Posets

Swapneel Mahajan

1 Directed graph

1.1 Relation on a set

Let X be any set.

A relation on X is a subset of $X \times X$.

We denote a relation by R, and write $(x, y) \in R$ for its elements.

We may read this as "x is related to y under R".

1.2 Directed graph

A directed graph is a pair (X, R) where X is a set and R is a relation on it.

We visualize elements of X as vertices, and draw an arrow (edge) from y to x whenever $(x, y) \in R$.

Warning: The above is a convention which we will strictly follow. One may instead equally well draw an arrow from x and y.

Example 1.1. Let $X = \{a, b, c\}$.

$$R_1 = \{(c, b), (b, a)\}$$

$$a \longrightarrow b \longrightarrow c$$

$$R_2 = \{(a, a), (a, b), (b, a), (b, c), (c, b)\}$$

$$\circlearrowright a \rightleftarrows b \rightleftarrows c$$

$$R_3 = \{(a, a), (b, c), (c, b)\}$$

$$\circlearrowright a \qquad b \rightleftarrows c$$

1.3 Relation-preserving maps

Let (X, R) and (Y, S) be directed graphs (sets equipped with relations).

A function $f : X \to Y$ is relation-preserving if $(x, y) \in R$ implies $(f(x), f(y)) \in S$.

In other words:

If there is an arrow from y to x in the first graph, then there is an arrow from $f(y)$ to $f(x)$ in the second graph.

Example 1.2. Let us go back to Example 1.1.

Define $f : (X, R_1) \to (X, R_2)$ by

$$f(a) = a, \quad f(b) = b, \quad f(c) = c.$$

This map is relation-preserving. It simply embeds a directed graph into a larger directed graph.

$$a \longrightarrow b \longrightarrow c \quad \longrightarrow \quad \circlearrowright a \rightleftarrows b \rightleftarrows c$$

Define $f' : (X, R_2) \to (X, R_1)$ by

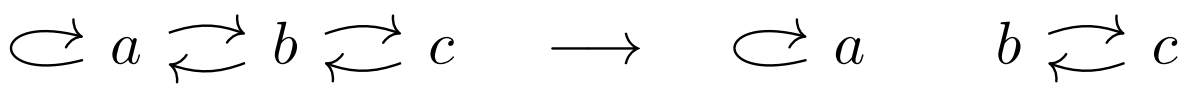
$$f'(a) = a, \quad f'(b) = b, \quad f'(c) = c.$$

This is not a relation-preserving map. For instance, $(a, a) \in R_2$, but $(f'(a), f'(a)) = (a, a) \notin R_1$.

Define $g : (X, R_2) \to (X, R_3)$ by

$$g(a) = a, \quad g(b) = a, \quad g(c) = a.$$

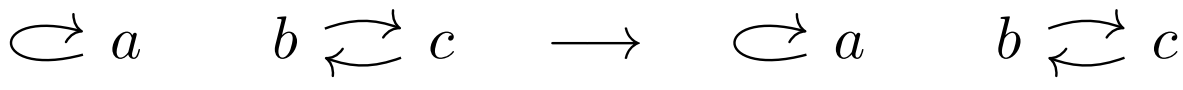
This map is relation-preserving.



Define $h : (X, R_3) \to (X, R_3)$ by

$$h(a) = a, \quad h(b) = c, \quad h(c) = b.$$

This map is relation-preserving.



1.4 Category of directed graphs

Observation 1.3. The composite of relation-preserving maps is again relation-preserving. That is, if $f : (X, R) \to (Y, S)$ and $g : (Y, S) \to (Z, T)$ are relation-preserving maps, then so is their composite $gf : (X, R) \to (Z, T)$.

Further, the identity map $(X, R) \to (X, R)$ is relation-preserving.

This yields a category whose objects are directed graphs and morphisms are relation-preserving maps. We denote this category by $\mathsf{Digraph}$.

1.5 Isomorphism of directed graphs

We say a relation-preserving map $f : (X, R) \to (Y, S)$ is an isomorphism of directed graphs if there is a relation-preserving map $g : (Y, S) \to (X, R)$ such that $gf = \mathrm{id}_X$ (the identity map on X) and $fg = \mathrm{id}_Y$.

We say that g is the inverse of f. It is also an isomorphism whose inverse is f.

In this case, $f : X \to Y$ and $g : Y \to X$ are inverse bijections.

An automorphism of a directed graph (X, R) is an isomorphism from (X, R) to itself.

Example 1.4. Let us go back to Example 1.2.

- $f : (X, R_1) \to (X, R_2)$ defines a bijection from X to itself, but it is not an isomorphism because $f' : (X, R_2) \to (X, R_1)$ is not relation-preserving.
- $g : (X, R_2) \to (X, R_3)$ is not an isomorphism since it is not even a bijection from X to itself.
- $h : (X, R_3) \to (X, R_3)$ is an automorphism. Its inverse is h itself.

1.6 Examples

Example 1.5. Let X be the set of all living humans.

- $(x, y) \in R_1$ if y is a parent of x.
- $(x, y) \in R_2$ if y is an ancestor of x.
- $(x, y) \in R_3$ if y is a sibling of x.

The map $f : (X, R_1) \to (X, R_2)$ defined by $f(x) = x$ is relation-preserving.

Example 1.6. Let X be the set of all students in class.

- $(x, y) \in R_1$ if y is in the same column as x.
- $(x, y) \in R_2$ if y is taller than x.
- $(x, y) \in R_3$ if y gets the same grade as x.

Fix a student $x_0 \in X$. The constant map $f : (X, R_2) \to (X, R_1)$ defined by $f(x) = x_0$ is relation-preserving. This is because $(x_0, x_0) \in R_1$.

Example 1.7. Let us look at number systems.

- For $X = \mathbb{Z}$, $(x, y) \in R$ if $y \neq x$.
- For $X = \mathbb{N}_+$, $(x, y) \in R$ if $gcd(x, y) = 1$.
- For $X = \mathbb{R}$, $(x, y) \in R$ if $|x| = |y|$.

Example 1.8. Let X be the set of all lines in $\mathbb{R}^2$.

- $(x, y) \in R$ if y is parallel to x.
- $(x, y) \in R$ if y is perpendicular to x.

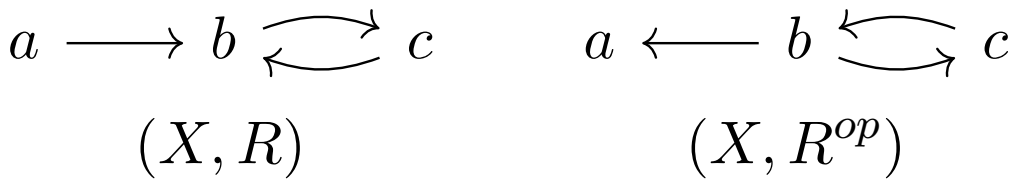
1.7 Opposite directed graph

Let X be a set. Let R be a relation on X. Then define another relation R^{op} on X as follows.

$$(x, y) \in R^{op} \iff (y, x) \in R. \tag{1.1}$$

We call R^{op} the relation opposite to R.

Thus, directed graph (X, R^{op}) is obtained from directed graph (X, R) by reversing direction of all arrows.



(X, R) $\qquad$ (X, R^{op})

1.8 Properties of relations

A relation R on a set X is

- reflexive if $(x, x) \in R$ for all $x \in X$.

  Every vertex has a loop pointing to itself.

- irreflexive if $(x, x) \notin R$ for all $x \in X$.

  There are no loops.

- symmetric if $(x, y) \in R$ implies $(y, x) \in R$ for all $x, y \in X$.

  If there is an arrow $y \to x$, then there is also an arrow $x \to y$.

- antisymmetric if $(x, y) \in R$ and $(y, x) \in R$ implies $x = y$ for all $x, y \in X$.

  Distinct vertices can have at most one arrow between them.

- transitive if $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$ for all $x, y, z \in X$.

  If there is an arrow $z \to y$ and an arrow $y \to x$, then there is also an arrow $z \to x$.

Each of these is a property that a relation either has or does not have.

1.9 Preposet, poset, partitioned set

A preposet is a set equipped with a relation which is transitive and reflexive. We call such a relation a preorder, and denote it by $\lesssim$.

A poset is a set equipped with a relation which is transitive, antisymmetric, reflexive. We call such a relation a partial order, and denote it by $\leq$.

A partitioned set is a set equipped with a relation which is transitive, symmetric, reflexive. We call such a relation an equivalence relation, and denote it by $\sim$.

This is summarized below.

Name	Type of relation
directed graph	arbitrary
preposet	transitive and reflexive
poset	transitive, antisymmetric, reflexive
partitioned set	transitive, symmetric, reflexive

Every poset is a preposet.

Every partitioned set is a preposet.

A preposet may not be a poset or a partitioned set.

Every preposet/poset/partitioned set is a directed set.

2 Preposet

2.1 Preposet

Recall: A preposet is a set P equipped with a relation which is transitive and reflexive.

Such a relation is usually denoted $\lesssim$.

In this notation, the preposet axioms are as follows.

For all $x, y, z \in P$,

- $x \lesssim x$,

  Every vertex has a loop pointing to itself.

- $x \lesssim y$ and $y \lesssim z$ implies $x \lesssim z$.

  If there is an arrow $z \to y$ and an arrow $y \to x$, then there is also an arrow $z \to x$.

2.2 Examples

The following are examples of preorders which are neither symmetric nor antisymmetric.

Example 2.1. Let $\mathbb{R}^{[0,1]}$ be the set of all functions $f : [0, 1] \to \mathbb{R}$. Fix a point $x_0 \in [0, 1]$.

Let $f \lesssim g$ if $f(x_0) \leq g(x_0)$.

Let us go over the properties one by one.

Reflexive: $f(x_0) \leq f(x_0)$.

Transitive: $f(x_0) \leq g(x_0)$ and $g(x_0) \leq h(x_0)$ implies $f(x_0) \leq h(x_0)$.

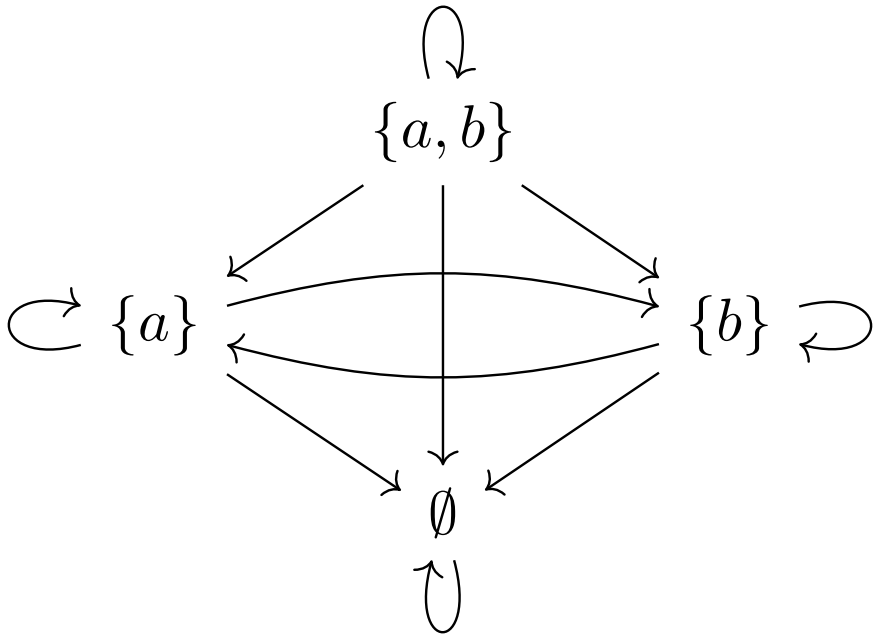
Not antisymmetric: Let f and g be such that $f \neq g$ but $f(x_0) = g(x_0)$. Then $f \lesssim g$ and $g \lesssim f$ but $f \neq g$.

Not symmetric: Let f and g be such that $f(x_0) < g(x_0)$. Then $f \lesssim g$ but $g \not\lesssim f$.

Example 2.2. Let X be any set and $P(X) = 2^X$ denote its power set. An element of $P(X)$ is a subset of X.

Let $A \lesssim B$ if $|A| \leq |B|$.

For $X = \{a, b\}$, $P(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Its preorder, viewed as a directed graph, is shown below.



Example 2.3. Fix $n \geq 1$. Let $M_n(\mathbb{R})$ be the set of all $n \times n$ matrices over $\mathbb{R}$.

Let $A \lesssim B$ if $\det(A) \leq \det(B)$.

