0 \Rightarrow A^T$ has $-\lambda_2$ as eigenvalue

(*) Such a basis is called a Flag (or some people call it fan) basis.

Contradiction. So $Xv_2 = 0$

Similarly $Xv_3 = \dots = Xv_n = 0 \quad \therefore X = 0$

proof of the lemma is complete.

Hints for the exercise: Take an eigen value λ

$\operatorname{Im}(B - \lambda I)$ has dim $\leq n-1$

Select any subsp V_{n-1} s.t.

$\dim V_{n-1} = n-1$

$V_{n-1} \supseteq (B - \lambda I)$

For $x \in V_{n-1}$

$Bx = (B - \lambda I)x + \lambda I x$

Both pieces on RHS belong to V_{n-1}

$\therefore V_{n-1}$ is B-invariant.

Take an eigen value μ of $B|_{V_{n-1}}$ and repeat the process.

We then have a seq of vect. subspaces

$V = V_n \supset V_{n-1} \supset V_{n-2} \supset \dots \supset V_1$

$\dim V_j = j$: Each V_j is B-invariant.

Select

$v_1 \in V_1, \quad v_2 \in V_2 - V_1 \; ; \; v_3 \in V_3 - V_2$ etc;

and $\{v_1, \dots, v_n\}$ is the desired basis. Check

Example: If $0 < r < 1$ then the origin is asymptotically stable for the Lorenz system.

Example: Write the equation for the damped pendulum as a first order system

Find the matrix $A = Df(0)$ and find the eigen values of A.

Example: Converse of Thm 7.7 is not true

If p is an asymptotically stable eq. point $\not\Rightarrow$ p is a sink

The simplest examples illustrate this. Case $n=1$ (!)

$\dot{x} = -x^3$ on $\mathbb{R}$

Origin is asymptotically stable. Well if $x(0) = c > 0$

$\dot{x}(0) < 0$ so $x(t)$ decreases for $t = 0$

and cannot cross origin. So $0 < x(t) \leq c \quad \forall t \geq 0$

$\lim_{t \to \infty} x(t) = 0$.

Similarly if $x(0) = c < 0$ then $\dot{x}(0) > 0$ and $x(t)$ increases monotonically with t unable to cross origin and again $\lim_{t \to \infty} x(t) = 0$ ...

Hyperbolic Equilibrium points: Consider a system of ODEs $\dot{x} = f(x)$ with an equilibrium point p such that

$Df(p)$ has no eigen values on the imaginary axis.

We say p is a hyperbolic equilibrium point

A $(n \times n)$ matrix over $\mathbb{R}$ is called hyperbolic if a none of the eigen-values of it lie on the imaginary axis.

Thm 7.9: Set of all hyperbolic matrices is dense in $GL_n(\mathbb{R})$ and open in $GL_n(\mathbb{R})$

The set of all hyperbolic matrices with distinct eigenvalues is dense in $GL_n(\mathbb{R})$

(Genericity of hyperbolic matrices)

Note that the second clause follows from the first since hyperbolic matrices w/ distinct eigen values is now the intersection two open dense subsets of $M_n(\mathbb{R})$

proof is given on p. 327

We supplement Thm 7.7 with the following

Thm 7.9: Suppose p is an equilibrium point of $\dot{x} = f(x)$ and $Df(p)$ has an eigen value with positive real part then p cannot be stable

Proof:

One can approach it in two different ways:

(i) Using geometrical ideas similar in spirit to the Lyapunov theory - finding invariant regions through which solutions starting near p must escape into and there on exit a fixed ball $B_\delta(p)$

The argument which is not difficult and geometrically appealing is available in Hirsch and Smale's book (Academic press 1974) pp 187-190.

Another much shorter but analytical approach is taken in E.A. Coddington and N. Levinson Theory of ODEs, Tata McGraw Hill 1987 pp 317-318. Proof uses the Jordan Canonical form - I believe this can be avoided by employing instead the more elementary triangular form discussed on p. 328.

There is one extra step one needs to go:

Let us assume A is any $(n \times n)$ matrix (real or complex entries). We know there is a ~~flag~~ Flag: $v_1, v_2, v_3, \ldots v_n$ (basis of $\mathbb{C}^n$) such that-

$$Av_1 = \lambda_1 v_1$$
$$Av_2 = \lambda_2 v_2 + \gamma v_1$$
$$Av_3 = \lambda_3 v_3 + \gamma' v_1 + \gamma'' v_2 \quad \text{etc;}$$

In the secnd equation ~~[illegible]~~ we can replace v_2 by ~~[illegible]~~ Rv_2 $(R >> 1)$ and divide by R

In the third, replace v_3 by $R^2 v_3$ and divide by R^2

This way we can arrange it that the "off diagonal entries" $\gamma, \gamma', \gamma''$ can be made as small as we please

Ex. Prove this in detail. So we get that-

$P^{-1}AP =$ Upper triangular matrix s.t. the n eigen values of A appear along the diagonal and off diag. entries $< \epsilon$ in absolute value.

It should be noted that the idea of making the off-diagonal entries smaller than any preassigned ϵ is very useful in many situations - See F. John, Lectures on ODE, NYU lect. notes Courant Insl. (1965) pp 121-122. We now prove the following theorem of Lyaponov (See F. Gantmacher, Theory of Matrices, Vol II)

Completion of lemma 7.8: We have to show that the unique sol. P of $A^TP + PA = -I$, is positive definite.

Pages 187-188

The idea is simple. It hinges on 2 things:

(i) $\frac{d}{dt}(v^T e^{tA^T} X e^{tA} v) = v^T e^{tA^T} L_A(X) e^{tA} v$ (7.17)

(ii) The Lagrange Sylvester technique shows

$$e^{tA} = \sum_{j=1}^{l} e^{-\lambda_j t}\left(E_{j,0} + tE_{j,1} + \dots + t^{k_j-1}E_{j,k_j-1}\right)$$

$E_{j,0}, \dots E_{j,k_j-1}$ are matrices depending only on A

So $e^{tA}v \to 0$ as $t \to \infty$ and also $v^T e^{tA^T} \to 0$ as $t \to \infty$

Call $\phi(t) = v^T e^{tA^T} X e^{tA} v$

$\phi(0) = v^T X v$. Now assume $v \neq 0$.

(Lyaponov's Thm) $L_A(X)$ neg def $\Rightarrow$ X positive definite

$L_A(X)$ neg def $\Rightarrow$ RHS of (7.17) < 0

$\Rightarrow$ $\phi(t)$ strictly decreasing as t increases

So if X fails to be pos. def,

$\exists\ v \neq 0$ s.t $v^T X v \leq 0$ i.e $\phi(t) \leq 0$ for $t=0$

$\therefore\ \phi(t) < 0$ for all $t > 0$

and $\phi(t)$ strictly $\downarrow$ $\Rightarrow \lim_{t\to\infty} \phi(t) \leq 0$.

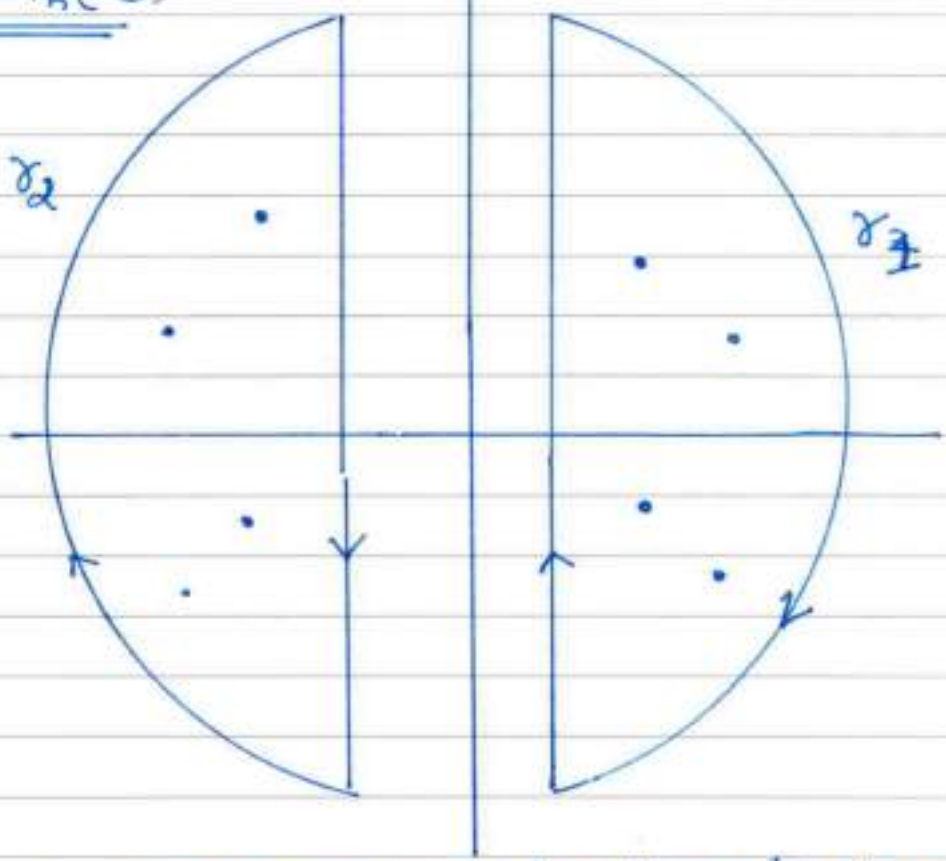
Contradiction.

