A Relation Between Pointwise Convergence of Functions and Convergence of Functionals

Haïm Brezis & Elliott Lieb (Proc. AMS, Vol. 88, No. 3, 1983)

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Outline

1. Motivation & Classical Background
2. General Functionals (Theorem 2)
3. Convexity, Lemma 3, and Limits of the Theory
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Motivation: Loss of Compactness

- Let (Ω, Σ, μ) be an arbitrary measure space.
- Consider a sequence $\{f_n\} \subset L^p(\Omega)$ with $\sup_n \|f_n\|_p \leq C < \infty$ and $f_n \to f$ a.e.
- **Classical Lower Semicontinuity:** By Fatou's Lemma,

$$\|f\|_p \leq \liminf_{n\to\infty} \|f_n\|_p.$$

- **The Core Problem in Variational Analysis:**
  - In critical Sobolev embeddings or unbounded domains, minimizing sequences rarely converge strongly: $\|f_n - f\|_p \not\to 0$.
  - High frequencies concentrate (blow-up) or mass "runs off to infinity" (evanescence).
  - Fatou's lemma throws away the escaping mass: $\liminf_{n\to\infty} \|f_n\|_p^p - \|f\|_p^p \geq 0$.
- **Question:** Can we identify the asymptotic deficit explicitly?

The Classical Brezis–Lieb Lemma (Theorem 1)

Theorem (Brezis & Lieb, 1983)

Let $0 < p < \infty$. Suppose $\{f_n\} \subset L^p(\Omega)$ satisfies:

1. $\sup_n \|f_n\|_p \leq C < \infty$,
2. $f_n \to f$ *pointwise almost everywhere.*

Then $f \in L^p(\Omega)$ and

$$\lim_{n\to\infty} \left(\|f_n\|_p^p - \|f_n - f\|_p^p \right) = \|f\|_p^p.$$

The Decoupling Principle

Setting the defect term $g_n := f_n - f \to 0$ a.e., we have:

$$\int_\Omega |f + g_n|^p \, d\mu = \int_\Omega |f|^p \, d\mu + \int_\Omega |g_n|^p \, d\mu + o(1) \quad \text{as } n \to \infty.$$

The macroscopic background f and microscopic defect g_n decouple without cross-terms!

Nuances in the Assumptions of Theorem 1

- **Case $0 < p \leq 1$:**
  - The uniform bound $\sup_n \|f_n\|_p < \infty$ is **unnecessary** if we assume $f \in L^p$.
  - For $t \in [0,1]$, $x \mapsto x^p$ is subadditive, yielding the inequality:

$$\big||a|^p - |b|^p\big| \leq |a-b|^p \implies \big||f_n|^p - |f_n - f|^p\big| \leq |f|^p \in L^1.$$

  - The result follows immediately by Lebesgue's Dominated Convergence Theorem.

- **Case $1 < p < \infty$:**
  - The uniform bound $\sup_n \|f_n\|_p \leq C$ is **strictly mandatory** (even if $f \in L^p$).
  - Since $t \mapsto t^p$ is strictly convex, $(M+1)^p - M^p \approx pM^{p-1} \to \infty$ as $M \to \infty$, so no dominating bound by $|f|^p$ exists without global control on $\{f_n\}$.

Topology: Why Weak Convergence is Not Enough

- **Weak convergence holds naturally:** By Banach–Alaoglu, uniform boundedness in reflexive L^p $(1 < p < \infty)$ combined with $f_n \to f$ a.e. implies:

$$f_n \rightharpoonup f \quad \text{weakly in } L^p.$$

- **The special case $p = 2$ (Hilbert space):**

$$\|f_n - f\|_2^2 = \|f_n\|_2^2 - 2\,\mathrm{Re}\langle f_n, f\rangle + \|f\|_2^2.$$

Since $f_n \rightharpoonup f \implies \langle f_n, f\rangle \to \|f\|_2^2$, the decoupling identity holds **purely from weak convergence**:

$$\|f_n\|_2^2 - \|f_n - f\|_2^2 = 2\,\mathrm{Re}\langle f_n, f\rangle - \|f\|_2^2 \to \|f\|_2^2.$$

- **Failure when $p \neq 2$:** Weak convergence alone fails to preserve the identity without pointwise convergence. Pointwise a.e. control is essential to prevent interference between oscillation frequencies.

