MA 417 Errata-Sheet for the notes including addenda

1. Page 68: Include problem number 23: $\dot{x} = F(x,t)$ where $|F(x,t)| \leq M(|x|)$ where $M : (0,\infty) \longrightarrow (0,\infty)$ is monotine increasing and

$$\int_c^\infty \frac{du}{M(u)} = +\infty$$

   Then solution exists for all times. Wintner's criterion for global existence.

2. Page 97: Problem 11. After the clause bounded above by c insert the following: c is greater than the real parts of the characteristic exponents.

3. Page 102: Definition following theorem 3.2: such that $\|A - cI\| < R$.

4. Page 102: Definition following theorem 3.2: ball centered at cI with radius R.

5. Page 111: Insert the following phrase after Runge Approx. Thm (line 6). The program is outlined on page 184 and declared as problem 11 on page 121.

6. Page 113. In theorem 3.11, Further $f(A) = r(A)$ (cor. 3.10)

7. Page 116:

$$q_1(z) = (z - \lambda_2)^{k_2} \dots (z - \lambda_t)^{k_t}$$

8. Page 118: Insert the following exercise: Prove directly the injectivity of the map Pol $\longrightarrow$ $\mathbb{R}^N$ given by $f \mapsto f(\Sigma_A)$. This was suggested by the students.

9. Page 121. Insert the following exercise: Prove the following Holomorphic functional calculus:

   Suppose Ω is a simply connected open set in the plane containing all the eigen-values of A and $H(\Omega)$ is the algebra of holomorphic functions on Ω the assignment $f \mapsto f(A)$ is an algebra homomorphism such that

   (i) If $f_n \longrightarrow f$ uniformly on compact subsets then $f_n(A) \longrightarrow f(A)$

   (ii) If $g : \Omega_1 \longrightarrow \mathbb{C}$ is holomorphic on a simply connected $\Omega_1 \supset f(\Omega)$ then

$$(g \circ f)(A) = g(f(A)).$$

10. Page 122: Insert problem 12. Suppose $\sum a_n z^n$ converges on $|Z| < R$ with sum $f(z)$ and all eigen values of A lie in the disc $|z| < R$ show that $\sum a_n A^n = f(A)$. Note that $\|A\| < R$ may fail.

11. Page 126: After equation (4.2) where $\phi(x)$ is continuously differentiable and non-vanishing

12. page 130: Proof of Wintner's theorem, After (4.13) insert the phrase Assume $y'(\xi) < 0$.

13. Page 130: In the last line after the contradiction, begin the new sentence with

    So $y' \geq 0$.

14. Page 136: Insert item (5): A noteworthy application of BVP for the Laplacian on a disc (circular harmonics) is towards reassembling a shattered fresco at Overtari Chapel in Padua (Italy). See the article of M. Fornasier and Domenico Toniolo.

15. Page 137: After equation 4.23 insert the phrase See pp 146-147)

16. Page 141: Delete the $\Delta'(\lambda_0) \neq 0$ in the statement of cor 4.9. Also in the proof add the CAUTIONARY note that it is NOT clear that $\Delta'(\lambda_0) \neq 0$ follows from the simplicity of eigen-values.

17. Page 148: Insert the following sentence (say six lines from below) Clearly T is defined on V.

18. Page 149: Fifth line from the top: with eigen-function ψ namely $T\psi = \lambda\psi$ and ψ not identically zero. From $\psi \in V$, delete symbols $\in V$. Insert a note with an asterisk: The solution ψ obviously cannot belong to V since the zeros of ψ are isolated. The space V although dense is TOO small. There is an optimal $\widetilde{V}$ strictly larger than V on which T is defined. For example the image of K given by (4.37) is larger than V.

19. Page 149: The formula for the Green's function is wrong. The correct formula is as follows:

$$\begin{aligned} G_0(x,\xi) &= -(x-1)\xi, \quad x \geq \xi \\ G_0(x,\xi) &= -(\xi-1)x, \quad \xi \geq x \end{aligned}$$

20. Page 149: The densely defined map is injective on $\widetilde{V}$.

21. Page 150: Line 5 from the top: Operator $K : L^2[0,1] \longrightarrow \widetilde{V}$ given by...

22. Page 149: Second line after (4.35): $L^2_\rho[0,1]$.

23. Page 149: Insert the exercise: Given $f \in L^2_\rho[0,1]$ show that $y(x)$ given by (4.35) lies in $L^2_\rho[0,1]$ and satisfies the estimate

$$\|y\|_{L^2_\rho} \leq \|f\|_{L^2_\rho} \Big(\int \int G_0(x,\xi)\rho(x)\rho(\xi)dxd\xi \Big)^{1/2}$$

24. Page 150: Add the following concluding note: Perforce we have avoided discussing the optimal domain $\widetilde{V}$ on which T is defined and is self-adjoint. Such matters are best relegated to some future course.