Exactly same argument gives

$L_A(X)$ positive def $\Rightarrow$ X neg definite

Hence P satisfying $A^TP + PA = -I$, must be pos. def.

Proof that hyperbolic matrices are Open dense in $M_n(\mathbb{C})$



Let A_0 be a hyperbolic matrix. We have to show $\exists$ a nbd of A_0 consisting of only hyperbolic matrices. Second, given any matrix B and any neighborhood U of B, U contains a hyperbolic matrix

First, we apply Cauchy's formula.

k = # eigen values with positive real parts

l = # eigen values with neg. real parts.

$$k+l=n$$

$$k = \frac{1}{2\pi i}\oint_{\gamma_1} \frac{q'_{A_0}(z)\,dz}{q_{A_0}(z)} \quad : \quad q_{A_0}(z) = \text{Characteristic polynomial of } A_0$$

γ_1, γ_2 respectively are to enclose all the positive (resp. neg) eigen values of A_0 See the displayed figure

$$l = \frac{1}{2\pi i}\int_{\gamma_2} \frac{q'_{A_0}(z)\,dz}{q_{A_0}(z)}$$

The integrals are evidently Cont. functions of the entries of A_0 and integer valued $\Rightarrow$ k, l locally Constant. Proof is Complete

The density assertion is easy.
Let $\lambda_1, \ldots, \lambda_n$ be the eigen values of B.
$\lambda_1, \ldots, \lambda_k$ lie on Img axis.
$\delta = \inf \{ \text{dist}(\lambda_j, \text{Img axis}) \mid j = k+1, \ldots, n \}$

Look at $B + \epsilon I$: $\epsilon < \delta/3$
These have eigen values $\lambda_1 + \epsilon, \ldots, \lambda_k + \epsilon$ and other eigen values get displaced by an amount $< \delta/3$ and so they cannot be on the imaginary axis. Thus $B + \epsilon I$ hyperbolic for every ϵ s.t $0 < \epsilon < \delta/3$.
proof is compete.

Finally we state without proof the Stable Manifold theorem for a system of ODEs in the nbd of a hyperbolic rest point. (See P. Glendinning's book p 84)

Thm 7.10 (local Stable manifold-thm):

Let p be a rest point of $\dot{x} = f(x)$ such that
$Df(p)$ is hyperbolic
In a nbd U of p, there exists smooth manifolds S_p and N_p of dimensions k and $n-k$

k = dim # of eigen values of $Df(p)$ with neg real parts
l = # of eigen values of $Df(p)$ with positive real parts.
Let $\mathbb{R}^n = V_+ \oplus V_-$ where V_+, V_- are the $Df(p)$ invariant subspaces such that
$A = A_+ \oplus A_-$; $A = Df(p)$; $A_+ = A|_{V_+}$, $A_- = A|_{V_-}$
Dim $V_+ = l$, Dim $V_- = k$
A_+ is Stable
A_+ = Sum of (gen) eigen subspaces of A with - eigen values with positive (resp negative) real parts.

(i) S_p and N_p are both invariant sets
(ii) S_p and N_p are tangent at p to the linear subspaces V_- and V_+ respectively
(Place the origin at the point p)
(iii) For any IC q in N_p with $q \neq p$ the trajectory $\phi(t,q)$ lies outside any preassigned nbd U of p for $t \geq t_U$.
For any IC q in S_p, $\phi(t,q) \to p$ as $t \to \infty$
S_p is called the local stable manifold around p
N_p is called the local unstable manifold at the point p.

For proofs as well as further details Consult the book of T. Matsumoto et.al; Bifurcations: Sights, Sounds and mathematics, Springer 1993 p.320 ff.

Poisson Stability and the Poincaré recurrence thm

One of the effective use of integral invariants was towards stability issues in celestial mechanics. See pp 57-67 of B.D. Popp's transl of Poincaré's memoir on the three body problem cited earlier (Springer, 2017). It is remarkable that a result arising in fluid mechanics - essentially spelling out the law of conservation of mass, could be employed in such a context as stability issues in 3 body problems.

We now come to the requisite definitions

Def 7.6 (Poisson stability):
Let Ω be open in $\mathbb{R}^n$ and $f: \Omega \to \mathbb{R}^n$ be smooth. Consider a system of ODEs

$$\dot{x} = f(x) \qquad \text{forward}$$

A point $p \in \Omega$ is said to be Poisson stable or recurrent point if $\omega(p)$ contains

points on the trajectory i.e $\phi(t,p) \in \omega(p)$ for some $t \geq 0$. One can define backwards Poisson stable p's... Note that if $p \in \omega(p)$ then certainly this condition is satisfied. On the other hand if $\phi(t,p) \in \omega(p)$ for some $t \geq 0$ then (since $\omega(p)$ is two sided invariant under the flow)

$$\phi_{-t}(\phi(t,p)) \in \omega(p) \text{ namely } p \in \omega(p)$$

So Poisson stability can be more succinctly described as $p \in \omega(p)$.

See p 22 of the important book:
V. V. Nemytskii and V. V. Stepanov: Qualitative Theory of differential equations, Princeton Univ. Press 1960.
and very importantly Chapt 6, p 425 ff.* which discusses these ideas in a more abstract setting of dynamical systems admitting an invariant measure.
We now prove

Thm 7.11 (Poincaré recurrence thm):

Suppose the autonomous system $\dot{x} = f(x)$ admits a Jacobi multiplier $\rho: \Omega \to \mathbb{R}$
Then, in any (two sided) invariant set S with finite integral $\mu(S) = \int_S \rho(x)\,dx_1 \dots dx_n$, (7.20)

The word Measure will refer to this measure (7.20).

almost all points are poisson stable.
That is to say almost all points of an invariant set S with finite measure, are recurrent.
Note that the measure here refers to the weighted Lebesgue measure $d\mu = \rho(x)\,dx_1 \dots dx_n$.
Note: We are tacitly assuming flow is global.

*) The book does not presume knowledge of Measth and develops ab initio theory of measure including a complete proof of Fubini's thm!

Before taking up the proof let us examine the significance of this result.

$p \in \omega(p)$ precisely means for every neighborhood U of p there exists a seq $t_k \to \infty$ such that for some $k_0 \in \mathbb{N}$, $\phi(t_k, p) \in U$ for all $k \geq k_0$.

In otherwords the trajectory $\phi(t,p)$ revisits U infinitely often no matter how small U is.

This justifies the use of the term recurrent point (popular in dynamical systems books)

Proof:

We denote by $\{\phi_t\}$ the flow generated by the system of ODEs. We shall assume that the flow is global.

Let $T = \phi_1$ that is the map $\Omega \to \Omega$ given by $p \mapsto \phi(1,p)$

note that T actually is a diffeo and in particular injective.

We have $T: S \to S$ and $T^{-1}: S \to S$ Since S is two sided invariant

Step I: Let B be any bounded Borel subset of S with positive meas (interesting case will be a ball with centre $p \in S$)

R = set of all $q \in B$ such that $T^l q \in B$ for some $l \geq 1$

i.e $q \in B \cap T^l(B)$ or $q \in B \cap T^{-l}(B)$

in particular $B \cap T^{-l}B \neq \phi$.

We shall show that R has full measure in B

So each trajectory $\phi(t,q)$ starting in B must return back to B atleast once (weak condition)

$\mu(R) = \mu(B)$

or equivalently $\mu(R^c \cap B) = 0$

To prove this, we first show

$T^{-l}(R^c) \cap T^{-m}(R^c) = \phi$ if $m > l$

Take $z \in T^{-l}(R^c) \cap T^{-m}(R^c)$: R^c refers to the complement in B for ease of notation.

$T^m z \in R^c$ so $T^{(m-l)}(T^l(z)) \in R^c$

but this is immediately a contradiction since $T^l z \in R^c$ which means $T^l z \in B$ and $T^\lambda(T^l z) \notin B$ for any $\lambda \geq 1$ but here we can take $\lambda = m - l$ and get $T^\lambda(T^l z) \in R^c \subseteq B$.