Example 2.4. Fix a prime p. For a positive integer n, let $v_p(n)$ be the exponent of p in the prime factorization of n. For example, $v_2(6) = 1$, $v_2(4) = 2$.

Let $\mathbb{N}_+$ be the set of positive integers.

Let $x \lesssim_p y$ if $v_p(x) \leq v_p(y)$.

2.3 Preorder-preserving map

Let P and Q be preposets.

A function $f : P \to Q$ is preorder-preserving if $x \lesssim y$ implies $f(x) \lesssim f(y)$.

Observe: This is the same as a relation-preserving map from P to Q.

2.4 Category of preposets

We have a category whose objects are preposets and morphisms are preorder-preserving maps. We denote this category by $\mathsf{Preposet}$. It is a full subcategory of $\mathsf{Digraph}$.

We say a preorder-preserving map $f : P \to Q$ is an isomorphism of preposets if there is a preorder-preserving map $g : Q \to P$ such that $gf = \mathrm{id}_P$ (the identity map on P) and $fg = \mathrm{id}_Q$.

An automorphism of a preposet P is an isomorphism from P to itself.

2.5 Opposite preposet

For a preposet $(P, \lesssim)$, define another relation $\lesssim^{op}$ on P by

(2.1) $$x \lesssim^{op} y \text{ if } y \lesssim x.$$

Check that this defines a preorder on P.

This is the opposite preorder and P equipped with this order is the opposite preposet.

We denote it by P^{op}.

Example 2.5. Consider preposet $P(X) = (2^X, \lesssim)$ in Example 2.2. Its opposite preposet $P(X)^{op} = (2^X, \lesssim^{op})$ is as follows.

Let $A \lesssim^{op} B$ if $|A| \geq |B|$.

For $X = \{a, b\}$, the directed graph of $P(X)^{op}$ is obtained by reversing the arrows of the directed graph shown there.

3 Partitioned set

3.1 Equivalence relation

Recall: An equivalence relation on a set X is a relation which is transitive, symmetric, reflexive.

Such a relation is usually denoted $\sim$. In other words, we write $x \sim y$ when x is related to y. We read this as "x is equivalent to y".

In this notation, the definition of an equivalence relation takes the following form.

For all $x, y, z \in X$,

- $x \sim x$,
- $x \sim y$ implies $y \sim x$,
- $x \sim y$ and $y \sim z$ implies $x \sim z$.

An equivalence relation allows us to write X as a disjoint union of equivalence classes.

The equivalence class of an element x under $\sim$, denoted $[x]$, is defined as

$$[x] := \{y \in X \mid x \sim y\}.$$

For any $x, y \in X$,

if $x \sim y$, then $[x] = [y]$,

else $[x]$ and $[y]$ are disjoint (that is, their intersection is the empty set).

Let

$$X_\sim := \{[x] \mid x \in X\}. \tag{3.1}$$

This is the set whose elements are the equivalence classes of $\sim$.

Note that there is a canonical quotient map

$$X \twoheadrightarrow X_\sim, \qquad x \mapsto [x]. \tag{3.2}$$

3.2 Examples

Example 3.1. Let $X = \mathbb{R}^2$.

Let $(x_1, y_1) \sim (x_2, y_2)$ if (x_1, y_1) and (x_2, y_2) are at the same distance from the origin, that is, $\sqrt{x_1^2 + y_1^2} = \sqrt{x_2^2 + y_2^2}$.

Check that this is an equivalence relation.

Let C_r denote the circle of radius r, that is,

$$C_r = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 = r^2\}.$$

The class $[(x, y)]$ equals C_r for $r^2 = x^2 + y^2$.

The set of equivalence classes is

$$X_\sim = \{C_r \mid r \in \mathbb{R}, r \geq 0\}.$$

The map

$$X \twoheadrightarrow X_\sim, \qquad (x, y) \mapsto C_{\sqrt{x^2+y^2}}.$$

Example 3.2. Let $\mathbb{R}^{[0,1]}$ be the set of all functions $f : [0, 1] \to \mathbb{R}$. Fix a specific point $x_0 \in [0, 1]$.

Let $f \sim g$ if $f(x_0) = g(x_0)$.

Check that this is an equivalence relation.

Let P_r be the set of all functions g in X with $g(x_0) = r$. The class $[f]$ equals P_r for $r = f(x_0)$. The set of equivalence classes is

$$X_\sim = \{P_r \mid r \in \mathbb{R}\}.$$

The map

$$X \twoheadrightarrow X_\sim, \qquad f \mapsto P_{f(x_0)}.$$

Example 3.3. Let $M_2(\mathbb{R})$ be the set of all 2×2 matrices over $\mathbb{R}$.

Let $A \sim B$ if A and B are similar, that is, there exists an invertible matrix P such that $B = P^{-1}AP$.

Check that this is an equivalence relation.

How do you describe the set of all equivalence classes?

3.3 Equivalence relation-preserving maps

Let X and Y be partitioned sets.

A function $f : X \to Y$ is equivalence relation-preserving if $x \sim y$ implies $f(x) \sim f(y)$.

It induces a function $f_\sim : X_\sim \to Y_\sim$ fitting into the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X_\sim & \xrightarrow[f_\sim]{} & Y_\sim. \end{array} \tag{3.3}$$

3.4 Category of partitioned sets

We have the category whose objects are partitioned sets and morphisms are equivalence relation-preserving maps. We denote this category by $\mathsf{Partition}$. It is a full subcategory of $\mathsf{Preposet}$.

We say an equivalence relation-preserving map $f : X \to Y$ is an isomorphism of partitioned sets if there is an equivalence relation-preserving map $g : Y \to X$ such that $gf = \mathrm{id}_X$ (the identity map on X) and $fg = \mathrm{id}_Y$.

4 Poset

4.1 Partially ordered set

Recall: A poset is a set P equipped with a relation which is transitive, reflexive, antisymmetric.

Such a relation is usually denoted $\leq$.

We read $x \leq y$ as "x is less than y", or "y is greater than x".

In this notation, the poset axioms are as follows.

For all $x, y, z \in P$,

- $x \leq x$,
- $x \leq y$ and $y \leq x$ implies $x = y$,
- $x \leq y$ and $y \leq z$ implies $x \leq z$.

Poset is a short form for partially ordered set. This term was coined by Birkhoff in the third edition of his famous book on Lattice Theory (1940, 1948, 1967).

If $x \leq y$ but $x \neq y$, then we write $x < y$, and say "x is strictly smaller than y" or "y is strictly greater than x".

We say y covers x whenever $x < y$ and there is no element strictly between x and y. We denote this by $x \lessdot y$.

We say x and y are comparable if either $x \leq y$ or $y \leq x$.

We say x and y are incomparable if neither $x \leq y$ nor $y \leq x$.

4.2 Order-preserving map

Let P and Q be posets.

A function $f : P \to Q$ is order-preserving if $x \leq y$ implies $f(x) \leq f(y)$.

In addition:

An order-preserving map is strictly order-preserving if $x < y$ implies $f(x) < f(y)$ for all $x, y \in P$.

An order-preserving map is cover relation-preserving if $x \lessdot y$ implies $f(x) \lessdot f(y)$ for all $x, y \in P$.

4.3 Category of posets

We have the category whose objects are posets and morphisms are order-preserving maps. We denote this category by Poset. It is a full subcategory of Preposet.

We say an order-preserving map $f : P \to Q$ is an isomorphism of posets if there is an order-preserving map $g : Q \to P$ such that $gf = \mathrm{id}_P$ (the identity map on P) and $fg = \mathrm{id}_Q$.

An automorphism of a poset P is an isomorphism from P to itself.

4.4 Linearly ordered set

A poset P is linearly ordered or totally ordered if for any $x, y \in P$ either $x \leq y$ or $y \leq x$, that is, x and y are comparable.

We can consider full subcategory Lin of Poset whose objects are linearly ordered sets, and whose morphisms are order-preserving maps.

Example 4.1. The sets of natural numbers $\mathbb{N}$, integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$, real numbers $\mathbb{R}$ are all linearly ordered under their usual order.

4.5 Hasse diagram

Since every partial order is a relation, every poset P is a directed graph.

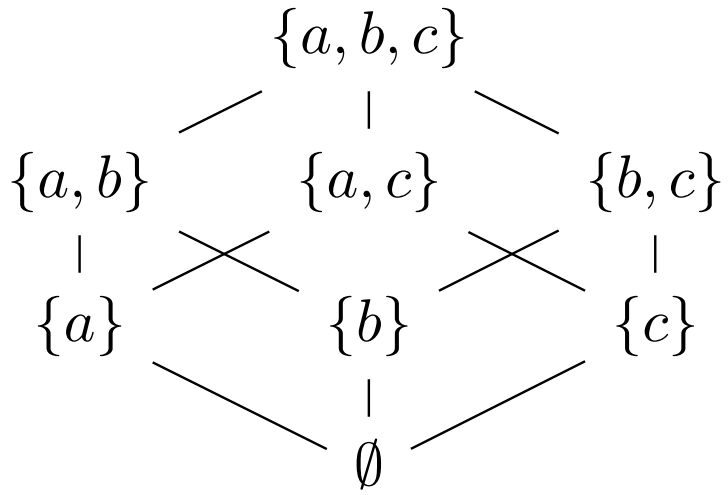
When P is finite or more generally locally finite (Section 9), we can draw a smaller directed graph called the Hasse diagram. We remove loops, and arrows implied by transitivity. In other words, we only draw arrows corresponding to cover relations.

Further, bigger elements are customarily written above smaller elements, which makes the arrows on the edges redundant.

4.6 Examples

Example 4.2 (Boolean poset)**.** Let X be any set and $P(X) = 2^X$ denote its power set. This is a poset under inclusion, that is, we say $A \leq B$ if A is a subset of B. This is the Boolean poset of X.

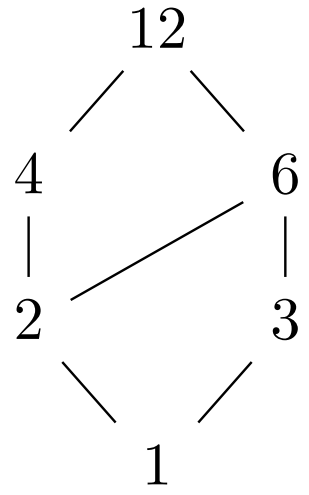
The Hasse diagram of the Boolean poset of $\{a, b, c\}$ is shown below.



Define $f : P(X) \to \mathbb{N}$ by $f(A) = |A|$. Then f is order-preserving since $A \subseteq B$ implies $|A| \leq |B|$.

Example 4.3 (Poset of divisors)**.** Fix a natural number $N \geq 1$. Let D_N be the set of all divisors of N. We say $a \leq b$ if a divides b (often written as $a|b$).

For $N = 12$, the Hasse diagram of D_{12} is shown below.



The map $f : D_N \to \mathbb{N}$ defined by $f(a) = a$ is order-preserving. This is because $a|b$ implies $a \leq b$.

Example 4.4. Let $X = \mathbb{R}^2$. We say $(x_1, y_1) \leq (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$.

The projection maps $p_1, p_2 : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$p_1(x, y) = x \quad \text{and} \quad p_2(x, y) = y$$

are order-preserving.

Example 4.5. An order-preserving map $f : \mathbb{R} \to \mathbb{R}$ is the same as an increasing function.

Similarly, a strictly order-preserving map $f : \mathbb{R} \to \mathbb{R}$ is the same as a strictly increasing function.

Example 4.6 (Poset of subspaces)**.** Fix a field $\mathbb{k}$. Let W be a vector space over $\mathbb{k}$, and $\mathrm{Sub}(W)$ be the set of all its subspaces.

This is a poset under inclusion, that is, we say $U \leq V$ if U is a subspace of V.

The map $f : \mathrm{Sub}(W) \to \mathbb{N}$ defined by $f(V) = \dim(V)$ is order-preserving.

Example 4.7. Let $\mathbb{R}^{[0,1]}$ be the set of all real-valued functions $f : [0, 1] \to \mathbb{R}$. We say $f \leq g$ if $f(x) \leq g(x)$ for all $x \in [0, 1]$.

The map $F : \mathbb{R}^{[0,1]} \to \mathbb{R}$ which sends f to $f(1/2)$ is order-preserving. This is because $f \leq g$ implies $f(1/2) \leq g(1/2)$.

4.7 Opposite poset

For a poset $(P, \leq)$, define another relation $\leq^{op}$ on P by

(4.1) $$x \leq^{op} y \text{ if } y \leq x.$$

Check that this defines a partial order on P.

This is the opposite partial order and P equipped with this order is the opposite poset.

We denote it by P^{op}.