25. Page 169: Add the reference:

    M. Fornasier and Domenico Toiolo: Fast Robust and efficient 2D patter recognition, Pattern Recognition, **38** (2005) 2074-2087.

26. Page 172: Insert the following exercise: Show that the Green's function (4.21) satisfies the boundary conditions as well as the ODE except at $x = \xi$. Show further that if ϕ is a smooth function with compact support in (a, b) then

$$\langle G, L\phi \rangle = \phi(\xi)$$

    This is usually expressed as $LG = \delta_\xi$.

27. Page 174: second last line: Replace $\xi = \xi(h) \to 0$ by $\xi = \xi(h) \to y(s, c_0)$.

28. Page 184:

    Holomorphic functional calculus for matrices. Details of the program outlined on pp 110-111.

    Theorem: Let Ω be a simply connected open set in the plane containing all the eigen-values of A. The map

$$f \mapsto f(A), \quad H(\Omega) \longrightarrow M_n(\mathbb{C})$$

    is unique subject to the following:

    (i) It is a ring homomorphism

    (ii) If f is a polynomial then $f(A)$ has its usual meaning.

    (iii) If $f_n \longrightarrow f$ uniformly on compact subsets of Ω then $f_n(A) \to f(A)$.

    (iv) If $g : \Omega_1 \longrightarrow \mathbb{C}$ is holomorphic where Ω_1 is simply connected open containing $f(\Omega)$ then

$$(g \circ f)(A) = g(f(A))$$

    Proof: Let γ be a simple closed curve containing all the eigen-values of A inside. Define

$$f(A) = \frac{1}{2\pi i} \oint_\gamma f(\zeta)(\zeta I - A)^{-1} d\zeta$$

The assignment $f \mapsto f(A)$ is obviously linear. To prove (ii) it obviously suffices to check that if $f(z) = z^k$ then

$$A^k = \frac{1}{2\pi i} \oint_\gamma \zeta^k (\zeta I - A)^{-1} d\zeta$$

Evidently we may replace γ by a large circle C (why?) and expand $(I - \zeta^{-1}A)^{-1}$ as a geometric series which converges when ζ varies over C and then integrate term by term and verify (ii). Condition (iii) is clear. We turn to the completion of (i). Let $f, g \in H(\Omega)$ and select polynomials $p_n(z), q_n(z)$ converging to $f(z), g(z)$ uniformly on compact subsets of Ω (Runge theorem). Then

$$(p_n q_n)(A) = p_n(A) q_n(A).$$

Let $n \to \infty$ and use (iii). Finally we prove (iv). From the last part

$$(f(A))^k = \frac{1}{2\pi i} \oint_\gamma f(\zeta)^k (\zeta I - A)^{-1} d\zeta$$

so that

$$p_n(f(A)) = \frac{1}{2\pi i} \oint_\gamma ((p_n \circ f)(\zeta))^k (\zeta I - A)^{-1} d\zeta$$

for every polynomial. Now take a sequence of polynomials converging to g and complete the argument.

Note that this was completely worked out in class but it was not in the notes.

MA 417 Problem set - I

1. Show that the minimal surface equation reduces to an ODE is the minimal surface is a surface of revolution. Can you integrate this ODE and find all minimal surfaces that are surfaces of rotation?

2. Suppose f is a real valued function defined on $[0, \infty)$ which is differentiable and such that $f(t)$ and $f'(t)$ have limits as $t \longrightarrow \infty$, then show that $f'(t) \to 0$ as $t \longrightarrow \infty$. Can you weaken the hypothesis only requiring differentiability and that $f(t)$ tends to a limit as $t \longrightarrow \infty$?

3. Consider the ODE $\dot{y} = ky(1-y)$. Show that all solutions with $y(0) > 0$ approach the equilibrium point 1.

4. Determine all rotation invariant solutions of $\Delta u = 0$.

5. Determine a travelling wave solution of the KdV equation of the form $C\text{sech}^2(Ax + Bt)$. There is a relationship between the speed of the wave and the height C. This is a particular solution of the ODE for elliptic functions we derived in class.

6. Determine a one parameter group of scaling symmetries of the heat equation $u_t = u_{xx}$ and find the corresponding group invariant solution and call it $G(x,t)$. Note that in general if u is a solution of the heat equation then so is u_x. Find $\frac{1}{4}G_x$ (the famous heat kernel).

7. (i) Does the heat equation have non-trivial travelling wave solutions? If so find them.
   (ii) What about $u_t = uu_x + u_{xx}$?
   (iii) Determine the ODE for the travelling wave solutions of the Fisher-Kolmogorov equation
   $$u_t - cu_{xx} = ru(1-u)$$
   The equation appears in connection with population dynamics, blood flow in Aeorta and many other applications.

8. We obtained a one parameter group of scaling transformations for the KdV equation. Perform a similar analysis with the modified KdV (abbreviated as mKdV) equation
   $$u_t = u^2 u_x + u_{xxx}$$

9. Let $p(t)$ be a smooth real valued function with period T. Show that all the solutions of the ODE $\dot{y} = p(t)y$ are periodic with period T if and only if $\int_0^T p(s)ds = 0$.