The sets $\{T^{-l}(R^c)\}_{l=0}^{\infty}$ all have the same measure and are pairwise disjoint. All of them contained in S of finite Meas. T is Measure preserving $\mu(T^{-l}R^c) = \mu(R^c)$ $\forall l$.

$\therefore \sum_{l=0}^{\infty} \mu(T^{-l}(R^c)) \leq \mu(S) < \infty$

All terms of the series are equal (!)

The infinite series would $= \infty$ if $\mu(R^c) > 0$

This shows $\mu(R^c) = 0$

Thus the solution $\phi(t, z)$ starting at $z \in B$ must return to B at least once in future for almost all z.

Step II: Now write $A_0 = R^c$

$A_l = \{x \in B \,/\, T^l x \in B$ and $T^m x \notin B$ for $m > l\}$

In other words if $x \in B$, $\phi(t, x)$ visits B for only finitely many times $t \in \mathbb{N}$ and l is the last entry.

Trivially $A_0, A_1, A_2, \ldots$ are pairwise disjoint.

A trivial verification (Straight from def)

Confirm $T^{-l}(A_0) = A_l$

$\therefore \mu(A_0) = \mu(A_l) \quad \forall\, l \geq 0$

Since T and hence T^{-l} is a measure preserving diffeomorphism.

So again, $\mu(A_l) = 0 \quad \forall\, l$

Because $\mu(\cup A_l) = \sum \mu(A_l)$ and LHS is finite. $\cup A_l \subseteq S$ (finite Measure)

So $\overset{\infty}{\underset{l=0}{\cup}} A_l$ has zero Measure

$B_0 = B - \overset{\infty}{\underset{l=0}{\cup}} A_l$ has full measure in B $(\mu(B_0) = \mu(B))$

Step III:

Interesting case would be to take S to be an open invariant set of finite Measure

B a ball contained in S.

For $q \in B_0$, the trajectory $\phi(t, B_0)$ must revisit B infinitely often through $t_j \in \mathbb{N}$ (a subseq)

We can take a countable collection B_j of open balls s.t each $x \in S$ is contained in infinitely many of these B_j with arb. small radii; $S = \cup B_j$.

$S_0 = \cup (B_j)_0$ has full Measure

$\mu(S) = \mu(S_0)$

If $q \in S_0$, let $\epsilon > 0$ be arbitrary

$B_\epsilon(q) \supset (B_j)_0$ and $q \in (B_j)_0$

Now $\phi(t, q)$ revisits $(B_j)_0$ infinitely often through times $t_j \in \mathbb{N}$

So $q \in \omega(q)$

That is to say, all points of S_0 are Poisson Stable or recurrent

Ex: A rest point or a periodic traj is obviously Poisson stable.

The Individual Ergodic Thm:

The Poincaré recurrence thm is a qualitative result. It gives no indication as to how long does one have to wait for the trajectory $\phi(t,q)$ $(q \in S_0)$ to revisit a given nbd of q_0. The individual Ergodic thm of Birkhoff-Khinchine quantifies this

We have assumed there is an invariant set S of finite volume. We say S is ~~indecomposable~~ irreducible if

$$S = A \cup B \quad ; \quad A \cap B = \phi$$

A, B both invariant $\Rightarrow \mu(A) = 0$ or $\mu(B) = 0$

(We are keeping this discussion informal and non-technical*) See Nemytskii-Stepanov p. 467.

This means (loosely) there are no preferred subsets A (with $\mu(A) > 0$ and $\mu(A^c) > 0$) into which the trajectories enter and stay there ... Or equivalently all regions in S are equally "preferred" by the system.

The average time of return to a given subset U of S is inversely proportional to the measure of the set: $\frac{1}{\mu(U)}$. So, for such irreducible systems

The individual ergodic thm asserts that the asymptotic average values of an observable (to borrow a terminology from physics. Or a measurable function $\psi : \Omega \to \mathbb{R}$ with $\int \psi \, d\mu < \infty$)

$$\lim_{T \to \infty} \frac{1}{T} \int^T \psi(\phi(t,q)) \, dt$$

along a trajectory exists and $= \int \psi \, d\mu$ almost everywhere

(*) The technical term is Ergodicity: The only invariant subsets have zero measure or full Meas. (complement has zero Meas)

The LHS gives the (long) time averages of the physical quantity χ along trajectories while RHS is the Space averages.

The Ergodic thm asserts the equality of Space averages and time averages.
Proof is found in many books for maps
$T: X \to X$ (X is a Measure Sp with a T-inv. measure μ and T is Ergodic)
However, less common is proofs for the Continuous Case for flows.
See for example pp 459 - 469 for a Complete proof and ramifications (for example pairs of Observables. See p 483 ff).

We shall have to close this discussion here and wrap up this chapter with a brief description of Poincaré Linearization which is also closely related to stability problems.

Problem of linearization and local Conjugacy.

It is useful to recall how a System of ODEs transf. under a change of Coordinates.
Here we consider an isolated rest point $p \in \Omega$

$$A = Df(p)$$

Where $f: \Omega \to \mathbb{R}^n$ is smooth and the System under Consideration is $\dot{x} = f(x)$: $f(p) = 0$ (7.21)
WLOG we may assume that $p = 0$
Issue is whether we can transform the System to a linear System $\dot{y} = By$ via a Coord transf.
$x = \chi(y)$. We shall see that the χ (if it exists) must satisfy a certain partial differential equation which is "Singular" at the origin.
We shall assume of course $\chi(0) = 0$.

Let us begin by assuming that f is real analytic namely f is the sum of its Taylor series in a nbd of origin:

$$f(x) = Ax + \dots$$ dots refer to quadratic and higher terms

Likewise $$\psi(y) = By + \dots \qquad (7.22)$$

putting $x = \psi(y)$ the ODE transforms to

$$(D\psi(y))\dot{y} = f(\psi(y))$$
$$= A\psi(y) + \dots$$
$$= A(By + \dots) + \dots$$
$$\therefore \psi * (B + Cy + \text{higher ord})\dot{y} = ABy + \dots$$
$$\therefore (I + B^{-1}Cy + \dots)\dot{y} = B^{-1}ABy + \dots$$

So the transformed ODE now reads:

$$\dot{y} = B^{-1}ABy + \dots \qquad (7.23)$$

We see that the two systems have linear parts that are conjugate matrices.

The problem is to choose ψ such that (7.23) reduces to $\dot{y} = (B^{-1}AB)y$ linear (7.23)'

But then we can perform a further coord transf. and arrange it so that (7.23)' reduces to

$$\dot{y} = Ay$$

This is then the Poincaré problem

Before looking into this problem and the obstructions to its solutions, it would be of interest to know what (if any) is the relevance of this problem to stability of the system $\dot{x} = f(x)$.

This is answered by the following

Thm 7.12 (Carton-Caratheodory)

A necessary and sufficient condition for the origin to be both forward and backwards

Stable rest point of $\dot{x} = f(x)$ (7.24) is that

(i) $A = Df(0)$ is diagonalizable and all its eigenvalues are purely imaginary

(ii) There is an analytic coordinate transf $x = \psi(y)$ fixing the origin

that transforms the given ODE (7.24) to

$$\dot{y} = Ay \qquad (7.24)'$$

See for example p 22 of the important ref.

J. Moser: Stable and random motions in dynamical systems, princeton Univ. press 1973.

Moser has given a derivation that the inverse transformation $y = \psi^{-1}(x) = \eta(x)$ must satisfy the following PDE

$$\sum_{j=1}^{n} f_j \frac{\partial \eta}{\partial x_j} = A\eta \qquad (7.25)$$

$f = [f_1, \ldots, f_n]^T$; η is of course a col. vector.

Moser has given a remarkable deduction of (7.25) via an intermediate step

$$\eta(p) = \lim_{T \to \infty} \frac{1}{T} \int_0^T e^{-tA} \phi(t,p)\, dt \qquad (7.26)$$

$\phi(t,p)$ is the unique sol. of (7.24) with IC p.

We cannot derive these here but the important point to note here is that η is supposed to be an analytic solution of a PDE (7.25) which is singular at the origin | Recall: $f_j(0) = 0$ $j = 1, 2, \ldots, n$.

There is no guarentee that the PDE (7.25) admits any analytic solutions !!