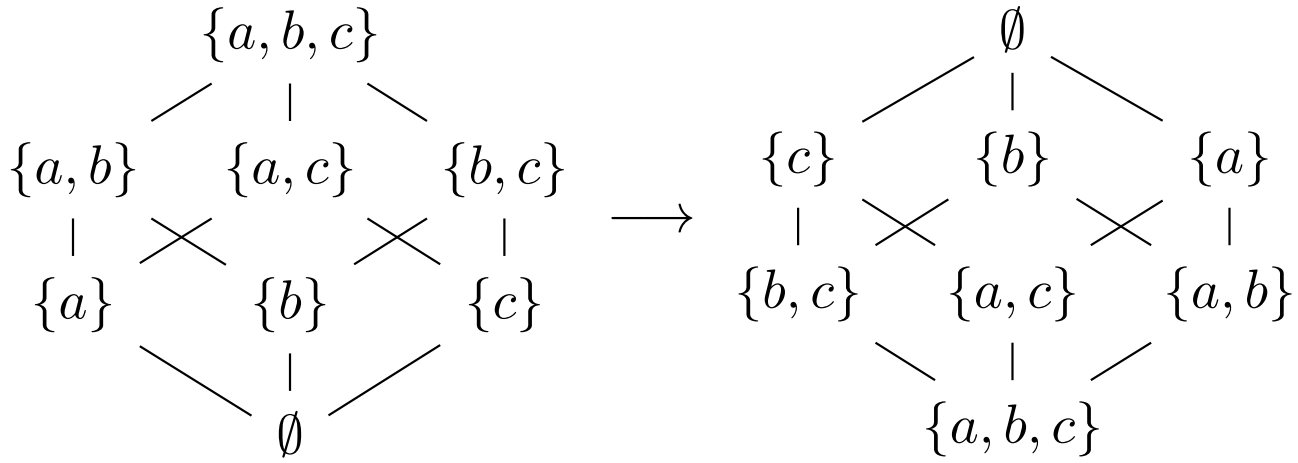
A poset is self-dual if it is isomorphic to its dual.

Example 4.8. The Boolean poset $P(X)$ is self-dual (under taking complements). That is, the map

$$P(X) \xrightarrow{\cong} P(X)^{op}, \qquad A \mapsto X \setminus A \tag{4.2}$$

is a poset isomorphism.

The case $X = \{a, b, c\}$ is illustrated below.

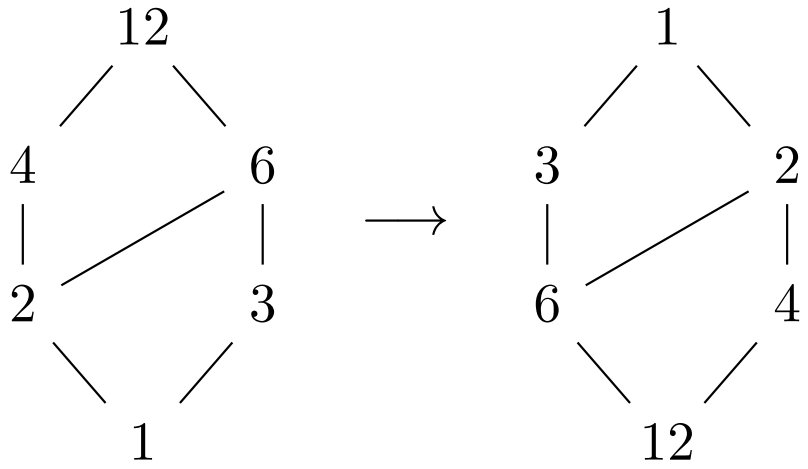


Example 4.9. Poset of divisors D_N is self-dual under the map

$$D_N \xrightarrow{\cong} D_N^{op}, \qquad d \mapsto N/d \tag{4.3}$$

is a poset isomorphism.

The case $N = 12$ is illustrated below.

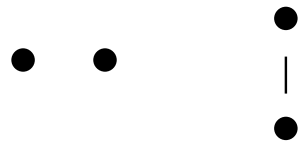


4.8 Posets up to isomorphism

We say two posets P and Q are isomorphic if there is an isomorphism $f : P \to Q$ between them.

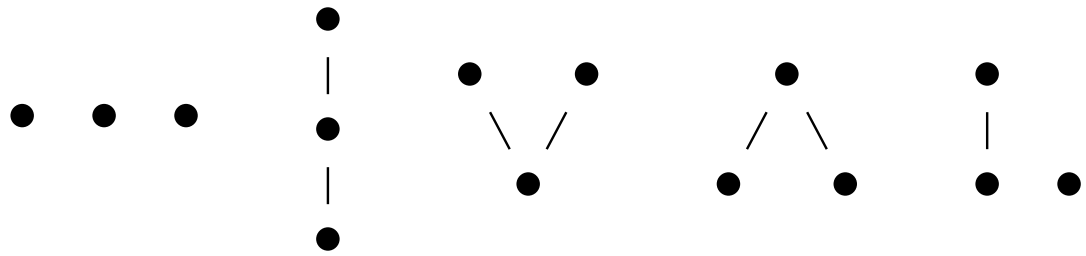
Let us classify posets up to isomorphism using number of elements.

- $n = 0$: Exactly 1 poset up to isomorphism.
  1. The empty poset (no elements, empty relation).
- $n = 1$: Exactly 1 poset up to isomorphism.
  1. A single isolated point.
- $n = 2$: Exactly 2 posets up to isomorphism.



  1. An antichain of 2 elements.
  2. A chain of 2 elements.

- $n = 3$: Exactly 5 posets up to isomorphism.



1. An antichain of 3 elements.
2. A chain of 3 elements.
3. One element below two incomparable elements.
4. Two incomparable elements below one element.
5. A 2-element chain next to 1 isolated element.

4.9 Subposet and full subposet

Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be posets, with P a subset of Q.

We say $(P, \leq_P)$ is a subposet of $(Q, \leq_Q)$ if

$$x \leq_P y \implies x \leq_Q y. \tag{4.4}$$

We say $(P, \leq_P)$ is a full subposet of $(Q, \leq_Q)$ if

$$x \leq_P y \iff x \leq_Q y. \tag{4.5}$$

Example 4.10. Under the usual linear orders, $\mathbb{N}$ is a full subposet of $\mathbb{Z}$ which is a full subposet of $\mathbb{Q}$ which is a full subposet of $\mathbb{R}$.

5 Problem set I

5.1 Test your understanding

Say whether each of the following statements is true or false.

1. There is a bijection between $\emptyset$ and $\{\emptyset\}$.
2. The power set of $\{\{1\}, \emptyset\}$ has 4 elements.
3. For (X, R) a directed graph, there is a unique relation-preserving map $(\emptyset, \emptyset) \to (X, R)$.
4. For (X, R) a directed graph, there is a unique relation-preserving map $(X, R) \to (\{*\}, \{(*, *)\})$.
5. Let X be a set and (Y, S) be a directed graph. Any function $f : X \to Y$ defines a relation-preserving map $(X, \emptyset) \to (Y, S)$.
6. For (X, R) a directed graph, there is a bijection between set X and set of all relation-preserving maps $(\{*\}, \emptyset) \to (X, R)$.
7. Let (X, R) be a directed graph and Y a set. A function $f : X \to Y$ defines a relation-preserving map $(X, R) \to (Y, f_* R)$, where $f_* R := \{(y, y') \in Y \times Y \mid \exists\, (x, x') \in R$ such that $f(x) = y,\ f(x') = y'\}$.
8. Let X be a set and (Y, S) a directed graph. A function $f : X \to Y$ defines a relation-preserving map $(X, f^* S) \to (Y, S)$, where $f^* S := \{(x, x') \in X \times X \mid (f(x), f(x')) \in S\}$.
9. A relation-preserving bijection is an isomorphism of directed graphs.
10. A Boolean poset is linearly ordered.
11. There is a Boolean poset which is linearly ordered.

5.2 Problems related to theory

1. Let R be a relation on X, and S a relation on Y. A map $f : (X, R) \to (Y, S)$ is relation-reversing if $(x, y) \in R$ implies $(f(y), f(x)) \in S$.

Suppose $f : (X, R) \to (Y, S)$ is relation-preserving. Then what can you say about $f : (X, R) \to (Y, S^{op})$, $f : (X, R^{op}) \to (Y, S)$, $f : (X, R^{op}) \to (Y, S^{op})$?

What happens if we compose a relation-preserving map with a relation-reversing map?

What happens if we compose two relation-reversing maps?

All of the above can also be asked in the context of preposets or posets. The basic definition is: A function between posets $f : P \to Q$ is order-reversing if $x \leq y$ implies $f(y) \leq f(x)$.

2. We defined the opposite preposet and opposite poset. What would happen if we try to define the opposite partitioned set?

3. Let $(P, \lesssim)$ be a preposet. Define a relation $\sim$ on P as follows.

$$x \sim y \text{ if } x \lesssim y \text{ and } y \lesssim x. \tag{5.1}$$

Check that $\sim$ is an equivalence relation. Show that $\lesssim$ induces a partial order on the set of equivalence classes $P_\sim$. So we can visualize a preposet by drawing the Hasse diagram of $P_\sim$ and explicitly writing out the elements in each equivalence class.

4. Let $f : X \twoheadrightarrow Y$ be a surjective map. Define a relation $\sim$ on X as follows.

$$x \sim y \text{ if } f(x) = f(y). \tag{5.2}$$

Check that $\sim$ is an equivalence relation. Moreover, $f : X \twoheadrightarrow Y$ factors as $X \twoheadrightarrow X_\sim \xrightarrow{\cong} Y$, where the first map is the canonical quotient map, and the second map is a bijection.

5. Suppose a relation $\sim$ on a set X is symmetric and transitive. We claim that $\sim$ is reflexive by the following argument. Suppose we are given any $x \in X$. Pick a y such that $x \sim y$. By symmetry, $y \sim x$, and by transitivity $x \sim x$. What is wrong with this argument?

6. Let P be a finite linearly ordered set. Prove that every order-preserving map $f : P \to P$ has at least one fixed point (an element $x \in P$ such that $f(x) = x$). Is this true if P is infinite?

7. Any poset $(P, \leq_P)$ is a full subposet of itself. Give an example of a set P with two different partial orders $\leq$ and $\leq'$ such that $(P, \leq')$ is a subposet of $(P, \leq)$, but is not a full subposet.

8. Let P be any poset. Show that P can be embedded into Boolean poset $P(X)$ for some set X. That is, P is isomorphic as a poset to a full subposet of $P(X)$.

5.3 Problems related to examples

1. On any set X, we have empty relation $R = \emptyset$. What properties does this relation satisfy? That is, is it reflexive, irreflexive, ...?

2. For each example of a directed graph (set equipped with a relation) that we have discussed, say whether it is reflexive, irreflexive, symmetric, antisymmetric, transitive.

   Do the same for the directed graph on $\mathbb{R}$ with relation defined by $(x, y) \in R$ if $|x - y| \leq 1$.

3. For a positive integer n, let $\omega(n)$ be the number of distinct prime factors of n. For example, $\omega(6) = 2, \omega(8) = 1$. Let X be the set of positive integers. Let $x \lesssim y$ if $\omega(x) \leq \omega(y)$. Check that this is a preorder which is neither symmetric nor antisymmetric.

4. Recall preorder $\lesssim_p$ on positive integers X from Example 2.4. Let p and q be distinct primes. Determine whether the identity map $f : (X, \lesssim_p) \to (X, \lesssim_q)$ defined by $f(x) = x$ is preorder-preserving?

5. Let $M_3(\mathbb{R})$ be the set of all 3×3 matrices over $\mathbb{R}$. We say $A \lesssim B$ if $\text{rank}(A) \leq \text{rank}(B)$. Show that $\lesssim$ is a preorder. Is it a partial order?

6. Let $C[0, 1]$ be the set of all continuous real-valued functions on $[0, 1]$. We say $f \lesssim g$ if $\int_0^1 f(x)\, dx \leq \int_0^1 g(x)\, dx$. Show that this yields a preposet but not a poset or a partitioned set.

7. Consider the following relation on $\mathbb{N} \times \mathbb{N}$:

   $$(m, n) \sim (m', n') \iff m + n' = n + m'. \tag{5.3}$$

   Check that this is an equivalence relation. Do you see that $(\mathbb{N} \times \mathbb{N})_\sim$ is the same as $\mathbb{Z}$? (One can in fact define $\mathbb{Z}$ in this manner.) How would you define the operations of addition and multiplication on these equivalence classes?