10. Assume that you DO NOT know ANYTHING about the functions $\sin x$ and $\cos x$. Now consider the ODE

$$y'' + y = 0$$

and $y(x)$ be the unique solution with initial conditions $y(0) = 0, y'(0) = 1$. Prove the following:

(i) Show that there exists a positive $T > 0$ such that $y(T) = 1$ and $y(x)$ monotone increases from 0 to 1 as x increases from 0 to T.

(ii) Show that $(y(x))^2 + (y'(x))^2 = 1$ whereby $y'(T) = 1$.

(iii) Show that $y(x)$ and $y'(x)$ form a basis of all solutions of the ODE.

(iv) Deduce that $y(x)$ is periodic with period $4T$.

(v) Deduce that $y(x + a) = y(x)y'(a) + y'(x)y(a)$ and $y(3x) = 3y(x) - 4(y(x))^3$.

Definition: $\pi = 4T$, $\sin x = y(x)$ and $\cos x = y'(x)$. Derive the entire trigonometry via ODEs :-)

11. Consider the pendulum equation (See pages 23-24 of the notes)

$$x'' + \frac{g}{l} \sin x = 0$$

with initial conditions $x(0) = 0$ and $x'(0) = c > 0$.

(i) Write it as a first order system and obtain a first integral. Equivalently obtain a relation between $x(t)$ and $x'(t)$.

(ii) Suppose $c < 2\sqrt{g/l}$ show that the pendulum performs oscillatory motion namely there is a positive time $T > 0$ such that $x(t)$ monotonically increases, reaches a maximum value that is less that π at $t = T$ and then returns back and goes to a minimum value at $t = 2T$. The solution is periodic with period $4T$

(iii) Find an expression for the period $4T$ as an integral.

(iv) Suppose $c > 2\sqrt{g/l}$ show that the pendulum reaches the summit at finite time $t = T$ namely $x(T) = T$ and performs a (non-uniform) circular motion.

(v) Discuss the borderline case $c = \sqrt{g/l}$. Also relate it to the critical value of the first integral that you found. Integrate the differential equation explicitly in this case and discuss the behaviour as time tends to $\pm\infty$.

12. Consider the system of ODEs: (page 19 of notes)

$$A\dot{x}_1 = (B - C)x_2x_3, \quad B\dot{x}_2 = (C - A)x_1x_3, \quad C\dot{x}_3 = (A - B)x_1x_2$$

Derive two independent first integrals and show that the solution exists for all times.

13. Consider the ODE

$$\dot{x} = x^2 - 1 - \cos t$$

What can you say regarding the existence/non-existence of periodic solutions?

MA 417 Problem set - II

1. A function $f(y,t)$ defined on $\Omega \subset \mathbb{R}^{n+1}$ is said to be locally Lipschitz if it satisfies the Lipschitz condition on each compact rectangle contained in Ω. Show that local Lipschitz suffices to ensure uniqueness.

2. (i) Consider the set B of all continuous functions $y(t)$ on $[t_0-h, t_0+h]$ such that $|y(t)-y_0| \leq b$ for all $t \in [t_0-h, t_0+h]$ with supremum norm. Show that B is a complete metric space and $T : B \longrightarrow B$ given by

$$Ty(t) = y_0 + \int_{t_0}^{t} f(y(s), s)ds$$

is a contraction mapping provided $f(y,t)$ is Lipschitz in the y variable and h besides being less than $a, b/M$ is also less than $1/L$ where L is the Lipschitz constant. Deduce the local existence theorem. Discuss the disadvantage of this argument.

(ii) Define a new metric d' as

$$d'(y,z) = \sup\{|y(t)-z(t)| \exp(-L|t-t_0|) \ : \ |t-t_0| \leq h\}$$

Show that this metric is complete and the map T in the last exercise is a contraction without having to restrict h depending on L.

3. Consider the IVP $\dot{y} = y^2 + t^2$ with $y(0) = 0$. Consider the rectangle $|y| \leq b$, $|t| \leq a$ and determine the interval $[-h, h]$ on which the solution exists as given by Picard's theorem. By varying a and b show that the best h we can get by this method is $1/\sqrt{2}$. However the solution lives on an interval containing $[-1, 1]$.

4. The variational equation: In each of the following exercises there is a parameter ϵ in the ODE and the solution $y(t,\epsilon)$ is expanded in a series $y(t,0) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \ldots$. Find the functions $y_1(t)$ and $y_2(t)$:

(i) $y' = \epsilon y^2 - t$, $y(0) = 1$.

(ii) $y'' + y = t^{-1}\sin(\epsilon t)$, $y(0) = 0, y'(0) = 1$.

(iii) $y'' + y - \epsilon y^3 = 0$, $y(0) = 0, y'(0) = 1$.