Consider for example the ODE

$$xy' = -3y$$

which has no analytic sol. except the trivial one in any nbd of the origin

The non existence of the analytic transf η or ψ can also be understood more easily and directly without having to derive (7.26)
See for example the important monograph by Y.N. Bibikov: Local theory of nonlinear analytic ODEs, Springer LNM. 702, Springer 1979
See p 7. The Direct map ψ (as opposed to η) satisfies the Singular PDE

$$(Dh)Ay - Ah = f(y+h) \qquad (7.25)'$$

Where $\psi(y) = y + h(y)$

The origin is a singular point for the nonlinear system (7.25)'.
However (7.25)' can be handled by seeking h as a power series and working stages in equating the homogeneous components of each degree on both sides
Details are worked out in p 78 ff of P. Glendinning's book.
Solvability of (7.25)' even at the level of formal (as opposed to convergent) power series leads to certain Obstructions called Resonance Conditions

Note that h involves quadratic and higher terms only. So we need to compare homogeneous components of deg ≥ 2 on both sides of (7.25)'. This explains the $\sum_{k=1}^{n} m_k \geq 2$ below. Note: Excludes the trivial relation $\lambda_s = \sum m_j \lambda_j$ with $m_j = 0$ if $j \neq s$, $m_s = 1$

Def 7.7 (Resonance Condition):
Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A = Df(0)$. The eigenvalues are in resonance (or satisfy the resonance cond) if for some s; $1 \leq s \leq n$,

$$\lambda_s = \sum_{j=1}^{n} m_j \lambda_j \quad ; \quad m_j \geq 0,\ m_j \in \mathbb{Z} \text{ and}$$

This excludes trivialities: $\lambda_s = \lambda_s$

$$\sum_{j=1}^{n} m_j \geq 2$$

If the resonance Cond. holds, we say A is resonant

Thm 7.13:

If A is non resonant - there is a formal Power series $h(y)$ satisfying (7.25)

The Convergence of the formal series is a delicate matter involving some Diophantine Conditions due to Siegel.

See p 82 of P. Glendinning's book.

We close this (already lengthy) Chapter here.

VIII Nonlinear Systems - periodic traj and limit cycles.

In this last (and hopefully short) Chapter we shall discuss a few results on periodic trajectories and limit cycles. Often we see that a non periodic traj has as its omega / α limit set a periodic solution such as the example

$$\dot{x} = -y + x(1-x^2-y^2) \qquad (8.1)$$
$$\dot{y} = x + y(1-x^2-y^2)$$

that we have encountered before more than once The unique periodic solution

$$x(t) = \cos t, \quad y(t) = \sin t \qquad (8.2)$$

is an example of a limit cycle. It is the omega limit set of every other solution except the equilibrium sol. Solutions spiral towards this (Except the const one) periodic traj. See the picture on p. 248. This example is illustrative but would appear to be artificially manufactured. One would like to see how to prove the existence of limit cycles and locate them in examples arising from physics. The classic example is the Van der Pol oscillator that arises in the study of electrical circuits. The popular technique is topological namely, appealing to the Poincaré-Bendixon theorem or fixed points of the Poincaré section.

Here we shall discuss this via Implicit function thm in the spirit of following the treatment in Kirkwood C. Chicone (ODEs, Springer 2006, p 368ff). We first begin with some one dimensional example.

Recall the example on p 17 (non autonomous)

$$\frac{dy}{dt} = f(y,t) \qquad (8\cdot3)$$

$f : [p,q] \times \mathbb{R} \to \mathbb{R}$ smooth,

$f(y, T+t) = f(y,t)$; $t \in \mathbb{R}$, $y \in [p,q]$ $(8\cdot3)'$

$f(p,t) > 0$ and $f(q,t) < 0$ $(8\cdot3)''$

See Hirsch-Smale-Devenay's book p.19 problem 16.

We used the fixed point thm to prove that

$\exists$ $y_0 \in [p,q]$ so that the solution of (8.3) with IC $y(0) = y_0$ which is periodic with period T.

Let us now look at a more delicate example again from Hirsch-Smale-Devenay, (p.19)

$$\dot{x} = x^2 - 1 - \cos t \qquad (8\cdot4)$$

It is natural to take $p = -\sqrt{2}$, $q = 0$

Instead of $(8\cdot3)''$ we have the weaker inequalities $f(-\sqrt{2}, t) = 2 - 1 - \cos t \geq 0$

$f(0,t) = -1 - \cos t \leq 0$

The above general result is not directly applicable

If you look at the discussion on p 17 the idea is to show that the map

$c \longmapsto y(T, c)$ where $y(t,c)$ is the unique sol. of the ODE with IC $y(0) = c$ sends $[p,q]$ to itself and if we take the fixed point of this as IC we get a periodic solution.

Here the issue is to obtain a conclusion that $c \longmapsto y(T,c)$ ($T = 2\pi$ here) $(8\cdot5)$ does indeed map $[-\sqrt{2}, 0]$ into itself so that the fixed pt idea can be applied.

In otherwords we need to show that solutions with IC in $[-\sqrt{2}, 0]$ cannot exit this interval for $0 \leq t \leq 2\pi$

Argument is elementary and so we shall keep it short (Compare with the discussion on p.310)

Let us Consider a solution $x(t)$ of the ODE (8.4) with IC in $[-\sqrt{2}, 0]$ and assume this exits the interval $[-\sqrt{2}, 0]$ at time τ and we shall arrive at a Contradiction.

Can it exit from left endpoint $-\sqrt{2}$?

Let $x(\tau) = -\sqrt{2}$ so that $\dot{x}(\tau) = 2 - 1 - \cos\tau \geq 0$ and so if $\tau \neq 0, 2\pi$, $x(t)$ will have to move back to the right of $-\sqrt{2}$ and stay within $[-\sqrt{2}, 0]$

This leaves the cases $\tau = 0, 2\pi$. Both are similar

$x(\tau) = -\sqrt{2}$ for $\tau = 0$

$\Rightarrow \dot{x}(\tau) = 0$, $\ddot{x}(\tau) = 2x(\tau)\dot{x}(\tau) + \sin t \Big|_{t=\tau} = 0$

$\dddot{x}(\tau) = \cos\tau > 0$ so again $x(t)$ is forced to move back into $[-\sqrt{2}, 0]$.

Can it exit from the right endpoint 0 ?

Situation is similar. Suppose $x(\tau) = 0$ for some $\tau \in [0, 2\pi]$ $\dot{x}(\tau) = x(\tau)^2 - 1 - \cos\tau = -1 - \cos\tau < 0$ so $x(t)$ gets pushed back into $[-\sqrt{2}, 0]$ unless $\tau = \pi$. This case needs Special Consideration: $\dot{x}(\tau) = 0$

$\ddot{x}(\tau) = 2x(\tau)\dot{x}(\tau) + \sin\tau = 0$

$\dddot{x}(\tau) = \cos\pi < 0$. So again we see that $x(t)$ Cannot exit $[-\sqrt{2}, 0]$ from right endpoint.

The map $[-\sqrt{2}, 0] \longrightarrow [-\sqrt{2}, 0]$

$x(0) \mapsto x(2\pi)$

has a fixed point c and for this choice of IC $x(0) = c$, we have $x(0) = x(2\pi)$. Check that the Sol is periodic (See p. 17 of these notes)

Example From C. Chicone: ODEs, Springer

Here the interval $[\alpha, \beta]$ such that the poincare map $x(0) \mapsto x(T)$ has both domain and codomain $[\alpha, \beta]$

is easily found. For $a, b > 0$ Consider the ODE $\dot{y} = (a\cos t + b)y - y^3$ (8.6)

We have the Const. solution $y(t) \equiv 0$

So Origin is certainly a barrier for any Solution $y(t)$ with IC $y(0) \geq 0$.

Claim: $y(0) \in (0, \sqrt{a+b}]$

$\Rightarrow y(t) \in (0, \sqrt{a+b}]$ for all $t \geq 0 : t \leq 2\pi$

and in particular the Poincaré map

$y(0) \mapsto y(2\pi)$ maps $[0, \sqrt{a+b}]$ into itself

To prove the claim we have to rule out exit from the right endpoint $\sqrt{a+b}$.

$$\dot{y}(0) = y(0)(a+b) - y(0)^3$$
$$= y(0)\left((a+b) - (y(0))^2\right) > 0$$

So $y(t)$ increases strictly at time $t=0$.

Assume T is the first instant >0 when it exits namely $y(T) = \sqrt{a+b}$; $y(t) < \sqrt{a+b}$ if $t < T$. Then $\dot{y}(T) \geq 0$.

But $\dot{y}(T) = \sqrt{a+b}\,(a\cos T - a) \leq 0$

So $\dot{y}(T) = 0$ and this means $T = 2\pi$

(Case $T=0$ has to be taken up). Then

$$\ddot{y}(T) = 0, \quad \dddot{y}(T) = -a\sqrt{a+b} < 0$$

Taylor exp^n shows $y(t)$ decreases at $t = T$ so that $y(t)$ moves back into $(0, \sqrt{a+b}]$ for $t > T$.

Finally if $y(0) = \sqrt{a+b}$. Again $\dot{y}(0) = \ddot{y}(0) = 0$ and $\dddot{y}(0) = -a\sqrt{a+b} < 0$ so $y(t)$ is pushed back into $(0, \sqrt{a+b}]$ for $t > 0$.