8. Fix a positive integer n. Take $X = \mathbb{Z}$. Let $x \sim y$ if $y - x$ is divisible by n. Check that this is an equivalence relation on $\mathbb{Z}$. Describe the equivalence classes and the quotient map $\mathbb{Z} \to \mathbb{Z}_\sim$.

9. Fix positive integers m and n. Let X be the set of all $m \times n$-matrices over $\mathbb{R}$. We say $A \sim B$ if there exist invertible $m \times m$ matrix P and $n \times n$ matrix Q such that $B = PAQ$. Check that $\sim$ is an equivalence relation. Describe set $X_\sim$ of equivalence classes.

10. Give examples to differentiate between order-preserving maps, strictly order-preserving maps, cover relation-preserving maps.

11. For $n \geq 1$, let $[n]$ denote the linearly ordered set $1 < 2 < \cdots < n$.
    (a) For $n = 3$, list all order-preserving maps from $[n]$ to itself. Draw Hasse diagram of the poset formed by these maps ordered pointwise ($f \leq g$ iff $f(i) \leq g(i)$ for all $i \in [n]$).
    (b) Show that for $n \geq 1$, number of such order-preserving maps is $\binom{2n-1}{n}$. (Count how many elements are mapped to each value $i \in [n]$ and apply a Stars and Bars counting argument.)

12. Describe all order-preserving maps from $\mathbb{N}$ to $\mathbb{Z}$ (or to any linearly ordered set).

13. What are all poset automorphisms of $\mathbb{N}$? of $\mathbb{Z}$?

14. How many posets with 4 elements are there up to isomorphism?

15. How many directed graphs, preposets, partitioned sets with $n = 0, 1, 2, 3$ elements are there up to isomorphism?

16. What are all poset automorphisms of Boolean poset $P(X)$?

17. Is it possible that D_N and $P(X)$ are isomorphic as posets for suitable N and X?

18. Let $\Bbbk := \mathbb{F}_2 = \{0, 1\}$ be the finite field with two elements. Make poset of subspaces $\mathrm{Sub}(W)$ in Example 4.6 explicit when W has dimension $0, 1, 2, 3$. What connection does this have to the Fano plane?

19. Show: Poset of subspaces $\mathrm{Sub}(W)$ is self-dual when W is finite-dimensional.

5.4 Problems related to category theory

We assume basic knowledge of categories and functors.

1. Consider the gadget whose objects are directed graphs and morphisms are relation-reversing maps. Does this yield a category under usual composition of functions? What happens if we

consider the gadget whose objects are directed graphs and morphisms are either relation-preserving or relation-reversing maps?

2. For objects a and b in a category C, we say $a \sim b$ if a is isomorphic to b. Show that this defines an equivalence relation on the set of objects of C. Can you describe the equivalence classes when $\mathsf{C} = \mathsf{Set}$ and when $\mathsf{C} = \mathsf{Vec}_{\mathbb{k}}$?

3. Describe the initial object, terminal object, product, coproduct in Poset.

4. Every set X is a poset with the discrete partial order: $x \leq y$ iff $x = y$. Check that this defines a functor from Set to Poset. How does this relate to the forgetful functor from Poset to Set?

   Can you extend the construction of a power set to a functor

   $$\mathsf{Set} \to \mathsf{Poset}, \qquad X \mapsto P(X)?$$

5. Check that the assignment mapping a directed graph (X, R) to its opposite directed graph (X, R^{op}) defines a functor from category $\mathsf{Digraph}$ to itself. In fact, it is an isomorphism from $\mathsf{Digraph}$ to itself whose inverse is itself.

   Similar statements hold for categories $\mathsf{Preposet}$, $\mathsf{Partition}$, Poset.

6. Give an example of an order-preserving bijection that is not an isomorphism in Poset.

7. Show that sending a partitioned set X to its set of equivalence classes $X_{\sim}$ defines a functor from category of partitioned sets $\mathsf{Partition}$ to category of sets Set.

8. Check:

   - A preposet is the same as a category with at most one morphism from one object to another.
   - A partitioned set is the same as a groupoid with at most one morphism from one object to another.
   - A poset is the same as a category with at most one morphism between any two objects.

   (Reflexivity leads to identity morphisms, and transitivity to composition of morphisms.)

   Further, a preorder-preserving/equivalence relation-preserving/order-preserving map is the same as a functor between the corresponding categories.

9. To each poset P associate a category C_P whose objects are elements of P, and there is a unique morphism from x to y whenever $x \leq y$. Check that the initial object, terminal

object, product, coproduct exists in C_P iff the bottom element, top element, meet, join exists in P (respectively). Do you see that an order-preserving map $P \to Q$ yields a functor $\mathsf{C}_P \to \mathsf{C}_Q$? Thus, in fact, we dealing here with a functor from Poset to the category of all categories.

Note: Notion of subposet corresponds to notion of subcategory, and that of full subposet corresponds to that of full subcategory.

5.5 Explorations

1. Let Δ denote the full subcategory of Lin consisting of finite linearly ordered sets of the form $0 < 1 < \cdots < n$ for $n \geq 0$. Thus, objects of this category are indexed by $\mathbb{N}$. This category forms the base for the notion of a simplicial set which plays an important role in homotopy theory and homological algebra.

6 Dilworth and Mirsky

6.1 Chain and antichain

Let P be a poset.

- A chain in P is a subset $C \subseteq P$ such that for all $x, y \in C$, either $x \leq y$ or $y \leq x$, that is, x and y are comparable.
- An antichain in P is a subset $A \subseteq P$ such that for all distinct $x, y \in A$, $x \not\leq y$ and $y \not\leq x$, that is, x and y are incomparable.

Note: P is a chain iff P is a linearly ordered set.

P is an antichain iff P is a discrete poset, that is, no two distinct elements are comparable.

6.2 Height and width

Let P be a finite poset.

- The height of P, denoted $h(P)$, is the maximum size $|C|$ of a chain C in P.
- The width of P, denoted $w(P)$, is the maximum size $|A|$ of an antichain A in P.

6.3 Dilworth's and Mirsky's theorems

Let P be a finite poset.

Theorem 6.1 (Dilworth, 1950)**.** *The width of P equals the minimum number of pairwise disjoint chains needed to partition P.*

Proof. See the proof of Theorem 8.2 in Jukna's book. □

Theorem 6.2 (Mirsky, 1970)**.** *The height of P equals the minimum number of pairwise disjoint antichains needed to partition P.*

Proof. See the proof of Theorem 8.1 in Jukna's book. □

Example 6.3. Let us look at poset D_{12} from Example 4.3. The chains of maximum size are

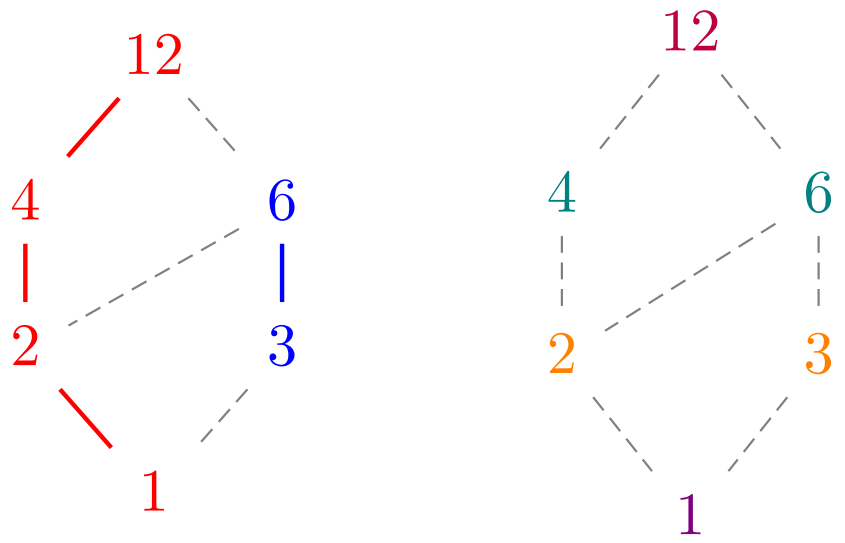
$$1 < 2 < 4 < 12, \quad 1 < 2 < 6 < 12, \quad 1 < 3 < 6 < 12.$$

So the height of the poset is 4.

The antichains of maximum size are

$$\{2, 3\}, \quad \{3, 4\}, \quad \{4, 6\}.$$

So the width of the poset is 2.



So we can partition the poset into two chains by Dilworth, and into four antichains by Mirsky.

7 Problem set II

7.1 Test your understanding

Say whether each of the following statements is true or false.

1. If P and Q are isomorphic posets, then they have the same height and width.
2. A discrete poset of size $n \geq 1$ has height 1 and width n.
3. If a finite poset has width k, then it can be partitioned into k pairwise disjoint antichains.
4. If P has n elements and contains no chain of size $r+1$, then $w(P) \geq \left\lceil \frac{n}{r} \right\rceil$.
5. If P has n elements and contains no antichain of size $s+1$, then $h(P) \geq \left\lceil \frac{n}{s} \right\rceil$.
6. If a finite poset P has a unique maximum chain, then $w(P) = 1$.
7. Suppose P has width k. If P is partitioned into k chains, then every maximum antichain intersects every chain of the partition in exactly one element.
8. Let $G = (X, E, Y)$ be a finite bipartite graph. If $|N(X)| \geq |X|$, then there exists a matching that saturates X.

7.2 Problems related to theory

1. Let P be a finite poset. Show: P and P^{op} have the same height, and P and P^{op} have the same width.
2. Let P be a nonempty finite poset. Show:
   - P is a chain iff P has width 1,
   - P is an antichain iff P has height 1.

   Deduce: There is a sequence $\{P_n\}$ of posets such that $\lim_{n\to\infty} |P_n| = \infty$, but $\{w(P_n)\}$ is a bounded sequence. And a similar statement with $h(P_n)$ instead of $w(P_n)$.
3. What happens to Dilworth's and Mirsky's theorems when P is the empty poset? Note: The empty poset has width 0 and height 0.

4. Let P be a finite poset, and $k \geq 0$. Show:
   - If P can be partitioned into k chains, then any antichain in P can have size at most k.
   - If P can be partitioned into k antichains, then any chain in P can have size at most k.

5. Let P be a finite graded poset with bottom element $\bot$. It has height $\mathrm{rk}(P) + 1$.
   (a) Show that grouping elements by their rank layers $L_i = \{x \in P \mid \mathrm{rk}(x) = i\}$ is an optimal partition of P into $\mathrm{rk}(P) + 1$ antichains.
   (b) The above rank-based partition may not be the only optimal partition into $\mathrm{rk}(P) + 1$ antichains. Give an example of P where this occurs.
   (c) What happens for Boolean posets? This is for exploration.

6. A bipartite graph G is a triple (X, E, Y), where X and Y are sets, and $E \subseteq X \times Y$. We view elements of X and Y as vertices (distinct from one another), and elements of E as edges. For $A \subseteq X$, define $N(A) \subseteq Y$ by

$$N(A) = \{y \in Y \mid \exists x \in A \text{ such that } (x, y) \in E\}.$$

   A matching in G is a subset of edges $M \subseteq E$ such that no two edges in M share a common vertex. It saturates (or covers) X if every single vertex in X is an endpoint of exactly one edge in M.
   Theorem 7.1 (Hall, 1935). *Let $G = (X, E, Y)$ be a finite bipartite graph. There exists a matching in G that saturates X iff for every $A \subseteq X$, we have $|N(A)| \geq |A|$.*
   Prove the above result, known as Hall's marriage theorem, by applying Dilworth's theorem to a suitable poset.
   For information related to this theorem, see Jukna, Chapter 5.

7.3 Problems related to examples

1. Consider the 'crown' poset of $2n$ elements $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ where $a_i \leq b_i$ and $a_i \leq b_{i+1}$ for all $1 \leq i \leq n$ (with $b_{n+1} = b_1$). Find its height, width, and explicitly write out all its minimal chain partitions and minimal antichain partitions.

2. Consider the 'fence' poset of n elements $\{x_1, x_2, \ldots, x_n\}$ where $x_i \leq x_{i+1}$ for all odd $i < n$ and $x_i \geq x_{i+1}$ for all even $i < n$. Find its height, width, and explicitly write out all its minimal chain partitions and minimal antichain partitions.

7.4 Explorations

1. What is the height and width of the Boolean poset? Give illustrations of Dilworth's and Mirsky's theorem. Look up Sperner's theorem.