5. For the pendulum equation $x'' + (g/l)\sin x = 0$, with initial conditions $x(0) = 0$ and $x'(0) = c$, find the variational equation for $\dfrac{\partial x}{\partial c}$ at $c = 2\sqrt{g/l}$.

6. Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth function such that $f(a_1) = 0 = f(a_2)$ and f has no zeros on (a_1, a_2) and positive on (a_2, ∞). Show that the IVP $\dot{y} = f(y)$ with $y(0) = c$ has infinite life span if $a_1 \leq c \leq a_2$ and if $c > a_2$ the life span is given by

$$\int_c^{\infty} \frac{du}{f(u)}$$

7. (i) Suppose $u(t)$ and $v(t)$ are solutions of a first order scalar ODE $\dot{y} = f(y,t)$ then show that $\max(u(t), v(t))$ and $\min(u(t), v(t))$ are also solutions.

   (ii) Prove Kneaser's theorem for the case of scalar first order ODEs.

   (iii) Show that a connected metric space with at least two points must be uncountable. Deduce that if an IVP for a system of ODEs has two distinct solutions then it has an uncountable family of distinct solutions.

8. (i) Determine the Peano funnel for the IVP $y' = y^{2/3}, y(0) = 0$ and the IVP $y' = \sqrt{y}$, $y(0) = 0$.

   (ii) Show that the IVP $y' = y^{2/3}$, $y(0) = -1$ has a unique solution on $[0, 3]$ but infinitely many solutions on $[0, 3 + \eta]$ for every $\eta > 0$.

   (iii) Find the solutions of the IVP $y' = y^{1/3}\sin(2t)$, $y(0) = 0$ satisfying the condition $y(\pi/2) = c\sqrt{8/27}$.

9. Problems on Gronwall's lemma: Let $f(y,t)$ be defined on $\mathbb{R}^2$ such that $|f(y,t)| \leq \phi(t)|y|$ where ϕ is a non-negative continuous on the real line such that $\int_a^\infty \phi(u)du$ is finite. Consider the first order scalar ODE $\dot{y} = f(y,t)$.

   (i) Suppose $y(t)$ is any solution, prove that $\sup\{|y(t)| : t \geq a\}$ is finite. In particular all solutions exist on $[a, \infty)$.

   (ii) Show that $y(t)$ tends to a limit as $t \longrightarrow \infty$.

   (iii) Determine these limits in the case of $\dot{y} = y(1 + t^2)^{-1}$.

   (iv) If $\int_a^\infty \psi(u)du$ then the solutions of $\dot{y} = -y + \psi(t)y$ grow like ce^t. Hint: put $y = e^t x$.

   (v) (More work is needed for this) Suppose that $|f(y,t) - f(z,t)| \leq \phi(t)|y - z|$ then there is a one-to-one correspondence between the set of all initial values and the set of limiting values (as $t \to \infty$).

10. Suppose ϕ is a non-negative real-valued continuous function such that $\int_{t_0-h}^{t_0+h} \phi(u)du$ is finite and $|f(y,t) - f(z,t)| \leq \phi(t)|y - z|$ then show that the solution of the IVP $\dot{y} = f(y,t)$, $y(t_0) = y_0$ is unique. Discuss the case $f(y,t) = \sqrt{t}\sin(y/t)$ which is evidently continuous. Does f satisfy a Lipschitz condition in y?

11. Osgood's uniqueness theorem: Suppose $\phi(u)$ is a positive continuous increasing function on $(0, a]$ such that $\int_0^a (\phi(u))^{-1}du = +\infty$ and f is a continuous function on the plane such that $|f(y,t) - f(z,t)| \leq \phi(|y - z|)$. Show that the initial value problem for the ODE $\dot{y} = f(y,t)$ has a unique solution.

12. (i) Let f on the real line be defined as $f(y) = y\log(y^{-1})$ if $y > 0$ and $f(y) = 0$ if $y \leq 0$. Show that f fails to be Lipschitz on any neighborhood of the origin. The zero function satisfies the ODE $\dot{y} = f(y)$ with zero initial condition. Are there any other solutions of this initial value problem?

(ii) Suppose $f(y,t)$ is continuous on $\mathbb{R}^n$ satisfies

$$|f(y,t) - f(z,t)| \leq L|y-z|\Big(\log|y-z|^{-1}\Big), \quad 0 < |y-z| < 1.$$

Show that the IVP for the ODE $\dot{y} = f(y,t)$ has a unique solution.

(iii) Can the Osgood's uniqueness condition be employed in this case?

13. Consider $y'' - u^3 = 0$ with initial conditions $u(0) = 1, u'(0) = 0$. Discuss the behaviour of solutions. Sketch the graph of the solution and determine the life span of solutions.