General Functionals: The Structural Hypothesis

Let $j : \mathbb{C} \to \mathbb{C}$ be continuous with $j(0) = 0$. We seek conditions under which:

$$\lim_{n\to\infty} \int_\Omega |j(f + g_n) - j(g_n) - j(f)|\, d\mu = 0 \quad \text{(strong } L^1 \text{ convergence)}.$$

Hypothesis (3) of Brezis & Lieb

For every sufficiently small $\epsilon > 0$, there exist continuous nonnegative functions $\varphi_\epsilon, \psi_\epsilon : \mathbb{C} \to [0, \infty)$ such that:

$$|j(a + b) - j(a)| \le \epsilon\varphi_\epsilon(a) + \psi_\epsilon(b) \quad \forall a, b \in \mathbb{C}.$$

Theorem (Theorem 2)

Let j satisfy the hypothesis above and let $f_n = f + g_n$ satisfy:

(i) $g_n \to 0$ *a.e.*,

(ii) $j(f) \in L^1(\Omega)$,

(iii) $\int_\Omega \varphi_\epsilon(g_n)\, d\mu \le C < \infty$ *(constant independent of ϵ and n)*,

(iv) $\int_\Omega \psi_\epsilon(f)\, d\mu < \infty$ *for all $\epsilon > 0$.*

Step-by-Step Proof of Theorem 2

Fix $\epsilon > 0$. Decompose the discrepancy using the positive-part cutoff:

$$W_{\epsilon,n}(x) := \Big[\,|j(f_n(x)) - j(g_n(x)) - j(f(x))| - \epsilon\varphi_\epsilon(g_n(x))\Big]_+, \quad \text{where } [t]_+ = \max(t, 0).$$

1. **Pointwise limit:** Since $g_n \to 0$ a.e. and j is continuous, $j(f_n) - j(g_n) \to j(f)$ a.e., which gives:

$$W_{\epsilon,n}(x) \xrightarrow{n\to\infty} 0 \quad \text{a.e.}$$

2. L^1 **Domination:** Substitute $a = g_n(x)$ and $b = f(x)$ into the structural hypothesis:

$$|j(f_n) - j(g_n) - j(f)| \le |j(g_n + f) - j(g_n)| + |j(f)| \le \epsilon\varphi_\epsilon(g_n) + \psi_\epsilon(f) + |j(f)|.$$

   Subtracting $\epsilon\varphi_\epsilon(g_n)$ yields the n-independent majorant:

$$W_{\epsilon,n}(x) \le \psi_\epsilon(f(x)) + |j(f(x))| \in L^1(\Omega).$$

3. **Lebesgue DCT:** By dominated convergence, $\lim_{n\to\infty} \int_\Omega W_{\epsilon,n}\, d\mu = 0$.

Completion of the Proof of Theorem 2

Define $I_n := \int_\Omega |j(f_n) - j(g_n) - j(f)|\, d\mu$.

- By definition of the positive part $[t]_+ \geq t$, we have:
$$|j(f_n) - j(g_n) - j(f)| \leq W_{\epsilon,n} + \epsilon\varphi_\epsilon(g_n).$$
- Integrate both sides over Ω:
$$I_n \leq \int_\Omega W_{\epsilon,n}\, d\mu + \epsilon \int_\Omega \varphi_\epsilon(g_n)\, d\mu \leq \int_\Omega W_{\epsilon,n}\, d\mu + \epsilon C.$$
- Take lim sup as $n \to \infty$ (recalling $\int W_{\epsilon,n} \to 0$):
$$\limsup_{n\to\infty} I_n \leq 0 + \epsilon C = \epsilon C.$$
- Since this holds for every arbitrarily small $\epsilon > 0$, taking $\epsilon \to 0$ gives:
$$\limsup_{n\to\infty} I_n = 0 \implies \lim_{n\to\infty} I_n = 0. \quad \blacksquare$$

Derivation of Theorem 1 as a Special Case

For $j(t) = |t|^p$, the condition holds with $\varphi_\epsilon(t) = |t|^p$ and $\psi_\epsilon(t) = C_\epsilon|t|^p$. Hypotheses (ii)–(iv) translate exactly to $f \in L^p$ and $\sup_n \|g_n\|_p < \infty$.

Convex Integrands: Lemma 3

Lemma (Lemma 3 – Brezis & Lieb)

Let $j : \mathbb{C} \to \mathbb{R}$ be convex with $j(0) = 0$. Fix $k > 1$. Then for every $\epsilon \in (0, 1/k)$ and $C_\epsilon := \frac{1}{\epsilon(k-1)}$:

$$|j(a+b) - j(a)| \le \epsilon[j(ka) - kj(a)] + |j(C_\epsilon b)| + |j(-C_\epsilon b)| \quad \forall a, b \in \mathbb{C}.$$

Proof Sketch.