2. Let P be a (possibly infinite) poset.

   - The height of P, denoted $h(P)$, is supremum of cardinalities of all chains in P:

   $$h(P) = \sup\{|C| \mid C \subseteq P \text{ is a chain}\}$$

   - The width of P, denoted $w(P)$, is supremum of cardinalities of all antichains in P:

   $$w(P) = \sup\{|A| \mid A \subseteq P \text{ is an antichain}\}$$

   It is possible that $h(P)$ or $w(P)$ is ∞. For example, $\mathbb{N}$ has height ∞ and width 1. An infinite discrete poset has height 1 and width ∞. Give examples:

   - Poset whose height is ∞, but each chain is finite.
   - Poset with height and width both ∞.

8 Lattice

8.1 Minimum and maximum

Let P be a poset. Let $x \in P$.

- We say x is a minimal element if $y \leq x$ implies $y = x$.
- We say x is a maximal element if $x \leq y$ implies $y = x$.

A poset may have many minimal and maximal elements.

A poset may have no minimal or maximal elements.

It is possible that an element is both a minimal and maximal element.

Let P be a poset.

- We say $\bot$ is the minimum element or bottom element of P if $\bot \leq x$ for all $x \in P$.
- We say $\top$ is the maximum element or top element of P if $x \leq \top$ for all $x \in P$.

A minimum element is a minimal element, but the converse is not true.

A maximum element is a maximal element, but the converse is not true.

Minimum, maximum elements may not exist, but they are unique whenever they exist.

We say P is bounded if both $\bot$ and $\top$ exist.

8.2 Meets and joins

Let P be a poset. Let $x, y, z \in P$.

- We say z is a lower bound of x and y if z is smaller than both x and y, that is, if $z \leq x$ and $z \leq y$.
- We say z is an upper bound of x and y if z is larger than both x and y, that is, if $x \leq z$ and $y \leq z$.

Elements x and y may have many lower and upper bounds.

Elements x and y may have no lower and upper bounds.

Let P be a poset. Let $x, y \in P$.

- The meet or greatest lower bound of x and y, denoted $x \wedge y$, is the largest element of P smaller than both x and y. That is,

  $x \wedge y \leq x, \quad x \wedge y \leq y, \quad$ and

  $z \leq x$ and $z \leq y$ implies $z \leq x \wedge y$.

- The join or least upper bound of x and y, denoted $x \vee y$, is the smallest element of P larger than both x and y.

Meets and joins may not exist, but they are unique whenever they exist.

8.3 Lattice

A poset is a lattice if meets and joins of any two elements exist.

A poset is a meet semilattice if meets of any two elements exist.

A poset is a join semilattice if joins of any two elements exist.

Note: A poset is a lattice iff it is a meet semilattice and a join semilattice.

Observe:

(8.1) $x \wedge x = x$ and $x \vee x = x$.

Whenever meets and joins exist,

(8.2) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$.

(8.3) If $x \leq y$, then $x \wedge y = x$ and $x \vee y = y$.

Thus, meets and joins of two elements exist in a linearly ordered set, so it is a lattice.

8.4 Examples

Example 8.1. Recall: The set of natural numbers $\mathbb{N}$ is a linearly ordered set.

It has a bottom element, namely 0, but no top element. So it is not bounded.

The meet of two numbers is the smaller of the two, while the join is the larger of the two. Thus, $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$. So $\mathbb{N}$ is a lattice.

Recall: The set of integers $\mathbb{Z}$ is also a linearly ordered set. It has no bottom or top element, while meets and joins are as above. So $\mathbb{Z}$ is also a lattice which is not bounded.

Example 8.2. Recall Boolean poset $P(X)$ from Example 4.2.

The minimum element is the empty set $\emptyset$,

the maximum element is X,

meet is given by intersection of sets,

join is given by by union of sets.

Thus, $A \wedge B = A \cap B$ and $A \vee B = A \cup B$.

So $P(X)$ is a lattice.

Example 8.3. Let $\mathbb{N}_+$ be the set of positive integers. We say $a \leq b$ if a divides b (often written as $a|b$).

The minimum element is 1,

there is no maximum element,

meet is given by gcd (greatest common divisor),

join is given by lcd (least common dividend).

Thus, $a \wedge b = gcd(a, b)$ and $a \vee b = lcd(a, b)$.

So $\mathbb{N}_+$ is a lattice.

Instead if we fix $N \geq 1$ and let D_N be the set of divisors of N as in Example 4.3, then N is the maximum element, and the rest is as before.

Example 8.4. Recall from Example 4.6 poset $\mathrm{Sub}(W)$ of subspaces of a $\Bbbk$-vector space W.

The minimum element is the zero subspace,

the maximum element is W,

meet is given by intersection of subspaces,

join is given by linear span of subspaces.

Thus, $U \wedge V = U \cap V$ and
$U \vee V = U + V = \{u + v \mid u \in U, v \in V\}$.

Note very carefully: Union of subspaces is not a subspace in general, so we take linear span.

More formally, $\mathrm{Sub}(W)$ is a full subposet of $P(W)$. The inclusion map from $\mathrm{Sub}(W)$ to $P(W)$ preserves meets but not joins in general.

Example 8.5. Recall from Example 4.7 poset $\mathbb{R}^{[0,1]}$ of all real-valued functions $f : [0, 1] \to \mathbb{R}$.

There is no minimum element,

there is no maximum element,

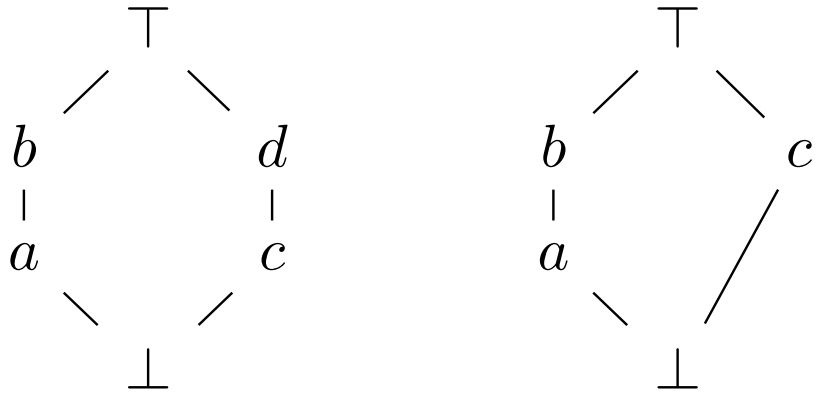
meet of f and g is given by $\min(f, g)$,

join of f and g is given by $\max(f, g)$.

$$\min(f, g)(x) := \min(f(x), g(x))$$
$$\max(f, g)(x) := \max(f(x), g(x))$$

So $\mathbb{R}^{[0,1]}$ is a lattice.

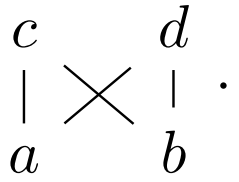
Example 8.6. Consider the following two posets.



Both of them are finite lattices.

The one on the right is called the pentagon lattice.

Example 8.7. Consider poset



It has two minimal elements a and b, and two maximal elements c and d.

So bottom element $\bot$ and top element $\top$ do not exist.

Meet or join of a and b does not exist.

Meet or join of c and d also does not exist.

8.5 Complete lattice

Let P be a poset and S be any subset of P.

Let $z \in P$.

- We say z is a lower bound for S if z is smaller than all elements of S.
- We say z is an upper bound for S if z is larger than all elements of S.

How would you interpret this statement when S is empty?

- The meet or greatest lower bound of S, denoted $\bigwedge S$ or $\bigwedge_P S$, is the largest element in P which is smaller than all elements of S.
- The join or least upper bound of S, denoted $\bigvee S$ or $\bigvee_P S$, is the smallest element in P which is larger than all elements of S.

A poset is a complete lattice if arbitrary meets and joins exist.

This is a stronger condition than what we had for a lattice. In other words, if P is a complete lattice, then P is also a lattice.

Fact 8.8. *A complete lattice has a top and bottom element.*

Proof. Take S to be the entire poset P. Then meet of S is the bottom element, and join of S is the top element:

$$\bigwedge P = \bot \qquad \text{and} \qquad \bigvee P = \top.$$

Alternatively, take S to be the empty set. Then meet of $\emptyset$ is the top element, and join of $\emptyset$ is the bottom element:

$$\bigwedge \emptyset = \top \qquad \text{and} \qquad \bigvee \emptyset = \bot.$$

□

Fact 8.9. *Every nonempty finite lattice is a complete lattice.*

We leave this as an exercise. Thus, there is a distinction between lattices and complete lattices only if the poset is either empty or infinite.

Example 8.10. Let X be any set.

Boolean poset $P(X)$ in Example 4.2 is complete.

Arbitrary meets and joins are given as follows.

$$\bigwedge S = \bigcap_{A \in S} A \qquad \text{and} \qquad \bigvee S = \bigcup_{A \in S} A.$$

Note:

$$\bigwedge \emptyset = X \qquad \text{and} \qquad \bigvee \emptyset = \emptyset.$$

When X is finite, we can also use Fact 8.9 to deduce that $P(X)$ is complete.

Now let X be an infinite set.

Consider poset $P_{fin}(X) = 2^X_{fin}$ whose elements are finite subsets of X ordered by inclusion.

It is a full subposet of $P(X)$.

Then $P_{fin}(X)$ is not a complete lattice.

- Meets of any non-empty family of subsets exist as above, but the empty meet does not exist since $P_{fin}(X)$ does not have a top element.
- Arbitrary joins may not exist. This is because union of an infinite number of finite subsets may not be finite. For instance, the join of all finite subsets is X which is infinite.

Example 8.11. Poset $\mathbb{Q}$ of rational numbers does not have a top element or a bottom element. So it is not a complete lattice. Can we fix this?

Let us consider its full subposet $\mathbb{Q} \cap [-10, 10]$. This poset has top element 10 and bottom element -10. But it is still not complete.

To see the problem:

Consider any increasing sequence of rational numbers converging to $\sqrt{2}$. Then the set S consisting of elements of this sequence does not have a least upper bound in $\mathbb{Q}$ (since $\sqrt{2}$ is an irrational number).

Summary: $\mathbb{Q}$ is incomplete in multiple ways!

How can we fix this? We need to get rid of the holes in $\mathbb{Q}$.

Example 8.12. Poset $\mathbb{R}$ of real numbers does not have a top element or a bottom element. So it is not a complete lattice.

Explicitly,

$$\bigwedge_{\mathbb{R}} \mathbb{N} = 0 \quad \text{and} \quad \bigvee_{\mathbb{R}} \mathbb{N}, \bigwedge_{\mathbb{R}} \mathbb{Z}, \bigvee_{\mathbb{R}} \mathbb{Z} \text{ do not exist.}$$

Let us consider full subposet $\mathbb{R} \cap [-10, 10]$. This is indeed a complete lattice.

One can do something even better:

Enlarge $\mathbb{R}$ to a bigger poset $\hat{\mathbb{R}}$ by adding two elements denoted $+\infty$ and $-\infty$, where $+\infty$ is the top element, and $-\infty$ is the bottom element. This is the extended real number system. It is a complete lattice.

$$\bigvee_{\hat{\mathbb{R}}} \mathbb{N} = \infty, \quad \bigwedge_{\hat{\mathbb{R}}} \mathbb{Z} = -\infty, \quad \bigvee_{\hat{\mathbb{R}}} \mathbb{Z} = \infty.$$

9 Locally finite poset

Let P be a poset.

For $x \leq y$ in P, let $[x, y]$ denote the set of all elements which lie between x and y, that is,

$$[x, y] = \{z \mid x \leq z \leq y\}. \tag{9.1}$$

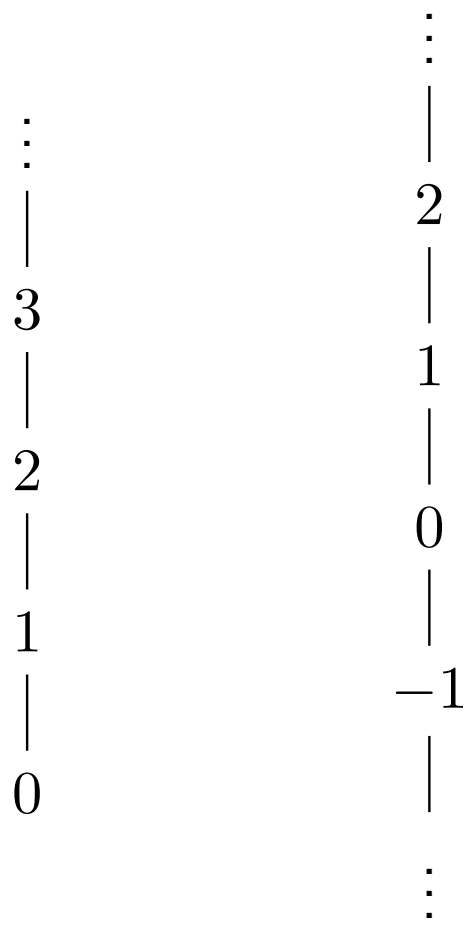
We refer to such a subset as an interval of P.