MA 417 Problem set - III

1. Suppose $v_1, v_2, \ldots, v_n$ are n vectors in $\mathbb{R}^n$ and A is an $n \times n$ matrix, show that

   $\mathrm{Det}[Av_1, v_2, \ldots, v_n] + \mathrm{Det}[v_1, Av_2, \ldots, v_n] + \ldots + \mathrm{Det}[v_1, v_2, \ldots, Av_n] = \mathrm{Trace}(A)\mathrm{Det}[v_1, v_2, \ldots, v_n]$.

   Do this in two ways:

   (i) Using linear algebra specifically observing that the two sides are multilinear alternating.

   (ii) Differentiating the identity

   $$\mathrm{Det}[\exp(tA)v_1, \exp(tA)v_2, \ldots, \exp(tA)v_n] = (\mathrm{Det}\exp(tA))\mathrm{Det}[v_1, v_2, \ldots, v_n]$$

   and putting $t = 0$.

2. Consider the system $\dot{y} = \Lambda(t)y$ where Λ is a skew-symmetric matrix with entries as continuous functions. Prove that if the fundamental matrix $\Phi(t)$ is orthogonal at some point $t = t_0$ then it is orthogonal for all values of t.

3. Let J be the $2n \times 2n$ matrix given by

   $$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

   (i) Show that the set G of all $2n \times 2n$ matrices S with real entries such that $S^T JS = J$ forms a subgroup of the non-singular matrices.

   (ii) The set of all $2n \times 2n$ matrices A with real entries satisfying $A^T J + JA = 0$ forms a real vector space V. What is its dimension?

   (iii) Suppose $H(t)$ is a $2n \times 2n$ matrices with entries as continuous functions and $H(t) \in V$ for all t. Then the fundamental matrix $\Phi(t)$ of the system $\dot{y} = H(t)y$ belongs to G for all t if it belongs to G for some time t_0.

4. Consider the unique solution $y(t, a)$ of system of ODEs $\dot{y} = A(t)y$ with initial conditions $y(0) = a$. The map $T : a \mapsto y(T, a)$ is a linear transformation. Is this injective? Suppose C is a bounded open set in $\mathbb{R}^n$ and for a fixed time t, $C_t = \{y(t, a) : a \in C\}$. Find the ratio of volume of C_t to the volume of C.

5. We have seen that the solution of the initial value problem $\dot{y} = A(t)y + b(t)$ with $y(t_0) = 0$ can be described as an integral

   $$\int_{t_0}^{t} M(t, s)b(s)ds$$

   where M is the transition matrix. Show that the solution operator $T : C[0, T] \longrightarrow C[0, T]$ given by $(Tb)x = \int_{t_0}^{t} M(t, s)b(s)ds$ is a compact operator with respect to supremum norm. Use the Ascoli-Arzela theorem.

6. Suppose $\Phi(t)$ is a fundamental matrix for a linear system $\dot{y} = A(t)y$. Find the linear system $\dot{z} = B(t)z$ whose fundamental is $\Phi(t)^{-1}$. The system $\dot{z} = B(t)z$ is called the adjoint system and is more conveniently written as $\dot{\zeta} = \zeta B(t)$ where $\zeta(t)$ is the row matrix traspose to $z(t)$.

7. A problem in linear algebra. Suppose A is a $m \times n$ matrix (of constants) and b is an $m \times 1$ column vector (constants) consider the problem of solving $Ax = b$. Discuss whether this problem is an overdetermined system. Show that the necessary and sufficient conditions for solvability of the linear system is that $z \cdot b = 0$ for all z in the null-space of A^T and the dot refers to the usual inner product on $\mathbb{R}^m$. Thus b must be orthogonal to the null space of A^T in order for the problem to be solvable.

8. Let us look at an analogous problem in ODEs. Consider the problem of solving $\dot{y} = A(t)y + b(t)$ with $y(0) = c$ and $y(T) = \alpha$. Explain why this is an overdetermined system and find compatibility conditions for its solvability. Hint: Use the adjoint system.

9. Suppose $w(t)$ is a row vector such that $w(t)y(t)$ is constant for all solutions of $\dot{y} = A(t)y = 0$ then $w(t)$ satisfies the adjoint equation and conversely. In this case show that the system of ODEs can be reduced to an $(n-1)$ dimensional system.

10. (i) Suppose H is a subgroup of the real line with respect to addition. Show that either H is dense in $\mathbb{R}$ or is discrete namely an infinite cyclic group.

    (ii) Suppose a continuous real valued function on the line has two periods that are linearly independent over rationals then the function is constant.

    (iii) Discuss for periodicity of solutions of $y'' + \omega^2 y = \sin\sqrt{2}t$.

11. Suppose that the smooth autonomous system $\dot{y} = f(y)$ with initial condition $y(0) = p$ has a periodic solution when $p = p_0$. Prove that the variational equation satisfied by the derivative $\frac{\partial y}{\partial p}$ at $p = p_0$ is a linear periodic system. Show that the monodromy matrix has 1 as an eigen-value.