Write $(a+b)$ as a convex combination of a, ka, and $C_\epsilon b$:

$$\alpha := 1 - k\epsilon, \quad \beta := \epsilon, \quad \gamma := (k-1)\epsilon \implies \alpha + \beta + \gamma = 1.$$

Then $a + b = \alpha a + \beta(ka) + \gamma(C_\epsilon b)$. By convexity of j:

$$j(a+b) \le \alpha j(a) + \beta j(ka) + \gamma j(C_\epsilon b) \implies j(a+b) - j(a) \le \epsilon[j(ka) - kj(a)] + |j(C_\epsilon b)|.$$

A dual combination expressing a in terms of $(a+b), ka, -C_\epsilon b$ yields the lower bound. □

Counterexample: Failure of Decoupling

What happens if j is convex, but $j(kg_n)$ is not bounded in L^1 for $k > 1$?

Construction on $\Omega = [0, 1]$ with Lebesgue Measure

Let $j(t) = e^{|t|} - 1$, $f(x) \equiv 1$, and define the concentrated sequence:

$$g_n(x) = \begin{cases} \ln(1+n) & \text{if } x \in (0, 1/n), \\ 0 & \text{otherwise.} \end{cases}$$

- $g_n \to 0$ pointwise a.e.
- L^1 norms of parts:

$$\int_0^1 j(f)\,dx = e - 1, \quad \int_0^1 j(g_n)\,dx = \frac{1}{n}\left(e^{\ln(1+n)} - 1\right) = \frac{n}{n} = 1.$$

- Combined function $f_n(x) = 1 + g_n(x)$:

$$\int_0^1 j(f_n)\,dx = \frac{1}{n}\left(e^{1+\ln(1+n)} - 1\right) + \frac{n-1}{n}(e-1) \xrightarrow{n\to\infty} e + (e-1) = 2e - 1.$$

- **The Gap:** $\lim_{n\to\infty} \int j(f_n) = 2e - 1 \neq (e-1) + 1 = e$

Application 1: Hardy–Littlewood–Sobolev Inequality

The Problem

Find the sharp constant and an extremal function achieving the supremum:

$$K = \sup_{f \in L^p \setminus \{0\}} \frac{\|Af\|_q}{\|f\|_p}, \quad \text{where } (Af)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^\lambda}\, dy.$$

Here $0 < \lambda < n$, $1 \le p \le q < \infty$, with balance condition $\frac{1}{p} + \frac{\lambda}{n} = 1 + \frac{1}{q}$.

- Let $\{f_n\}$ be a maximizing sequence: $\|f_n\|_p = 1$ and $\|Af_n\|_q \to K$.
- Using symmetrization (Riesz rearrangement inequality) and scaling, one extracts a subsequence:

$$f_n \to f \neq 0 \text{ a.e.}, \quad Af_n \to Af \text{ a.e.}$$

- **Difficulty:** Weak limit satisfies $\|f\|_p \le 1$ and $\|Af\|_q \le K$. Why can't mass leak to infinity and drop the Rayleigh quotient $\|Af\|_q / \|f\|_p < K$?

Application 1: Proof of Attainment via Theorem 1

Write $f_n = f + g_n$ with $g_n \to 0$ a.e. Apply Theorem 1 to **both** L^p and L^q:

$$\|f_n\|_p^p = \|f\|_p^p + \|g_n\|_p^p + o(1), \quad \|Af_n\|_q^q = \|Af\|_q^q + \|Ag_n\|_q^q + o(1).$$

Raise the maximizing quotient to the power p:

$$K^p = \lim_{n\to\infty} \frac{\|Af_n\|_q^p}{\|f_n\|_p^p} = \lim_{n\to\infty} \frac{\left(\|Af\|_q^q + \|Ag_n\|_q^q\right)^{p/q}}{\|f\|_p^p + \|g_n\|_p^p}.$$

Since $1 \le p \le q$, $t := p/q \le 1$. By subadditivity $(u+v)^t \le u^t + v^t$:

$$K^p \le \limsup_{n\to\infty} \frac{\|Af\|_q^p + \|Ag_n\|_q^p}{\|f\|_p^p + \|g_n\|_p^p}.$$

Using the operator bound $\|Ag_n\|_q^p \le K^p \|g_n\|_p^p$:

$$K^p \left(\|f\|_p^p + \|g_n\|_p^p\right) \le \|Af\|_q^p + K^p \|g_n\|_p^p \implies K^p \|f\|_p^p \le \|Af\|_q^p.$$

Since $f \neq 0$, dividing gives $\frac{\|Af\|_q}{\|f\|_p} \ge K$. Thus, f is an exact maximizer!