We say P is locally finite if every interval of P is finite.

Note: Any finite poset is locally finite.

Example 9.1. Posets $\mathbb{N}$ and $\mathbb{Z}$ are locally finite, while $\mathbb{Q}$ and $\mathbb{R}$ are not locally finite.

In particular, $\mathbb{N}$ and $\mathbb{Z}$ have Hasse diagrams as below.



However, $\mathbb{Q}$ and $\mathbb{R}$ do not have Hasse diagrams. In fact, there are no cover relations at all in $\mathbb{Q}$ and $\mathbb{R}$. This is because between any two distinct rational/real numbers, there is always another rational/real number.

Example 9.2. Let us go back to Example 8.10.

If X is finite, then $P(X)$ is finite, and hence locally finite.

So suppose X is infinite. Then Boolean poset $P(X)$ is not locally finite. For instance, interval $[\emptyset, X]$ is infinite. It contains singleton subset $\{x\}$ for each $x \in X$.

In contrast, $P_{fin}(X)$ is an infinite poset which is locally finite. For any finite subsets $A, B \in P_{fin}(X)$ with $A \subseteq B$, interval $[A, B]$ has exactly $2^{|B \setminus A|}$ elements.

10 Problem set III

10.1 Test your understanding

Say whether each of the following statements is true or false.

1. The empty poset $\emptyset$ is a complete lattice.
2. A linearly ordered set is a lattice, but not necessarily a complete lattice.
3. A finite lattice has a top element $\top$ and a bottom element $\bot$.
4. A full subposet of a locally finite poset is also locally finite.
5. A bounded lattice is complete.
6. A full subposet of a finite lattice is a finite lattice.
7. Let $f : P \to Q$ be an order-preserving map between two join semilattices. Then, for any $x, y \in P$, we have $f(x \vee y) = f(x) \vee f(y)$.
8. If every nonempty subset of a poset P has a join, then P is a complete lattice.
9. For elements x, y, z in a lattice P,

   $$x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z) \quad \text{and} \quad x \wedge (y \vee z) \geq (x \wedge y) \vee (x \wedge z).$$

10. Every finite lattice P with $|P| \geq 2$ has height at least 2 and width strictly smaller than $|P|$.

10.2 Problems related to theory

1. Let P be a lattice. Show: $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$ for all $x, y \in P$.
2. Suppose P is a lattice. For any $x, y, z \in P$ show that $(x \vee y) \vee z = x \vee (y \vee z)$. Further show that this element is the least upper bound of the subset $S = \{x, y, z\}$.

   Extend this result to any nonempty finite set S. It may not work when S is empty since P could be an infinite lattice which is not bounded.

Show that if P is a nonempty finite poset then P is a lattice iff P is a complete lattice.

Deduce that a nonempty finite lattice is bounded.

Can you give an example of an (infinite) bounded lattice which is not complete?

3. Show:
   - A finite meet semilattice with a top element is a lattice.
   - A finite join semilattice with a bottom element is a lattice.

   Show that the statements are false if we drop the finite condition.

4. Let P be a poset. Show:
   - If arbitrary meets exist in P, then so do arbitrary joins.
   - If arbitrary joins exist in P, then so do arbitrary meets.

   Thus, existence of either meets or joins is enough to guarantee that P is a complete lattice.

5. Show: If P is a (complete) lattice, then so is any interval $[x, y]$ in P (viewed as a full subposet) with the same meets and joins.

6. Prove the result below using the outline provided.
   Theorem 10.1 (Knatser–Tarski, $\sim$ 1940)**.** *Let P be a complete lattice and $f : P \to P$ be an order-preserving map. Let $F = \{x \in P \mid f(x) = x\}$ denote the set of all fixed points of f. Then F is a complete lattice with partial order induced from P.*

   (a) Define $A = \{x \in P \mid f(x) \leq x\}$. Show that A is nonempty by utilizing Fact 8.8. Prove that $u = \bigwedge A$ belongs to A and satisfies $u \leq f(u)$, thereby concluding that u is the unique least fixed point of f.

   (b) Dually, define $B = \{x \in P \mid x \leq f(x)\}$. Show that B is nonempty and prove that $v = \bigvee B$ is the unique greatest fixed point of f.

   (c) Let $S \subseteq F$ be a subset of fixed points, and let $x = \bigvee S$ evaluated in lattice P. Show that interval $[x, \top]$ is a complete lattice mapping into itself under f. Deduce that the least fixed point of f within this interval is the least upper bound of S in F. Similarly, we argue for the greatest lower bound of S.

   Here are some illustrations of the theorem.
   - Every order-preserving map $f : P(X) \to P(X)$ has a fixed point. Take f to be $f(A) = A \cap K$, or $f(A) = f(A) \cup K$, or $f(A) = f(A) \setminus K$ for a fixed set K. These are all order-preserving. Their composites yield further examples. In each case, determine the complete lattice of fixed points.

- Every increasing map $f : [0, 1] \to [0, 1]$ has a fixed point. Give an example where f has a unique fixed point and where f has infinitely many fixed points.
- Fix a linear map $T : W \to W$, and define $f : \mathrm{Sub}(W) \to \mathrm{Sub}(W)$ by $f(U) = T(U) = \{T(u) \mid u \in U\}$. It is order-preserving. The complete lattice of fixed points F consists of T-invariant subspaces of W. For $W = \mathbb{R}^3$ and T a counterclockwise rotation around z-axis (by an angle not a multiple of π), F consists of $\{0\}$, z-axis, xy-plane, $\mathbb{R}^3$, forming a 4-element diamond lattice.

7. Let $\{a_n\}$ be a sequence in $\hat{\mathbb{R}}$. For $k \geq 0$, put $b_k := \sup\{a_k, a_{k+1}, \dots\}$. Note that $b_0 \geq b_1 \geq b_2 \dots$. Put $\beta := \inf\{b_0, b_1, \dots\}$. We write $\beta = \limsup_{n\to\infty} a_n$. Similarly, we define $\liminf_{n\to\infty} a_n$ by interchanging roles of $\sup$ and $\inf$. These notions make sense in any complete lattice. (Thus we can make sense of convergence of a sequence in any complete lattice!)

   For Boolean poset $P(X)$, for a sequence of subsets $A_1, A_2, \dots,$

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \quad \text{and} \quad \liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k.$$

   (These are important in probability theory, the A_i are interpreted as events.) In a complete lattice P, $\limsup$ and $\liminf$ are defined as above, with union replaced by join, and intersection replaced by meet, with $A_i \in P$. Show:

   (a) $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$ for any sequence $\{a_n\}$ in P.

   (b) If $\{a_n\}$ is an increasing sequence in P, then $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \bigvee_{n=1}^{\infty} a_n$. There is a similar statement for a decreasing sequence in P.

10.3 Problems related to examples

1. Consider the 5 posets with three elements (up to isomorphism). What are the minimal and maximal elements in these posets? Which of these posets have a top element $\top$ or a bottom element $\bot$? Which of these are lattices, meet semilattices, join semilattices?

2. Consider linearly ordered set $S = \{x \in \mathbb{Q} \mid x^2 < 2\}$ (with linear order induced from $\mathbb{Q}$). Show that S does not have a top element. Further, show that S does not have a least upper bound in poset $\mathbb{Q}$.

3. Let W be a vector space over a field $\mathbb{k}$. Show: Poset of subspaces $\mathrm{Sub}(W)$ in Example 4.6 is complete.

4. Recall notion of interval from (9.1). Show:
   (a) Every interval in Boolean poset $P(X)$ is isomorphic to a Boolean poset.
   (b) Every interval in divisor poset D_N is isomorphic to a divisor poset.
   (c) Every interval in subspace poset $\mathrm{Sub}(W)$ is isomorphic to a subspace poset.

5. Let P be a bounded lattice with bottom element $\bot$ and top element $\top$. A complement of $x \in P$ is an element $y \in P$ such that $x \wedge y = \bot$ and $x \vee y = \top$. Lattice P is complemented if every $x \in P$ has at least one complement. It is uniquely complemented if every $x \in P$ has exactly one unique complement. Show:
   (a) $P(X)$ with $\bot = \emptyset$ and $\top = X$ is uniquely complemented. The unique complement of A given by $X \setminus A$.
   (b) D_N with $\bot = 1$ and $\top = N$ is uniquely complemented iff N is square-free. More precisely, writing $N = p_1^{a_1} \ldots p_m^{a_m}$, $d \in D_N$ has a (unique) complement iff for all i, the highest power of p_i dividing d is either 0 or a_i.
   (c) $\mathrm{Sub}(W)$ of an n-dimensional vector space W over a field $\mathbb{k}$ with $\bot = \{0\}$ and $\top = W$ is complemented. It is uniquely complemented iff $n = 1$. If $\mathbb{k}$ is an infinite field and $n \geq 2$, then every proper subspace U has infinitely many complements. If $\mathbb{k} = \mathbb{F}_q$, then subspace U of dimension k has $q^{k(n-k)}$ complements.

   Illustrate the above with $X = \{a, b, c\}$, $N = 12$, $W = \mathbb{R}^2$ and $W = \mathbb{F}_2^3$.

6. Let P be a bounded lattice with bottom element $\bot$ and top element $\top$. An element $a \in P$ is an atom if it covers the bottom element ($\bot \lessdot a$). An element $c \in P$ is a coatom if it is covered by the top element ($c \lessdot \top$). The lattice P is atomic if every $x \in P$ can be written as the join of the atoms it contains: $x = \bigvee\{a \in \text{Atoms}(P) \mid a \leq x\}$. Dually, P is coatomic if every $x \in P$ can be written as the meet of the coatoms above it: $x = \bigwedge\{c \in \text{Coatoms}(P) \mid x \leq c\}$. Show:
   (a) $P(X)$ is both atomic and coatomic. (Atoms are singletons $\{x\}$ and coatoms are $X \setminus \{x\}$.)
   (b) D_N is atomic iff D_N is coatomic iff N is square-free. Illustrate for D_{30} and D_{12}.
   (c) $\mathrm{Sub}(W)$ (for W of dimension n) is both atomic and coatomic. (Atoms are 1-dimensional lines, and coatoms are $(n-1)$-dimensional hyperplanes.) For $\mathbb{k} = \mathbb{F}_q$, number of atoms equals number of coatoms which equals $\frac{q^n - 1}{q - 1}$.

Note: In the above statements, we allow $X = \emptyset$, $N = 1$, $n = 0$.

7. Consider the following four posets ordered pointwise ($f \leq g$ iff $f(x) \leq g(x)$ for all $x \in [0, 1]$):
   (a) $P_1 = \mathbb{R}^{[0,1]}$ (all real-valued functions on $[0, 1]$).
   (b) $P_2 = C([0, 1], [0, 1])$ (all continuous functions mapping $[0, 1]$ to $[0, 1]$).
   (c) $P_3 = [0, 1]^{[0,1]}$ (all functions mapping $[0, 1]$ to $[0, 1]$).
   (d) $P_4 = \hat{\mathbb{R}}^{[0,1]}$ (all functions mapping $[0, 1]$ to $\hat{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$).

poset	$\bot$	$\top$	lattice	complete lattice
P_1	no	no	yes	no
P_2	yes	yes	yes	no
P_3	yes	yes	yes	yes
P_4	yes	yes	yes	yes

   Justify the entries in the above table.

8. Consider set of extended natural numbers $\hat{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$. It is a poset with usual order on $\mathbb{N}$, with ∞ larger than any natural number. Similarly, consider set of extended integers $\hat{\mathbb{Z}} := \mathbb{N} \cup \{-\infty, \infty\}$. Are $\hat{\mathbb{N}}$ and $\hat{\mathbb{Z}}$ complete lattices?

10.4 Explorations

1. In a finite poset P, there exists a minimal element and a maximal element. To show the latter, use the following algorithm. Start with any element x in P. If x is maximal, then we are done. If not, then pick $y \in P$ with $y > x$, and so on. Since P is finite, this process terminates, and we end up with a maximal element.

   What happens when P is infinite? Take a look at Zorn's lemma.

2. Show: For any poset, there is a "smallest" complete lattice which contains the given poset. This is the Dedekind–MacNeille completion of the poset. This construction applied to $\mathbb{Q}$ yields $\hat{\mathbb{R}}$. This is Dedekind's approach to the construction of the real numbers.