So the Poincaré map $y(0) \mapsto y(2\pi)$

$[0, \sqrt{a+b}] \to [0, \sqrt{a+b}]$ has a fixed point c. If $c > 0$ then surely the Sol. with IC $y(0) = c$ is a periodic sol.

However, if $c = 0$ then we do not necessarily get the trivial Solution (!) So, are there (non trivial) periodic Solutions in this case ?

(akin to)

This is a "one dimensional Floquet theory" except that the ODE is nonlinear!

Let us compute the derivative of the Poincaré map $P : [0, \sqrt{a+b}] \to [0, \sqrt{a+b}]$

$$c \mapsto y(2\pi, c)$$

$y(t,c)$ is the unique sol. of the ODE with IC c

Let us compute $P'(c)$ at $c = 0$

well, use the variational equation (Chapt I)

Diff the ODE wrt c:

$$\frac{d}{dt}\left(\frac{\partial y}{\partial c}\right) = \frac{\partial y}{\partial c}\left(a\cos t + b - (y(t,c))^2\right)$$

$$+ y(t,c)\left(-2y(t,c)\cdot \frac{\partial y}{\partial c}\right)$$

put $\forall t$ $c = 0$

$$\frac{d}{dt}\left(\frac{\partial y}{\partial c}\Big|_{c=0}\right) = \frac{\partial y}{\partial c}\Big|_{c=0} (a\cos t + b) \quad (8.7)$$

Since $y(t,0) \equiv 0$.

Solving the linear ODE (8.7) we get

taking into the IC $\frac{\partial y}{\partial c}\Big|_{t=0} = \frac{\partial}{\partial c}\left(y|_{t=0}\right)$

$$= \frac{\partial}{\partial c}\, c = 1$$

$$\frac{\partial y}{\partial c}\Big|_{c=0} = \exp(a\sin t + bt)$$

$$\therefore \quad \frac{\partial y}{\partial c}\Big|_{c=0,\ t=2\pi} = \exp 2\pi b > 1$$

So we have found

$$P'(0) > 1 \qquad (8.8)$$

So, even if the fixed point of the Poincaré map happens to be origin (the unlucky situation!) we shall use (8.8) to show that there must be at least ONE MORE fixed point of P and we do get the existence of non-trivial periodic solutions.

We are in the following situation

Thm 8.1: Let P be a smooth map

$$P : [A, B] \to [A, B]$$

with a fixed point c.

Suppose $P'(c) > 1$ there is atleast one more fixed point c_0 in $[A, B]$: $c_0 \neq c$.

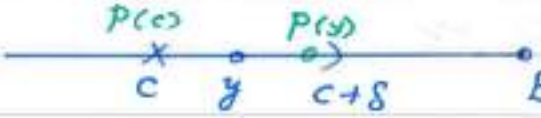
proof. $\exists\ \delta > 0$ s.t

$$P'(x) > 1 \quad \text{for} \quad c - \delta < x < c + \delta.$$

$P(y) - P(z) = (y - z) P'(x) > (y - z)$ if $c + \delta > y > z > c - \delta$

In particular with $z = c$

$P(y) - c > (y - c)$ if $y > c$: $y \in (c - \delta, c + \delta)$



$(P(c) = c)$

Suppose $P(x)$ P pushes points $y \in (c, c + \delta)$ farther away from c. That is c is a repelling point.

So in particular $P(y) > y$ for any $y \in (c, c + \delta)$

$P(B) \leq B$

So B is a fixed point or else $P(x) - x$ changes sign in $[c, B]$ and we have a fixed point in (c, B).

Similarly work on the interval $y \in (c - \delta, c)$ and we would conclude either $P(A) = A$ or else there is a fixed point in (A, c)

In any event there is a fixed point of P other than c.

proof is Complete

Cor 8.2: The ODE $\dot{y} = (a \cos t + b) y - y^3$ has a nontrivial periodic solution with period 2π with $|c|$ in $(0, \sqrt{a + b}\,]$.

We now discuss stability of periodic solutions obtained via a Poincaré map for scalar nonautonomous ODE $\dot{y} = f(t, y)$.

Thm 8.3: Consider the scalar first order ODE

$$\dot{y} = f(t,y) \quad : y(0) = c \qquad (8.9)$$

$$f(t+T, y) = f(t,y) \text{ for all } (t,y) \in \mathbb{R}^2$$

$f : \mathbb{R}^2 \to \mathbb{R}$ smooth $\quad (T > 0)$

$y(t,c)$ is the unique sol. of the ODE with IC $y(0) = c$

Let P be the poincare map

$$c \mapsto y(T, c) \qquad (8.10)$$

Fixed points of P corresp. to periodic solutions

(Assume (8.10) is defined on some interval I and $P : I \to I$)

If $|P'(p)| < 1$ then the periodic solution $y(t,p)$ is asymptotically stable.

If $|P'(p)| > 1$ the periodic orbit is unstable

proof: We first dispose off the last part. As in Thm 8.1, $\exists$ a nbd N of p such that

$$x \in N \Rightarrow |P(x) - p| > |p - x|(1+h) \qquad (8.11)$$

for some $h > 0$. Take h so small that $|P'(\xi)| > 1+h$ for all $\xi \in N$.

This means

$$|y(T,p) - y(T,x)| > |p-x|(1+h)$$

If $P(x) \in N$ we can apply (8.11) again!

$$|P^2(x) - p| \geq |p-x|(1+h)^2$$

and further iteration gives, if $P^j(x) \in N$ for $j = 0, 1, 2, \ldots, k-1$

$$|P^k(x) - p| \geq |p-x|(1+h)^k \quad ; (p \neq x)$$

Ultimately $P^k(x)$ must exit N for some $k \in \mathbb{N}$. But $P^k(x) = y(kT, x)$ (Exercise. Prove this)

and $P^k(p) = p = y(kT, p)$

We see that solutions starting at x, p with $x \in N - \{p\}$ must eventually diverge ...

Let us dispose off the little exercise on p346

$$P^k(x) = y(kT, x)$$

For $k=2$ this reads $y(T, y(T,x)) = y(2T, x)$

Note that since the ODE is non autonomous the group property $y(s, y(t,p)) = y(s+t, p)$ fails.

But when $s = T$

$y(t+T, p)$: $y(t, q)$ both satisfy the ODE

So if we take $q = y(T, p)$ both agree at $t=0$

So by uniqueness

$$y(t+T, p) = y(t, y(T, p)) \qquad (8\cdot12)$$

This general version is quite useful.

More gen. for $k \in \mathbb{Z}$

$$y(t+kT, p) = y(t, y(kT, p)) \qquad (8\cdot12)'$$

Idea is to select k such that

$$\tau = t + kT \in [0, T)$$

So that $(8\cdot12)'$ reads:

$$y(\tau, p) = y(t, y(kT, p)) = y(t, P^k(p))$$

This is true for any p

In particular replacing p by x and taking as a choice for p the fixed point.

So now,

Write $t = (t - kT) + kT = s + kT$; $0 \le s < T$

$$y(t, x) = y(s, y(kT, x))$$
$$= y(s, P^k x) \text{ . Likewise}$$
$$y(t, p) = y(s, P^k p) = y(s, p)$$

$$\therefore |y(t,x) - y(t,p)| = |y(s, P^k(x)) - y(s, p)|$$
$$\le |P^k(x) - p| \exp LT$$

(using Gronwall lemma.)

Now if $|P'(p)| < 1$ then in place of $(8\cdot11)$ we have the opp. situation:

$$|P(x) - p| < |x - p|(1-h) \text{ and } \} \ (8\cdot11)'$$

more gen. $|P^k(x) - p| \le |x-p|(1-h)^k$

So we can choose $k \in \mathbb{N}$ so large that $|p-x|(1-h)^k$ is as small as we please and then

$$|y(t,x) - y(t,p)| \leq |p-x|(1-h)^k \exp LT \to 0$$

as $t \to \infty$

Since the requisite $k \in \mathbb{N}$ s.t. $t = s + kT$ with $0 \leq s < T$, also $\to \infty$. Proof is complete.

Example: Let us now examine whether the non trivial periodic solution of the ODE

$$\dot{y} = (a\cos t + b)y - y^3$$

which was shown to exist, is asymptotically stable.

For the solution $y(t,c)$, $\frac{\partial y}{\partial c}$ satisfies

$$\frac{d}{dt}\left(\frac{\partial y}{\partial c}\right) = \frac{\partial y}{\partial c}\left\{(a\cos t + b) - 3(y(t,c))^2\right\} \qquad (8.13)$$

with initial cond'n. Solving this linear ODE

$$\frac{\partial y}{\partial c} = \exp\left(a\sin t + bt - 3\int_0^t (y(s,c))^2 ds\right)$$

So, $P'(c) = \left.\frac{\partial y}{\partial c}\right|_{t=2\pi}$ namely

$$= \exp\left(2\pi b - 3\int_0^{2\pi} (y(s,c))^2 ds\right) \qquad (8.14)$$

We are interested in that c as the IC that gives us non trivial periodic sol.