Application 2: Brezis–Nirenberg Problem Setup

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) be bounded. Consider the critical Sobolev quotient:

$$K_\lambda = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} R_\lambda(u), \quad R_\lambda(u) := \frac{\int_\Omega |\nabla u|^2\,dx - \lambda \int_\Omega |u|^2\,dx}{\|u\|_{2^*}^2}, \quad 2^* = \frac{2n}{n-2}.$$

Loss of Compactness at the Critical Exponent

The Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is **not compact**.

- For $\lambda = 0$, K_0 is the sharp Sobolev constant on $\mathbb{R}^n$, which is never achieved on any bounded domain Ω.
- **Theorem (Brezis–Nirenberg, 1983):** If $K_\lambda < K_0$ (which holds for $\lambda > 0$ when $n \geq 4$), the infimum K_λ **is achieved**.

Let $\{u_n\}$ be a minimizing sequence normalized so that $\|u_n\|_{2^*} = 1$:

$$\int_\Omega |\nabla u_n|^2\,dx - \lambda \int_\Omega |u_n|^2\,dx \to K_\lambda.$$

Application 2: Step 1 – Non-triviality of the Weak Limit

Since $\{u_n\}$ is bounded in $H_0^1(\Omega)$, there exists a subsequence such that:

$$u_n \rightharpoonup u \text{ in } H_0^1(\Omega), \quad u_n \to u \text{ in } L^2(\Omega) \text{ (Rellich-Kondrachov compact)}, \quad u_n \to u \text{ a.e.}$$

Why is $u \not\equiv 0$?

By definition of the optimal Sobolev constant K_0:

$$\int_\Omega |\nabla u_n|^2 \, dx \geq K_0 \|u_n\|_{2^*}^2 = K_0.$$

Subtracting this from the minimizing relation:

$$\lambda \int_\Omega |u_n|^2 \, dx = \int_\Omega |\nabla u_n|^2 \, dx - K_\lambda + o(1) \geq K_0 - K_\lambda + o(1).$$

Using strong L^2 convergence $\int_\Omega |u_n|^2 \to \int_\Omega |u|^2$:

$$\lambda \int_\Omega |u|^2 \, dx \geq K_0 - K_\lambda > 0 \implies u \neq 0.$$

Application 2: Step 2 – Proving u is a Minimizer

Let $v_n := u_n - u \rightharpoonup 0$ weakly in H_0^1.

1. **Dirichlet integral decoupling (Hilbert structure):**

$$\int_\Omega |\nabla u_n|^2 \, dx = \int_\Omega |\nabla u|^2 \, dx + \int_\Omega |\nabla v_n|^2 \, dx + o(1).$$

2. **L^{2^*} decoupling via Brezis–Lieb (Theorem 1):**

$$1 = \|u_n\|_{2^*}^{2^*} = \|u\|_{2^*}^{2^*} + \|v_n\|_{2^*}^{2^*} + o(1).$$

Since $2^* \geq 2$, the inequality $(1-t)^{2/2^*} \leq 1 - t^{2/2^*}$ yields:

$$1 = \|u_n\|_{2^*}^{2} \leq \|u\|_{2^*}^{2} + \|v_n\|_{2^*}^{2} + o(1).$$

Combine the bounds using $\int |\nabla v_n|^2 \geq K_0 \|v_n\|_{2^*}^2$ and $K_\lambda < K_0$:

$$\int |\nabla u|^2 - \lambda \int |u|^2 + K_0 \|v_n\|_{2^*}^2 \leq K_\lambda \left(\|u\|_{2^*}^2 + \|v_n\|_{2^*}^2 \right) + o(1).$$

Since $K_0 > K_\lambda$, we obtain:

$$\int_\Omega |\nabla u|^2 \, dx - \lambda \int_\Omega |u|^2 \, dx \leq K_\lambda \|u\|_{2^*}^2 \implies R_\lambda(u) \leq K_\lambda.$$

Summary & Lasting Legacy

Core Takeaways:

- Exact calculation of the missing Fatou defect:

$$\|f_n\|_p^p - \|f_n - f\|_p^p \to \|f\|_p^p.$$

- Decouples functionals without requiring compactness.
- The proof technique (dominated cutoff $W_{\epsilon,n}$) generalizes to a wide class of nonlinearities $j(\cdot)$.

Modern Impact:

- Cornerstone of P.-L. Lions' **Concentration-Compactness Principle** (1984).
- Standard tool for critical-exponent PDEs (Yamabe problem, non-linear Schrödinger, harmonic maps).
- Provides the foundational profile decomposition in modern dispersive equations.

Thank You! Questions?