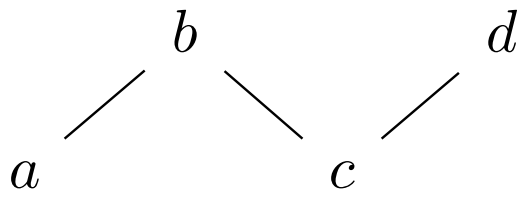
3. The Knatser–Tarski fixed point theorem can be used to prove the Banach decomposition theorem and Schroder–Bernstein theorem. See Davey–Priestley, Exercise 2.32.

11 Connected poset

A nonempty poset P is connected if for any two elements, there is a finite sequence of relations linking them.

Example 11.1. Poset $P = \{a, b, c, d\}$ with $a < b$, $c < b$, $c < d$ is connected.

Its Hasse diagram is shown below.



Elements a and d are incomparable but linked via $a \leq b \geq c \leq d$.

Here are some general observations.

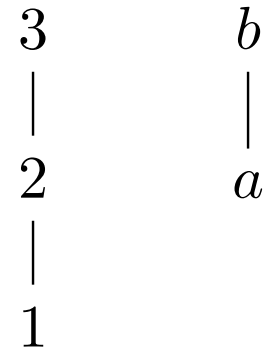
- A nonempty linearly ordered set is connected.
- A poset with bottom or top element is connected.
- A nonempty meet semilattice or join semilattice is connected.

Example 11.2. Posets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ are connected since they are linearly ordered.

Boolean poset $P(X)$ of any set X is connected since it has a bottom and top element ($\emptyset$ and X). Its full subposet of finite subsets is also connected since it has bottom element $\emptyset$.

The poset of divisors of $\mathbb{N}_+$ is connected since it has a bottom element 1.

Example 11.3. This is a nonexample. The poset containing two or more isolated chains (such as $1 < 2 < 3$ and $a < b$) with no relations between them.



More simply, we can consider a poset containing two or more isolated points.



This is an example of a discrete poset.

12 Graded poset

12.1 Graded poset

A locally finite poset P is graded if there exists a function

$$\mathrm{rk} : P \to \mathbb{N} \tag{12.1}$$

such that

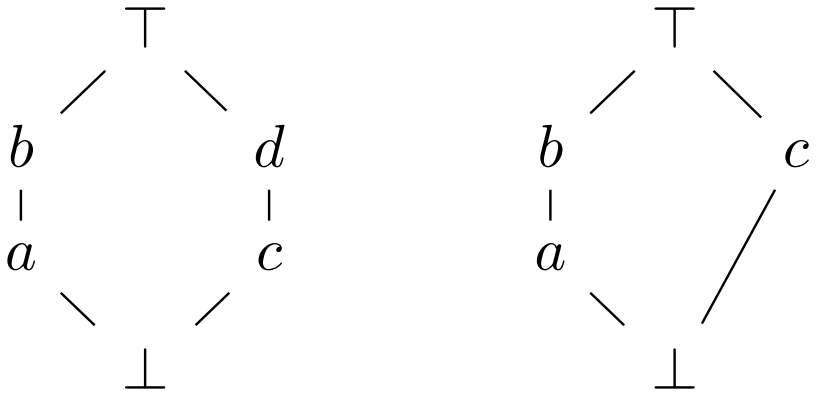
$$x \lessdot y \quad \text{implies} \quad \mathrm{rk}(x) + 1 = \mathrm{rk}(y) \tag{12.2}$$

for all $x, y \in P$.

We say that rk is a rank function of P.

Once this is fixed, we refer to $\mathrm{rk}(x)$ as the rank of x.

Example 12.1. Recall the following posets from Example 8.6.



The poset on the left is graded. It has rank 3 (as we will see).

The poset on the right is not graded.

12.2 Length of maximal chain

Let C be a chain in P.

The length of C is defined to be $|C| - 1$. It is one less than the size of C.

Thus, $a < b$ has length 1, $a < b < c$ has length 2, and so on.

Note: a has length 0, and $\emptyset$ has length -1.

Observation 12.2. Let P be a locally finite poset.

- If P is graded and $x \leq y$, then all maximal chains from x to y have the same length.
- If P is finite with a bottom and top element, and all maximal chains from $\bot$ to $\top$ have the same length, then P is graded.

12.3 Rank of a poset

Let P be a finite, connected, graded poset.

The rank of P, denoted $\mathrm{rk}(P)$, is the difference between the maximum and minimum values of any rank function of P.

If P is empty, its rank is defined to be -1.

12.4 Conventions

For a graded poset with bottom element $\bot$, we take the rank function for which $\mathrm{rk}(\bot) = 0$. Then $\mathrm{rk}(z)$ is the length of any maximal chain in $[\bot, z]$.

For a finite graded poset with top element $\top$, we take the rank function for which $\mathrm{rk}(\top) = \mathrm{rk}(P)$.

When both $\bot$ and $\top$ exist, the choices $\mathrm{rk}(\bot) = 0$ and $\mathrm{rk}(\top) = \mathrm{rk}(P)$ are consistent.

12.5 Examples

Example 12.3. Poset $\mathbb{N}$ is graded and the identity is a rank function. Note: $\mathrm{rk}(0) = 0$ consistent with our convention on rank functions.

In contrast, poset $\mathbb{Z}$ is not graded. Why?

Since $\mathbb{Q}$ and $\mathbb{R}$ are not locally finite, they fall outside the scope of graded posets.

Example 12.4. Let X be a finite set and $P(X)$ its Boolean poset as in Example 4.2.

This poset is graded with $\mathrm{rk}(A) = |A|$.

If B covers A ($A \lessdot B$), then B is obtained by adding exactly one element to A.

Thus, $|B| = |A| + 1$, which implies
$\mathrm{rk}(B) = \mathrm{rk}(A) + 1$.

Example 12.5. Recall poset of divisors D_N from Example 4.3.

This poset is graded with $\mathrm{rk}(n) = \Omega(n)$, where $\Omega(n)$ is the number of prime factors of n counted with multiplicity.

If $m \lessdot n$, then $n = m \cdot p$ for some prime p.

Hence, the number of prime factors increases by 1, that is, $\mathrm{rk}(n) = \mathrm{rk}(m) + 1$.

Example 12.6. Let W be a finite-dimensional vector space over a finite field $\mathbb{k}$.

Recall its poset of subspaces $\mathrm{Sub}(W)$ from Example 4.6.

This poset is graded with $\mathrm{rk}(V) = \dim(V)$.

If U is covered by V ($U \lessdot V$), then U is a maximal proper subspace of V, and $\dim(V) = \dim(U) + 1$.

Thus, $\mathrm{rk}(V) = \mathrm{rk}(U) + 1$.

Remark. The above example can be made to work for any (infinite) field $\mathbb{k}$ if we suitably enlarge our definition of 'locally finite' to allow more posets.

13 Semimodular lattice

13.1 Lower and upper semimodular lattices

Let P be a finite lattice.

It is lower semimodular if the following property holds.

(13.1) If $x \vee y$ covers x and y,

then x and y cover $x \wedge y$.

It is upper semimodular if the dual property holds.

(13.2) If x and y cover $x \wedge y$,

then $x \vee y$ covers x and y.

13.2 Reformulation in terms of graded posets

Fact 13.1. *A finite (lower or upper) semimodular lattice is graded.*

Proof. This result is given for instance in Gratzer's book in Section IV.2, Theorem 1. It is also give in THA, Proposition B.2. □

In fact, a finite lattice is lower semimodular iff it is graded and the rank function satisfies

$$\mathrm{rk}(x) + \mathrm{rk}(y) \leq \mathrm{rk}(x \wedge y) + \mathrm{rk}(x \vee y) \tag{13.3}$$

for all $x, y \in P$.

Similarly, a finite lattice is upper semimodular iff it is graded and the rank function satisfies

$$\mathrm{rk}(x) + \mathrm{rk}(y) \geq \mathrm{rk}(x \wedge y) + \mathrm{rk}(x \vee y) \tag{13.4}$$

for all $x, y \in P$.

Example 13.2. Recall the pentagon lattice from Example 8.6.

It is not lower semimodular. To see this: $b \vee c = \top$, which covers b and covers c. Now $b \wedge c = \bot$. While c covers $\bot$, b does not cover $\bot$. Thus, the lower semimodular condition fails.

Similarly, it is not upper semimodular.

In fact, we saw in Example 12.1 that the pentagon lattice is not graded. So we can also use Fact 13.1 to deduce that it is not lower or upper semimodular.

Now consider the other lattice in Example 8.6 with 6 elements. It is graded. However it is still not lower or upper semimodular.

We can show this by a similar argument to the one above.

Alternatively, we can also see that inequalities (13.3) and (13.4) are both violated.

13.3 Modular lattice

A finite lattice is modular if it is both lower and upper semimodular.

In other words, a finite lattice is modular iff it is graded and the rank function satisfies

$$\mathrm{rk}(x) + \mathrm{rk}(y) = \mathrm{rk}(x \wedge y) + \mathrm{rk}(x \vee y). \tag{13.5}$$

13.4 Examples

Example 13.3. Let X be a finite set. Boolean poset $P(X)$ from Example 4.2 is a modular lattice. The modular equality (13.5) reduces to

$$|A| + |B| = |A \cap B| + |A \cup B|$$

for any subsets A and B of X.

Example 13.4. Let W be a finite-dimensional vector space over a finite field $\mathbb{k}$. Poset of subspaces $\mathrm{Sub}(W)$ from Example 4.6 is a modular lattice. The modular equality (13.5) reduces to

$$\dim(U) + \dim(V) = \dim(U \cap V) + \dim(U + V)$$

for any subspaces U and V of W.

14 Problem set IV

14.1 Test your understanding

Say whether each of the following statements is true or false.

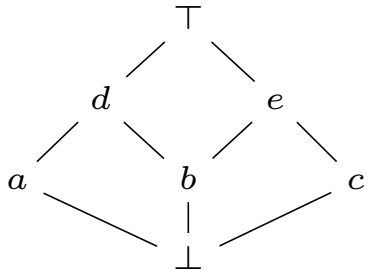
1. A nonempty poset is connected iff every two elements are comparable.
2. A locally finite poset with a bottom element is graded.
3. Poset $\mathbb{Z}$ with its usual order is graded.

14.2 Problems related to theory

1. Show: If P is a locally finite graded poset, then so is any interval $[x, y]$ in P (with the induced partial order).
2. Check: A finite linearly ordered set is a modular lattice.
3. Let P be a finite graded lattice. Show that (13.3) implies (13.1), and (13.4) implies (13.2).
4. This is more of an observation. Check: For poset P:
   - P has $\bot$ iff P^{op} has $\top$.
   - P has meets iff P^{op} has joins.
   - P is (uniquely) complemented iff P^{op} is (uniquely) complemented.
   - P is atomic iff P^{op} is coatomic.
   - P is (locally finite) graded iff P^{op} is (locally finite) graded.
   - P is lower semimodular iff P^{op} is upper semimodular.

14.3 Problems related to examples

1. Consider the following poset.



   Show that it is a graded poset and also a lattice. Check whether it is lower or upper semimodular by analyzing conditions (13.1) and (13.2) and inequalities (13.3) and (13.4).

   What happens if we replace the above poset by its opposite poset?

2. Give an example of a finite connected graded poset which does not have either a bottom element or a top element.

3. Check explicitly that any Boolean poset satisfies (13.1) and (13.2).

4. Show that divisor poset D_N from Example 4.3 is a modular lattice.

5. Which is the smallest lattice (in terms of number of elements) which is not modular?

14.4 Explorations

1. You may explore modular lattices further. There is also a notion of distributive lattices.

15 Product and coproduct of posets

15.1 Product of posets

Let P and Q be two posets.

Define a new poset $P \times Q$ with partial order

(15.1) $$(x_1, y_1) \leq_{P \times Q} (x_2, y_2)$$
$$\text{if } x_1 \leq_P x_2 \text{ and } y_1 \leq_Q y_2.$$

We refer to $P \times Q$ as the product of P and Q.

The projection maps

$$p_1 : P \times Q \to P \quad \text{and} \quad p_2 : P \times Q \to Q$$

defined by

$$p_1(x, y) = x \quad \text{and} \quad p_2(x, y) = y$$

are order-preserving.

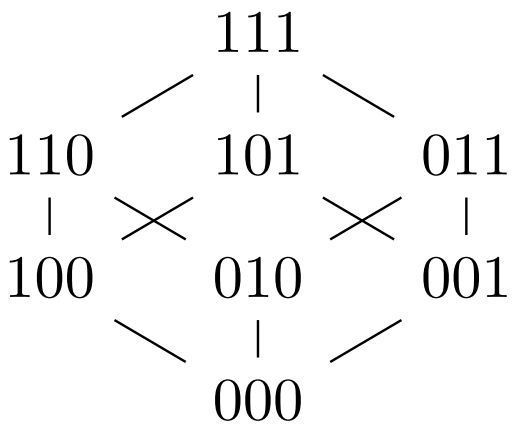
If we take $P = Q = \mathbb{R}$, then we recover Example 4.4.