12. Consider a linear system $\dot{y} = A(t)y$ with periodic $A(t)$. Suppose all the characteristic exponents have real parts strictly less than c, prove that there exists $R > 0$ such that the estimate $|y(t)| \leq R|y(0)| \exp ct$ for all $t \geq 0$ holds for all solutions. What does this say if $c < 0$?

13. Suppose that $y, y_1, y_2, \ldots, y_n$ are n smooth functions defined on the real line, show that

    (i) $W(yy_1, yy_2, \ldots, yy_n) = y^n W(y_1, y_2, \ldots, y_n)$.

    (ii) $W(1, 2y, 3y^2, \ldots, ny^{n-1}) = n!(n-1)! \ldots 3!2! \left(\frac{dy}{dx}\right)^{n(n-1)/2}$.

    (iii) $W(y_1, y_2, \ldots, y_n)(\phi')^{n(n-1)/2} = W(y_1 \circ \phi, y_2 \circ \phi, \ldots, y_n \circ \phi)$.

MA 417 Problem set - IV

1. Show that if all eigen-values of A have negative real parts then all solutions of $\dot{y} = Ay$ tend to zero as $t \longrightarrow \infty$. Note that A need not be diagonalizable and you need to look at the Lagrange-Sylvester interpolation polynomial.

2. Calculate $\exp(tA)$, $\sin \pi A$ and $\log A$ for the following 5×5 matrix:

$$\begin{pmatrix} 7 & -2 & 1 & 2 & -3 \\ 8 & -2 & 2 & 3 & -4 \\ 6 & -2 & 2 & 2 & -3 \\ -8 & 3 & -1 & -1 & 4 \\ -1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

   given that the minimal polynomial is $(z-1)^3(z-2)^2$.

3. Show that the Lagrange-Sylvester interpolating polynomial reduces to the Lagrange interpolation polynomial when the minimal polynomial has distinct roots. One can actually recover the Lagrange-Sylvester interpolation polynomial from the Lagrange polynomial by allowing the roots to coalesce in groups but carrying out this process needs some care.

4. Under what conditions is it true that ALL solutions of $\dot{y} = Ay$ are periodic?

5. Suppose A is a non-singular REAL matrix with NO eigen-values on the negative real axis then show that A has a REAL logarithm.

6. Let A be a given $n \times n$ matrix and $R(z) = (zI - A)^{-1}$. This function is defined on the complex plane minus the set of eigen-values of A. Show that $R(z)$ is differentiable as a function of z. Find its derivative. The function is first of all continuous. Simplify $R(z) - R(w)$.

7. **The real-canonical form:**

   (i) Suppose that A is a 2×2 real matrix with complex eigen values $a \pm ib$. Let $Z \in \mathbb{C}^n$ be an eigen-vector and X, Y be the real and imaginary parts of Z. Show that X and Y are linearly independent over $\mathbb{R}$ and find $P^{-1}AP$ where $P = [X, Y]$. The special form $P^{-1}AP$ is called the real canonical form of A. Generalize this to the case when A is diagonalizable with list of eigen values

$$\lambda_1, \ldots, \lambda_k, a_1 \pm ib_, \ldots, a_s + \pm ib_s.$$

   The first k eigen-values in this list are real.

(ii) Find the real canonical form of the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -c & 0 \end{pmatrix}$$

8. Use the real canonical form to show that $SO(3,\mathbb{R})$ is path connected.

9. Let V be the vector space of all 3×3 skew-symmetric matrices. Show that $A \mapsto \exp(A)$ maps V into a connected subgroup of $SO(3,\mathbb{R})$. Is this injective? Look at the principal fundamental matrix of $\dot{y} = Ay$, where A is the matrix in exercise 7(ii).

10. In class we proved that the map $f \mapsto f(\Sigma_A)$ from Pol to $\mathbb{R}^N$ is surjective in order to produce the polynomials $\phi_{j,k}(z)$. Prove directly that the map is injective and deduce that it is surjective.

11. Let A be a given $n\times n$ matrix with complex entries and Ω be an open set in $\mathbb{C}$ containing all the eigen-values of A. We have developed the functional calculus that assigns to each holomorphic function $f(z)$ on Ω, a matrix $f(A)$ with certain properties. Suppose that $g : G \longrightarrow \mathbb{C}$ is holomorphic on a simply connected open set G containing $f(\Omega)$ then

$$(g\circ f)(A) = g(f(A)).$$

In other words our holomorphic functional calculus respects compositions also. Hint: The result is easy if g is a polynomial.

12. Suppose $a_0 + a_z + a_2 z^2 + \ldots$ is a power series converging on a disc $|z| < R$ to a function $f(z)$ and this disc contains all the eigen-values of A. Can you infer that the series of matrices

$$a_0 I + a_1 A + a_2 A^2 + \ldots$$

converges to $f(A)$? Note that $\|A\| < R$ may fail. We are not demanding absolute convergence. Can you use a crucial property of the functional calculus?