Divide (8.13) by $\frac{\partial y}{\partial c}$ and integrate (explain why is it legal to divide? Can it vanish?) from 0 to 2π. Periodicity would make LHS vanish

$$0 = 2\pi b - 3\int_0^{2\pi} (y(s,c))^2 ds \qquad (8.14)'$$

Subst. into (8.14) now gives

$$P'(c) = \exp(-4\pi b) < 1$$

So the non trivial periodic sol. is asymp. stable

Finally, an exercise:

Exercise: Show that there is only one non trivial periodic sol. of the ODE

$$\dot{y} = (a\cos t + b)y - y^3.$$

Well, if there are two then the Poincaré map has 3 fixed points and so $P'(c)$ must equal 1 at two distinct values $c = c_1$, $c = c_2$.

$$\Rightarrow \int_0^{2\pi} \left((y(s, c_1))^2 - (y(s, c_2))^2 \right) ds = 0$$

Integrand doesn't vanish identically (Why?) so must change sign

$\therefore \exists\ s_0$ such that $y(s_0, c_1)^2 = (y(s_0, c_2))^2$

But $\rho_c = (y(s, c))^2$ satisfies the ODE

$$\dot{\rho} = 2(a\cos t + b)\rho - \rho^2$$

so ρ_{c_1}, ρ_{c_2} are two sol. that agree at s_0

$$\Rightarrow \rho_{c_1}(s) \equiv \rho_{c_2}(s).$$

Fill in the details and complete the sol.

We now pass on to planar systems. We shall look at only one example — the Van der Pol Oscillator.

Locating the limit cycle of the Van der Pol Oscillator

From C. Chicone's book cited earlier (p368)

The system of ODEs is

$$\dot{x} = y \qquad (8.15)$$
$$\dot{y} = x - \epsilon(x^2-1)y$$

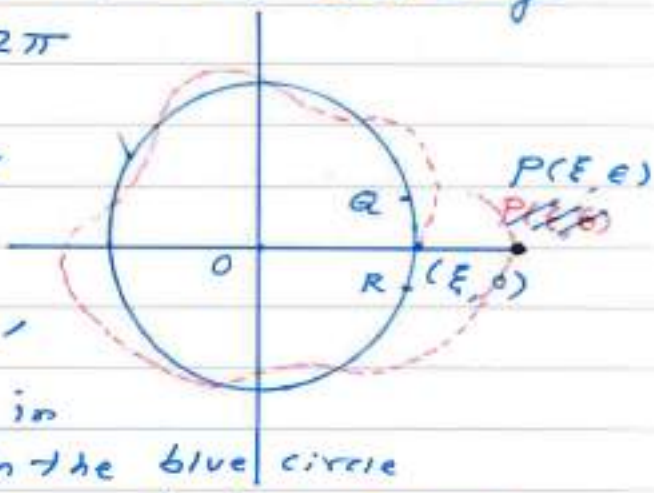
prescribe IC: $x(0) = \xi$, $\dot{x}(0) = y(0) = 0$ (8.15)'

Equation (8.15) reduces to the harmonic oscillator when $\epsilon = 0$ and so we regard (8.15) as a perturbation of the harmonic oscillator; we have for the latter the periodic solution

$$t \longmapsto (\xi \cos t, \xi \sin t) \qquad (8.16)$$

For small ϵ the solution to (8.15) - (8.15)' would stay close to that of (8.16) over say

$$[0, T] \; ; \; T > 2\pi$$



Blue circle is (8.16) periodic. for unperturbed eqn

The dotted red curve is the sol. of (8.15) with same IC (8.15)'

Q, R are points close to $(\xi, 0)$ in the upper/lower half planes, on the blue circle

The red curve must remain close to blue curve and so must get close to the points Q and R in a tiny ball $B_\delta(\xi, 0)$. This means the dotted red curve Must cross the x-axis at some

point $P(\xi, \epsilon)$ (8.17)

at time $T(\xi, \epsilon)$ (8.17)'

The map $(\xi, \epsilon) \longmapsto P(\xi, \epsilon)$ (8.18)

is called parametrized Poincaré map.

Since the ODE is autonomous

A periodic solution of (8.15) would correspond to fixed point of (8.18)

namely, an IC $\xi = \xi_0$ such that

$$P(\xi_0, \epsilon) = \xi_0.$$

Corresp. to this fixed point ξ_0, with IC $x(0) = \xi_0$, $y(0) = 0$, the periodic sol. has period $T(\xi_0, \epsilon)$

Define the displacement function

$$\delta(\xi, \epsilon) = P(\xi, \epsilon) - \xi \qquad (8.19)$$
$$= x(T(\xi, \epsilon), \epsilon) - \xi$$

Since $T(\xi, \epsilon)$ is (by def) the time at which the trajectory crosses the x-axis, the condition

$$y(\xi, \epsilon) = 0 \qquad (8.19)'$$

is guarenteed to hold. So we are looking for zeros of (8.19). When $\epsilon = 0$, $\delta(\xi, \epsilon) \equiv 0$ and there is degeneracy. $\delta(\xi, 0) \equiv 0$.

It is natural to solve for the equation

$$\delta(\xi, \epsilon) = 0 \qquad (8.20)$$

as a function of ϵ appealing (of course) to the Implicit function theorem.

$\delta(\xi, 0) \equiv 0 \Rightarrow \frac{\partial}{\partial \xi}\delta(\xi, 0) = 0$ and the hypothesis of IFT as such fails. To circumvent this difficulty let us perform a Taylor expansion around $\epsilon = 0$:

$$\delta(\xi, \epsilon) = \underbrace{\delta(\xi, 0)}_{=0} + \epsilon \delta_\epsilon(\xi, 0) + \epsilon^2 R(\xi, \epsilon)$$

R is a smooth function

$$= \epsilon\left(\delta_\epsilon(\xi, 0) + \epsilon R(\xi, \epsilon)\right)$$

So for $\epsilon \neq 0$,

$\delta(\xi, \epsilon) = 0$ iff $\Delta(\xi, \epsilon) := \delta_\epsilon(\xi, 0) + \epsilon R(\xi, \epsilon)$ vanishes.

So we must instead solve the equation

$$\Delta(\xi, \epsilon) = 0 \qquad (8.21)$$

instead of (8.20)

For this we need the Seed Solution

$\Delta(\xi, 0) = 0$ and at this Seed Sol

$\Delta_\xi(\xi_0, 0) \neq 0$. Hypothesis needed for IFT.

-Well, $\Delta(\xi, 0) = 0$ translates to the condition

$$\delta_\epsilon(\xi, 0) = 0 \quad \text{at some } \xi_0 \tag{8.22}$$

$$\delta_{\epsilon\xi}(\xi, 0) \neq 0 \quad \text{for this } \xi_0.$$

The IFT then gives us for each ϵ near 0 and $\epsilon \neq 0$, a unique ξ close to ξ_0 for which $\Delta(\xi_0, \epsilon)$ and consequently $\delta(\xi, \epsilon) = 0$ i.e a fixed point of (8.18) a periodic sol. of the Vander Pol. Equation.

Computing $\delta_\epsilon(\xi, 0)$: and then its root ξ_0.

Start with the def (8.19):

$$\delta_\epsilon(\xi, 0) = \dot{x}(T(\xi, \epsilon), \epsilon)\, T_\epsilon(\xi, \epsilon)\Big|_{\epsilon=0} + x_\epsilon(T(\xi, \epsilon), \epsilon)\Big|_{\epsilon=0}$$

$$= \dot{x}(T(\xi, 0), 0)\, T_\epsilon(\xi, 0) + x_\epsilon(T(\xi, 0), 0)$$

$$= -y(T(\xi, 0), 0)\, T_\epsilon(\xi, 0) + x_\epsilon(T(\xi, 0), 0) \tag{8.23}$$

But $y(T(\xi, 0), 0) = 0$ by def of $T(\xi, 0)$ (8.23) and we need to Compute $x_\epsilon(T(\xi, 0), 0)$ and Obviously need to use the variational equation.