13. Show that if A is a skew-Hermitian matrix then $\exp(A)$ is unitary. If A is a trace zero matrix then $\exp(A)$ has determinant one. If A is a Hamiltonian matrix then $\exp(A)$ is symplectic.

MA 417 Problem set - V

1. Show that every non-trivial solution of the Airy's equation $y'' + xy = 0$ has infinitely many zeros on $(0, \infty)$ and that if p_n denotes the $n-$th zero, $|p_{n+1} - p_n| \longrightarrow 0$ as $n \to \infty$.

2. Show that if N_k is the number of zeros of a non-trivial solution of the Airy's equation lying in $(k, k+1)$ then
$$\left[\frac{\sqrt{k}}{\pi}\right] \leq N_k \leq \left[\frac{\sqrt{k+1}}{\pi}\right]$$
Deduce that if $\zeta_1, \zeta_2, \ldots$ is the list of zeros in $(0, \infty)$ then
$$\sum_{n=1}^{\infty} \frac{1}{\zeta_n^p}$$
converges if $p > 3/2$ and diverges if $p \leq 3/2$. Hint: Compare with $\sin\sqrt{k}(x-k)$ and $\sin\sqrt{k+1}(x-k)$ on $[k, k+1]$.

3. Discuss the zeros of $y'' + \rho x^{-2} y = 0$ and use it to discuss whether the condition in Wintner's theorem is necessary. Here ρ is a constant.

4. Transform the Hermite's equation $y'' - 2xy' + 2\lambda y = 0$ into self-adjoint form. Transform the Laguerre equation $xy'' + (1-x)y' + py = 0$ into self adjoint form.

5. (i) Suppose that the continuous function $\rho(x) \leq 0$ on $[a, b]$ show that the operator $L : V \longrightarrow C[a, b]$ given by
$$Ly = y'' + \rho(x)y$$
is injective where V is the vector space of all twice continuously differentiable functions vanishing at a and b. Hint: Multiply by y and integrate by parts.

   (ii) Let Ω be a smooth bounded domain in $\mathbb{R}^3$ and $\rho : \Omega \longrightarrow \mathbb{R}$ be continuous on $\overline{\Omega}$ such that $\rho(x) \leq 0$ on Ω. Show that the operator $L : V \longrightarrow C(\overline{\Omega})$ given by
$$Ly = \Delta y + \rho(x)y$$
is injective where V is the vector space of all twice continuously differentiable functions on $\overline{\Omega}$ vanishing along the boundary. Hint: What is the rule for integration by parts in several variables?

6. **Green's function for an intial value problem:** In class we have discussed Green's function for a BVP for a second order ODE where ρ is smooth. Consider now the IVP
$$y'' + \rho(x)y = f(x), \quad y(a) = 0, y'(a) = 0.$$

Show that the solution can be expressed as

$$y(x) = \int_a^b G(x,\xi) f(\xi) d\xi$$

for some function $G(x,\xi)$. Show that this function satisfies the following properties:

(i) $G(x,\xi)$ is continuous and satisfies $G(a,\xi) = 0, G'(a,\xi) = 0$.

(ii) $G'(\xi+,\xi) - G'(\xi-,\xi) = 1$

(iii) $G'' + \rho(x)G = \delta_\xi$

Here δ_ξ is the Dirac mass at the point ξ namely $\langle \delta_\xi, \phi \rangle = \phi(\xi)$.

7. Let $G(x,\xi)$ be the Green's function for $y'' + \rho(x)y = f(x)$ with $y(a) = y(b) = 0$ we saw by looking at the formula derived for $G(x,\xi)$ that $G(a,\xi) = G(b,\xi) = 0$. Prove this directly without any reference to an explicit representation of the Green's function. Likewise prove the symmetry of the Green's function without using appealing to the explicit formula.

MA 417 Hints: Selected Problems in set - II

Osgood's uniquness theorem: Suppose that $y(t)$ and $z(t)$ are two solutions of the ODE with the same initial condition. The two solutions may not only agree at the intial time t_0 they may agree on say $[t_0, t_1]$ and then separate out. We have to show that they NEVER separate but agree throughout.

Put $u(t) = y(t) - z(t)$ and we have to show that $u(t)$ is identicall zero. If we suppose not then we may assume that $u(t_0) = 0$ and $u(t) > 0$ on $t_0 < t < c$. We have

$$\begin{aligned} u(t) &= \int_{t_0}^{t} (f(y(s)) - f(z(s))ds \\ 0 < u(t) &\leq \int_{t_0}^{t} \phi(u(s))ds = U(t) \text{ say.} \end{aligned}$$

$\dot{U} = \phi(u) \geq \phi(U)$. Imitate the proof of Gronwall lemma. It is better to exercise care and integrate from $t_0 + \epsilon$ to c rather than from t_0 to c and then allow $\epsilon \to 0+$.

MA 417 Problems on Legendre polynomials and Bessel functions:

1. Show that if n is a non-negative integer then one of the power series for the ODEs

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

terminates leading to a polynomial solution $y(x)$. Show that the degree of this polynomial is exactly p and that $y(1) \neq 0$. Can the ODE have two linearly independent polynomial solutions? The normalized solution $y(x)/y(1)$ is called the Legendre polynomial of degree n denoted by $P_n(x)$.

(i) Show that if $m \neq n$ then

$$\int_{-1}^{1} P_m(x)P_n(x)dx = 0.$$

(ii) Prove that if $Q_n(x) = D^n(x^2-1)^n$ then the polynomials $Q_n(x)$ also satisfies the condition

$$\int_{-1}^{1} Q_m(x)Q_n(x)dx = 0, \quad m \neq n.$$

2. Suppose that V is an innerproduct space and $\{v_1, v_2, v_3, \ldots\}$ and $\{w_1, w_2, w_3, \ldots,\}$ are two sets of non-zero orthogonal vector such that for EVERY $k = 1, 2, 3, \ldots$

$$\text{span}\{v_1, v_2, \ldots, v_k\} = \text{span}\{w_1, w_2, \ldots, w_k\}$$

then there is a sequence of scalars $c_1, c_2, c_3, \ldots$ such that $v_k = c_k w_k$. Deduce that

$$P_n(x) = c_n D^n(x^2-1)^n$$

By setting $x = 1$ derive the formula for $P_n(x)$. Called Rodrigues formula.

3. Use Rodrigues formula to prove that $P_n(x)$ has precisely n distinct zeros in $(-1, 1)$.

4. Consider the Bessel's ODE

$$x^2y'' + xy' + (x^2 - p^2)y = 0.$$

We can think of this as $x^2y'' + xy' - p^2y = -x^2y$ and so as a perturbation of the Cauchy Euler equation $x^2y'' + xy' - p^2y = 0$. For the latter we have solutions $x^{\pm p}$. So it is natural to seek a solution of the Bessel's ODE in the form

$$y(x) = x^p(a_0 + a_1x + a_2x^2 + \ldots), \quad a_0 \neq 0.$$

Carry this out and show that there is a solution with one arbitrary constant a_0. Taking $a_0 = 1/(2^p\Gamma(p+1))$ the special solution is denoted by $J_p(x)$ and is given by

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(x/2)^{2n+p}}{n!\Gamma(n+p+1)}$$

5. Prove the following:

$$(x^p J_p(x))' = x^p J_{p-1}(x), \quad (x^{-p} J_p(x))' = -x^{-p} J_{p+1}(x)$$

Discuss the interlacing of the zeros of J_p, J_{p-1} and J_{p+1}.

6. Show that $J_0(z)$ is an entire function. Suppose y_1, y_2 are two linearly independent solutions of the Bessel's ODE (with $p = 0$) and both y_1 and y_1 are of class C^1 near the origin then they are linearly dependent. Show that

$$J_0(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin \theta) d\theta.$$

Qualifying examination on Differential Equations (December 2019)

Total Marks 30. All questions are compulsory

(1a) Suppose that $f(t, y)$ is continuous on the real line and satisfies

$$|f(t, y)| \leq \psi(t)|y|$$

where ψ is continuous and $\int_0^\infty \psi(t)dt < \infty$. Show that the solution of the ODE $\dot{y} = f(t, y)$ with $y(0) = p$ has a limit as $t \longrightarrow \infty$. Hint: Use Gronwall lemma. You need to show first that the solution exists on $[0, \infty)$. [5]

(1b) Let $f \in C^1(\mathbb{R})$ and suppose that a solution $y(t)$ of the ODE

$$\frac{dy}{dt} = f(y)$$

exists on $\mathbb{R}$ and has a limit p as $t \to \infty$. Prove that $f(p) = 0$. [5]

(2a) Let $f \in C^1(\mathbb{R}^n)$ and $y(t)$ be a solution of the system of ODEs $\dot{y} = f(y)$ with maximal interval of existence (α, β). Suppose there is a compact set $K \subset \mathbb{R}^n$ such that $y(t) \in K$ for all $t \in (\alpha, \beta)$ then prove that $\beta = +\infty$. [5]

(2b) Suppose Λ is a 6×6 skew-symmetric matrix with real entries. Is the zero solution of the system $\dot{y} = \Lambda y$ always stable? [5]

(3a) Is the equilibrium point $(\frac{\gamma}{\delta}, \frac{a}{b})$ for the following system stable? Give reasons. Note the constants a, b, γ, δ are all positive.

$$\dot{x} = -ax + bxy, \quad \dot{y} = \gamma y - \delta xy$$ [5]

(3b) Show that the solution of $\dot{y} = x^2 + y^4$ with $y(0) = 0$ is an odd function. Suppose $(-A, A)$ is its maximal interval of existence then show that A must be finite. [5]

Note: Precise and rigorous justification is needed for all the questions.