Note that the ODE is equivalent to the 2nd order:

$$\ddot{x} + x + \epsilon \dot{x}(x^2 - 1) = 0. \tag{8.24}$$

So the variational eqn obtained by diff (8.24) wrt and Setting $\epsilon = 0$:

$$\frac{d^2}{dt^2}\left(\frac{\partial x}{\partial \epsilon}\Big|_{\epsilon=0}\right) + \left(\frac{\partial x}{\partial \epsilon}\right)_{\epsilon=0} + \dot{x}(x^2-1)\Big|_{\epsilon=0} = 0 \tag{8.25}$$

original

With Zero Initial Conditions (IC do not dep. on ϵ)

But we know that for $\epsilon = 0$, $x = \xi \cos t$

$$\therefore \dot{x}(x^2-1)\Big|_{\epsilon=0} = -\xi \sin t\,(1 - \xi^2 \cos^2 t)$$

$$= -\frac{1}{4}(\sin t)(4\xi - \xi^3) + \frac{\xi^3}{4}\sin 3t$$

So (8.25) is:

$$\left(\frac{\partial x}{\partial \epsilon}\Big|_{\epsilon=0}\right)'' + \left(\frac{\partial x}{\partial \epsilon}\right)_{\epsilon=0} = \frac{1}{4}\sin t\,(4\xi - \xi^3) - \frac{\xi^3}{4}\sin 3t \quad (8.25)'$$

Solving this elementary ODE (with zero ic) is a routine undergrad exercise!

$$x_\epsilon(t) = \frac{1}{8}(\sin t - t\cos t)(4\xi - \xi^3) + \frac{\xi^3}{32}\sin 3t$$

and $T(\xi, 0) = 2\pi$ (obviously)

$$\therefore x_\epsilon(T(\xi,0), \xi) = -\frac{\pi}{4}\xi(4 - \xi^2)$$

We see that $\xi_0 = 2$ is a simple zero of the ~~displacement~~ function $\delta_\epsilon(\xi, 0)$ (i.e the reduced displacement function $\Delta(\xi, 0)$)

We have our seed solution $\xi_0 = 2$. for (8.22)

We now have to check:

$$\delta_{\epsilon\xi}(\xi_0, 0) \neq 0 \text{ for } \xi_0 = 2 \quad (8.22)'$$

But this must trivially hold since $\xi_0 = 2$ is a ~~this~~ simple zero of the equation $\delta_\epsilon(\xi, 0) = 0$ as we have just seen. The validity of IFT has now been checked

Thm 8.4: For small positive ϵ the ~~S~~ Vander Pol Oscillator equations (8.15) admits a periodic solution.

Ex: Show that the origin is an unstable equilibrium point by computing the eigen values of $Df(0,0)$ (f = RHS of (8.15))

In fact $x\dot{x} + y\dot{y} = \epsilon y^2(1 - x^2)$

Use ~~Lyap~~ $f(x,y) = x^2 + y^2$ to show that the periodic sol. is orbitally stable and in fact is a limit cycle. It is a periodic traj which is the ω-limit set of other distinct neighboring periodic solutions.

Van der Pol Oscillator again: Topological Methods

The analytic method described above is known as Poincaré Continuation. The linearization or variational eqn was a linear system with periodic matrix of the form $\dot{x} = A(t)x$ (we worked with the scalar 2nd order ODE instead) directly

This is closely tied up with Floquet theory that we have discussed in part I of this course.
See F. John: Lectures on ODEs, Courant lect. notes (1965) NYU. pp 138-150

We shall briefly touch upon a second method of proving the existence of a limit cycle with a topological flavour. This discussion will be extremely short without much details. More details can be found in C. Chicone's book already referred to earlier. Other refs will be provided.

For the van der Pol System:

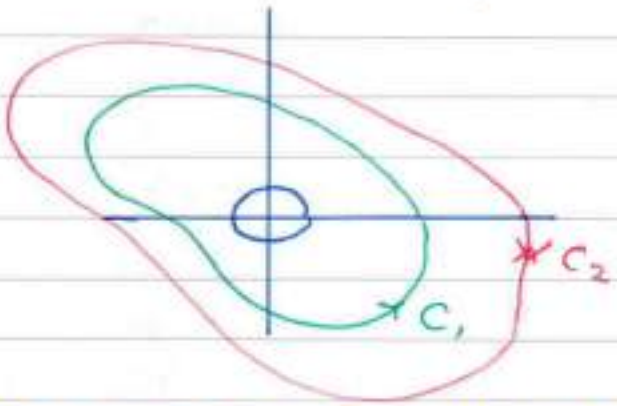
$$\dot{x} = -y \qquad (8.26)$$
$$\dot{y} = x - \epsilon(x^2-1)y$$

Write $\rho(x,y) = x^2+y^2$ and we see
$\dot{\rho} = 2\epsilon(y(t))^2 \geq 0$ at points on the y-axis and ρ is strictly increasing away from the x-axis. So the solution starting in the disc $D = \{(x,y) / x^2+y^2<1\}$ cannot be periodic (we are not looking at the Const. Sol)
What about outside $x^2+y^2=1$. The region is not simply Conn. The Dulac-Bendixon thm cannot be directly applied (see Chapt 5, p 234).

Ex: Take $B(x,y) = (x^2+y^2-1)^{-\frac{1}{2}}$
Check $\mathrm{Div}(B.f) \neq 0$; $f = (-y, x-\epsilon(x^2-1)y)$
See C. Chicone p 105.

There cannot be any periodic solution lying in any simply conn. region in $\{x^2+y^2>1\}$
Periodic Sol. if any must wind around $(0,0)$

Prove that there cannot be two such periodic Solutions. Hint: Imitate the proof of Dulac-Bendixon thm but in the region between two periodic trajectories γ_1, γ_2
Div (Bf) maintains a fixed sign. Applying Green's thm would now give TWO boundary int. Both are zero!



$\exists\ \eta>0$ s.t for any Sol. with IC outside $B_\eta(0,0)$ stays outside this disc for $t \geq t_0(\eta)$

Idea is to find a trapping region which is a Compact subset of $\mathbb{R}^2 - B_\eta(0,0)$ which is an annulus (or homeomorphic to an annulus)
$\{\eta < x^2+y^2 < R\}$
This is where the details get slightly messy and the details uninteresting*
See the details worked out in pp 8-10 in the book of Yan-Qian Ye: Theory of limit cycles, AMS 1986.
We shall have occasion to refer to this very important memoir soon.
We now appeal to the Celebrated

Thm 8.5 (Poincaré Bendixon thm): Consider a planar system $\dot{x} = P(x,y)$; $\dot{y} = Q(x,y)$ (8.27)

*) Not appropriate to discuss at this very end of the course!

P, Q are smooth on Ω open in $\mathbb{R}^2$

Assume that K is a compact subset of Ω that is forward invariant; K has no rest points. Any trajectory starting in K has as its Omega limit set a closed subset homeomorphic to the circle i.e a periodic trajectory.

For a proof see pp 74-79 of
J. John: Lect. in ODEs, NYU lect. notes 1965

(Hilbert's Sixteenth problem: Of the 23 problems presented by D. Hilbert in the International Math. Congress of 1900, ~~three~~ four of them concern differential equations !

Hilbert's 16th problem concerns estimating the number of limit cycles and their relative positions for polynomials P, Q in (8.27). (understanding)
Assume $\deg P, Q \leq n$. Estimate # of limit cycles in terms of n. Unlike other problems in the list this one received relatively less attention in the decades that followed except for a detailed and investigation due to Dulac who was a student of Poincaré. Dulac in ~~1912~~ 1923 published a long memoir proving the finiteness of the # of limit cycles for (8.27) with polynomial RHS.
Dulac's work was completely understood only by the 1980s. An error was discovered by in Dulac's work and the work of Ilyashenkov was a culmination of ~~many~~ other efforts by several mathematicians besides himself (including work of I.G. Petrowski. Dulac's papers significantly develop local analytic th. of ODEs
For a detailed commentary see the book of
Yu. S. Ilyashenkov: Finiteness thms for limit cycles, AMS, 1991. The work uses ideas from complex analysis, algebraic geometry and other ideas.

Book of M. Farkas: Periodic motions, Springer 1994 Contains some details for quadratic vector fields.

The book of Ye. Yan-Qian: Theory of limit cycles cited earlier Contains an excellent account of the history of the subject, proofs of thms with several illuminating examples and is highly recommended for getting into these territories of ODEs.

We shall close this chapter here and also close this course MA417.

Good luck with further investigations!

Appendix: Smooth dep. on IC and parameters

This will not be video recorded.

Material made available on the website

www.math.iib.ac.in/~gopal

Appendix:

Differentiability of Solutions with respect to IC and parameters

We need to go back to Chapt I, p38 and notations are as in that Chapter.

We consider a system of ODEs

$$\frac{dx}{dt} = F(x, t, c_1, \ldots, c_k) \qquad -1-$$

with IC $x(t_0) = x_0$

Where $F: \Omega \times \Pi \to \mathbb{R}^n$ is continuous : Ω connected domain in $\mathbb{R}^{n+1}$ containing (x_0, t_0).

Π is a parameter domain in $\mathbb{R}^k$. We may assume Π is a compact box in $\mathbb{R}^k$.

Assume that

$$\bar{B}_R(x_0) \times [t_0 - a, t_0 + a] \times \Pi \subset \Omega \times \Pi$$

We know from chapt 1 that $\exists\ h > 0$ such that- for any $y_0 \in \bar{B}_R(x_0)$, $(c_1, \ldots, c_k) \in \Pi$, the ODE IVP

$$\frac{dx}{dt} = F(x, t, c) \qquad -2-$$

with IC $x(t_0) = y_0 \in \bar{B}_R(x_0)$

has a unique solution $y(t, y_0, c)$ defined on $[t_0 - h, t_0 + h]$

Thm-1: The map $(t, c) \mapsto y(t, c)$ is smooth on $[t_0 - h, t_0 + h] \times \Pi$.

Further, the derivatives $z_j = \frac{\partial y}{\partial c_j}(t, c_0) : c_0 \in \Pi$ which exist and satisfy the variational equations

$$\dot{z}_j = D_y F(y(t, c_0), t, c_0)\, z_j + D_c F(y(t, c_0), t, c_0)\, \hat{e}_j \qquad -3-$$

with

$$z_j\big|_{t=t_0} = 0 \quad \text{(zero IC)} \qquad (\hat{e}_j \in \mathbb{R}^k)$$

Proof: We need to ~~simply~~ simplify notations and in particular avoid subscripts.

(a) $D_c F$ of course denotes the $n \times k$ matrix $\left[\frac{\partial F_i}{\partial c_j}\right]$

(b) $\hat{e}_j$ appearing in the last line is the jth unit vector in $\mathbb{R}^k$ (column vector)

(c) $D_y F$ is the $(n \times n)$ matrix $\left[\frac{\partial F_i(y, t, c)}{\partial y_j}\right]$

The system of ODEs for z_j $(j = 1, 2, \ldots, k)$ or the variational Equations can be written as

$$\left.\begin{aligned} \dot{z} &= A(t)z + b(t); \\ IC: \quad z(t_0) &= 0 \end{aligned}\right\} \quad (4)$$

Where $A(t) = D_y F(y(t, c_0), t, c_0)$

and $b(t) = D_c F(y(t, c_0), t, c_0)$.

Note that $A(t)$ is a $n \times k$ matrix
$b(t)$ is a $n \times k$ matrix
z is a $n \times k$ matrix

In other words each z_j is a $n \times 1$ col. and these as stacked as $[z_1, \ldots, z_k]$ to form a $n \times k$ matrix.

Equation (4) has a solution defined throughout the Compact interval $[t_0 - h, t_0 + h]$

We have to show:

$$\frac{y(t, c_0 + v) - y(t, c_0) - z(t, c_0)v}{\|v\|} \to 0 \quad \text{as } v \to 0 \text{ in } \mathbb{R}^k \qquad (5)$$

(5) is then the Desideratum.

This would show that $c \mapsto y(t, c)$ is differentiable and its derivative at $c = c_0$ is precisely $z(t, c_0)$.

One small detail:

Suppose $f: G \to \mathbb{R}^N$ is a function of class C^1 on an open set G in $\mathbb{R}^M$.

Let K be a compact subset of G

$$\lim_{v \to 0} \frac{f(x+v) - f(x) - Df(x)v}{\|v\|} = 0$$ uniformly wrt x (as x varies over K)

Exercise: Prove this. The key observation is the uniform continuity of Df on any compact subset of G. Here is a hint. Work in a compact ball (or any convex compact will do)

$$f(x+v) - f(x) = \int_0^1 \frac{d}{dt} f(x+tv)\,dt$$

$$= \int_0^1 Df(x+tv)v\,dt$$

$$= \int_0^1 (Df(x+tv) - Df(x))v\,dt + Df(x)v$$

$$\therefore f(x+v) - f(x) - Df(x)v = \int_0^1 (Df(x+tv) - Df(x))v\,dt$$

Let $\epsilon > 0$ be arb. Unif. Cont. gives us a δ ...
Take abs. value and estimate

The only other imp. ingredient in the proof is Gronwall's lemma. Rest is just routine manipulations adding and subtracting terms, MVT etc;

We must begin by writing out the expression

$y(t, c_0+v) - y(t, c_0) - Z(t, c_0)v$ as an integral over $[t_0, t]$. That is convert the ODEs into integral equations as was done in Chapt I.

$$y(t, c_0+v) - y(t, c_0) \qquad (6)$$
$$= \int_{t_0}^{t} \left[F(y(s, c_0+v), s, c_0+v) - F(y(s, c_0), s, c_0) \right] ds$$

and

$$z(t, c_0) = \int_{t_0}^{t} \left(A(s) z(s, c_0) + b(s) \right) ds \qquad (6')$$

Add and subtract terms in (6) to use MVT

$$F(y(s, c_0+v), s, c_0+v) - F(y(s, c_0), s, c_0+v)$$
$$+ F(y(s, c_0), s, c_0+v) - F(y(s, c_0), s, c_0)$$

this equals $D_c F(y(s, c_0), s, c_0) v + R_1$

R_1 is the remainder; $\frac{R_1}{\|v\|} \to 0$ unif. by the exercise

$$= D_y F(\xi, s, c_0+v) \left(y(s, c_0+v) - y(s, c_0) \right)$$
$$+ R_1 + D_c F(y(s, c_0), s, c_0) v$$

but notice that the last term is exactly $b(s) v$

~~Sol~~ ~~F(y(s, c_0+v), s, c_0+v) - F~~

So expression (6) reads:* (see foot note for ξ)

$$y(t, c_0+v) - y(t, c_0) \qquad (7)$$
$$= \int_{t_0}^{t} \left[D_y F(\xi, s, c_0+v) \left(y(s, c_0+v) - y(s, c_0) \right) ds \right.$$
$$\left. + R_1 + \underbrace{D_c F(y(s, c_0), s, c_0) v}_{= b(s) v} \right] ds$$

Note the reappearance of the diff. $y(t, c_0+v) - y(t, c_0)$ which suggests that Gronwall lemma will soon be invoked!

Now we need to consider (7) − (6')v:

$$y(t, c_0+v) - y(t, c_0) - z(t, c_0) v \qquad (8)$$
$$= \int_{t_0}^{t} \left[D_y F(\xi, s, c_0+v) \left(y(s, c_0+v) - y(s, c_0) \right) \right.$$
$$\left. + R_1 - A(s) z(s, c_0) \right] ds$$

Cont. wrt IC gives. $\xi \to y(s, c_0)$ as $v \to 0$.

(*) ξ lies on the line seg joining $y(s, c_0+v)$ and $y(s, c_0)$

Call $y(t, c_0+v) - y(t, c_0) - Z(t, c_0)\, v = \Delta(t, c_0, v)$

The boxed term in red on p361.

$$D_y F(\xi, s, c_0+v) = D_y F(\xi, s, c_0+v) - D_y F(y(s, c_0), s, c_0) + D_y F(y(s, c_0), s, c_0)$$

(this is $A(s)$)

$$= R_2 + A(s)$$

$R_2 \to 0$ unif. as $v \to 0$. ($D_y F$ cont. and so is unif. cont.)

So we get from (8) that

$$\Delta(t, c_0, v) = \int_{t_0}^{t} (R_2 + A(s))\, \Delta(s, c_0, v) + \int_{t_0}^{t} (R_1 + R_2 Z(s, c_0) v)\, ds$$

Now we take norms and get an ineq:

$$\| \Delta(t, c_0, v) \| \leq \int_{t_0}^{t} \alpha \| \Delta(s, c_0, v) \| ds + \beta (t - t_0) \| v \|$$

$$\alpha = \sup (\| R_2 \| + \| A(s) \|)$$

$$\beta = \sup \| R_1 + R_2 Z(s, c_0) v \|_h \to 0 \text{ as } v \to 0.$$

(and we have used unif. of conv. Exercise on p 360)

By Gronwall lemma

$$\| \Delta(t, c_0, v) \| \leq \| v \| \beta \exp \alpha h$$

Since $\beta \to 0$ as $v \to 0$; we get that

$$\frac{\| \Delta(t, c_0, v) \|}{\| v \|} \to 0 \text{ as } v \to 0$$

proof is Complete.

There is no need to prove diff wrt IC:

we can convert IC into parameters. Let us illustrate this

$$\frac{dy}{dt} = f(y, t, c) : \quad y(t_0) = z \quad (IC)$$

$$\frac{dy}{dt} = f(y,$$

Define $Y = y - z$

So that $Y(t_0) = 0$ and Y satisfies the system of ODEs

$$\frac{dY}{dt} = f(Y+z, t, c) = g(Y, t, c, z)$$

with zero IC

So by this trick the IC have been incorporated along with the parameters $c_1, \ldots, c_K$ or the parameter domain has been augmented.