More generally, for posets $P_1, \ldots, P_k$, we define their product $P_1 \times \cdots \times P_k$, with partial order defined componentwise.

Example 15.1. Let $C_2 = \{0, 1\}$ denote the 2-element chain ordered by $0 < 1$.

Consider poset $C_2 \times \cdots \times C_2 = C_2^n$.

For $n = 3$, its Hasse diagram is shown below.



For $X = \{x_1, \ldots, x_n\}$, consider its Boolean poset $P(X)$ from Example 4.2. Define poset isomorphism

$$\varphi : P(X) \xrightarrow{\cong} C_2^n \tag{15.2}$$

by

$$\varphi(A) = (v_1, \ldots, v_n) \text{ where } v_i = \begin{cases} 1 & \text{if } x_i \in A, \\ 0 & \text{if } x_i \notin A. \end{cases}$$

Example 15.2. Recall poset of divisors D_N from Example 4.3. Write

$$N = p_1^{a_1} \dots p_k^{a_k}.$$

Write every divisor $d \in D_N$ as $d = p_1^{m_1} \dots p_k^{m_k}$ with $0 \leq m_j \leq a_j$ for each j.

Let $C_{a_j+1} = \{0, 1, \dots, a_j\}$ denote the chain $0 < 1 < \cdots < a_j$.

Define poset isomorphism

$$\psi : D_N \xrightarrow{\cong} C_{a_1+1} \times \cdots \times C_{a_k+1} \tag{15.3}$$

by

$$\psi(d) = (m_1, \dots, m_k)$$

with $d = p_1^{m_1} \dots p_k^{m_k}$.

15.2 Coproduct of posets

Let P and Q be two posets.

Define a new poset $P \sqcup Q$ with partial order

(15.4) $x \leq_{P \sqcup Q} y$ if

- $x, y \in P$ and $x \leq_P y$, or
- $x, y \in Q$ and $x \leq_Q y$.

If $x \in P$ and $y \in Q$, then x and y are incomparable.

We refer to $P \sqcup Q$ as the coproduct or disjoint union of P and Q.

This poset is not connected when P and Q are both nonempty.

The inclusion maps

$$i_P : P \to P \sqcup Q \quad \text{and} \quad i_Q : Q \to P \sqcup Q$$

are order-preserving.

More generally, for posets $P_1, \ldots, P_k$, we define their coproduct $P_1 \sqcup \cdots \sqcup P_k$.

We have seen some examples in Example 11.3.

16 Linear extensions of a poset

Let $(P, \leq)$ be a finite poset.

A linear extension of P is a linearly ordered set $(P, \leq_L)$ such that

$$x \leq y \quad \text{implies} \quad x \leq_L y.$$

Observe: This is an order-preserving bijection from P to a chain of the same size.

Let $e(P)$ denote the number of linear extensions of P.

Example 16.1. Let $P = \{a, b, c\}$ with $a \leq c$ and $b \leq c$. This poset has exactly two linear extensions:

$$a \leq_{L_1} b \leq_{L_1} c \quad \text{and} \quad b \leq_{L_2} a \leq_{L_2} c.$$

Example 16.2. Recall Boolean poset $P(X)$ for $X = \{a, b, c\}$ from Example 4.2.

One way to obtain a linear extension is to sort subsets first by size, and then alphabetically:

$$\begin{aligned}\emptyset \leq_L \{a\} \leq_L \{b\} \leq_L \{c\} \leq_L \{a, b\} \\ \leq_L \{a, c\} \leq_L \{b, c\} \leq_L \{a, b, c\}.\end{aligned}$$

Since $A \subseteq B$ implies $|A| \leq |B|$, ordering subsets by size and breaking ties alphabetically yields a linear extension of any power set.

Another example is

$$\begin{aligned}\emptyset \leq_L \{a\} \leq_L \{b\} \leq_L \{a, b\} \leq_L \{c\} \\ \leq_L \{a, c\} \leq_L \{b, c\} \leq_L \{a, b, c\}.\end{aligned}$$

17 Problem set V

17.1 Test your understanding

Say whether each of the following statements is true or false.

1. The empty poset has exactly one linear extension.
2. For finite posets P and Q, we have $w(P \times Q) = w(P)w(Q)$.
3. If $x_1 \lessdot x_2$ in P and $y_1 \lessdot y_2$ in Q, then $(x_1, y_1) \lessdot (x_2, y_2)$ in $P \times Q$.
4. Let P be the discrete poset on n elements. The number $e(P)$ of linear extensions of P is $n!$.
5. Let $C_2 = \{a_1 < a_2\}$ and $C_3 = \{b_1 < b_2 < b_3\}$. The number of linear extensions of $C_2 \sqcup C_3$ is 6.
6. For finite posets P and Q, $e(P \times Q) = e(P)e(Q)$.

17.2 Problems related to theory

1. Let P and Q be posets. Show:
   (a) If P and Q are bounded with bottom elements $\bot_P$, $\bot_Q$ and top elements $\top_P$, $\top_Q$, then $P \times Q$ is bounded with bottom element $(\bot_P, \bot_Q)$ and top element $(\top_P, \top_Q)$.
   (b) If P and Q are lattices, then $P \times Q$ is a lattice with meets and joins given componentwise, that is,

   $$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2) \tag{17.1a}$$

   $$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2). \tag{17.1b}$$

   (c) In a product of bounded lattices, $(x, y) \in P \times Q$ has a complement (x', y') iff x' is a complement of x in P and y' is a complement of y in Q. Deduce that $P \times Q$ is uniquely complemented iff both P and Q are uniquely complemented.

(d) $(x_1, y_1) \lessdot (x_2, y_2)$ in $P \times Q$ iff either ($x_1 \lessdot x_2$ in P and $y_1 = y_2$) or ($x_1 = x_2$ and $y_1 \lessdot y_2$ in Q).

2. Check: If P and Q are (locally-finite) graded posets, then so is $P \times Q$ with

$$\mathrm{rk}(x, y) := \mathrm{rk}(x) + \mathrm{rk}(y). \tag{17.2}$$

In addition, if P and Q are finite and connected, then so is $P \times Q$ with

$$\mathrm{rk}(P \times Q) = \mathrm{rk}(P) + \mathrm{rk}(Q). \tag{17.3}$$

3. Check: If P and Q are finite lattices which are modular, then so is $P \times Q$. (Use (17.1) and (17.2) to check (13.5).)

4. Check: If P and Q are (locally-finite) graded posets, then so is $P \sqcup Q$ (by combining the rank functions of P and Q).

5. Show: For posets P and Q, there are poset isomorphisms

$$(P \times Q)^{op} \cong P^{op} \times Q^{op} \quad \text{and} \quad (P \sqcup Q)^{op} \cong P^{op} \sqcup Q^{op}.$$

Remark: This follows from the fact that the functor which sends P to P^{op} is an isomorphism from category Poset to itself.

6. Let P be a finite poset. Show: P is linearly ordered iff P has exactly one linear extension.

7. Show: For finite posets P and Q,

$$h(P \sqcup Q) = \max\{h(P), h(Q)\} \quad \text{and} \quad w(P \sqcup Q) = w(P) + w(Q).$$

8. Let P and Q be finite posets with $|P| = m$ and $|Q| = n$. Show: $e(P \sqcup Q) = \binom{m+n}{m} e(P) e(Q)$.

17.3 Problems related to examples

1. What are all the linear extensions of the poset in Example 8.7?

   For which N does divisor poset D_N have exactly one linear extension?

2. For $n \geq 0$, let C_n denote the chain of size n, or equivalently, length $n-1$. Show:
   (a) Coproduct $C_m \sqcup C_n$ has $\binom{m+n}{m}$ linear extensions.
   (b) Product $C_n \times C_2$ has $\frac{1}{n+1}\binom{2n}{n}$ linear extensions. This is the n-th Catalan number. It is the number of
   - valid ways of parenthesizing n pairs of brackets.
   - ways of going from $(0, 0)$ to (n, n) by moving right or up without crossing the diagonal and staying below it.
   - triangulations of a convex $(n+2)$-gon into n triangles by $n-1$ diagonals that do not intersect in their interiors.
   - rooted planar binary trees with $2n+1$ vertices (with n internal nodes and $n+1$ leaves).

   It has many other interpretations. See Stanley, Volume II, Exercise 6.19.

3. For $n \geq 0$, let T_n denote the n-th Catalan number. Check that they satisfy the recursion

$$T_n = \sum_{i=0}^{n-1} T_i T_{n-1-i} \tag{17.4}$$

   for $n \geq 1$, with initial condition $T_0 = 1$.

   For sequence $\{T_n\}$, its generating function is written as

$$T(x) = \sum_{n=0}^{\infty} T_n x^n. \tag{17.5}$$

   Show that

$$T(x) = \frac{1-\sqrt{1-4x}}{2x}. \tag{17.6}$$

   (Use recursion (17.4) to first show that $xT(x)^2 = T(x) - 1$, and then solve this equation for $T(x)$.)

4. Show:
   (a) There is a poset isomorphism $P(X) \times P(Y) \xrightarrow{\cong} P(X \sqcup Y)$ for any sets X and Y. It sends (A, B) to $A \sqcup B$. How does this isomorphism relate to (15.2)?
   (b) There is a poset isomorphism $D_M \times D_N \cong D_{M \cdot N}$ iff M and N are coprime, that is, $gcd(M, N) = 1$. For the coprime case, how does the

isomorphism relate to (15.3)? (For the noncoprime case, think of why $D_p \times D_p$ cannot be isomorphic to D_{p^2}.)

(c) There is an order-preserving map $\mathrm{Sub}(U) \times \mathrm{Sub}(V) \hookrightarrow \mathrm{Sub}(U \oplus V)$ for any vector spaces U and V over a field $\Bbbk$. It sends a pair of subspaces (U_1, V_1) to their direct sum $U_1 \oplus V_1$. This map is a poset isomorphism iff either U or V is the zero vector space. (To understand why we do not get an isomorphism when U and V are both nonzero, think of what happens when $U = V = \mathbb{R}$ and $U \oplus V = \mathbb{R}^2$.)

17.4 Explorations

1. How many linear extensions does Boolean poset $P(X)$ have for $|X| = n$? Show that $\prod_{k=0}^{n} \binom{n}{k}!$ is a lower bound.

2. Here is an algorithm to generate all linear extensions of a finite poset P. Start with a minimal element x in P. Then take a minimal element y in $P \setminus \{x\}$. Then take a minimal element z in $P \setminus \{x, y\}$, and so on.

   Try this algorithm on some of our examples to see how it works.

18 q-calculus

18.1 q-number

For integer $i \geq 0$, define the i-th q-number by

$$(i)_q := 1 + q + \cdots + q^{i-1}. \tag{18.1}$$

Convention: $(0)_q := 0$.

Thus,

$$(1)_q = 1,$$
$$(2)_q = 1 + q,$$
$$(3)_q = 1 + q + q^2,$$

and so on.

For $q = 1$, we recover the usual number i.

18.2 q-factorial

For integer $n \geq 0$, define q-factorial of n by

$$(n)_q! := (n)_q(n-1)_q \dots (1)_q. \tag{18.2}$$

Convention: $(0)_q! := 1$.

Thus,

$$\begin{aligned}(1)_q! &= 1,\\ (2)_q! &= 1+q,\\ (3)_q! &= (1+q)(1+q+q^2)\\ &= 1+2q+2q^2+q^3,\end{aligned}$$

and so on.

For $q = 1$, we recover the usual factorial.

18.3 q-binomial coefficient

For integers $0 \leq i \leq n$, define q-binomial coefficient

$$\binom{n}{i}_q := \frac{(n)_q!}{(i)_q!(n-i)_q!}. \tag{18.3}$$

It is a polynomial in q.

For integers $i > n$, we define this to be zero.

For example,

$$\begin{aligned}
\binom{4}{2}_q &= \frac{(4)_q!}{(2)_q!(2)_q!} = \frac{(4)_q \cdot (3)_q \cdot (2)_q \cdot (1)_q}{(2)_q \cdot (1)_q \cdot (2)_q \cdot (1)_q} \\
&= \frac{(1+q+q^2+q^3)(1+q+q^2)}{(1+q)(1)} \\
&= (1+q^2)(1+q+q^2) \\
&= 1+q+2q^2+q^3+q^4.
\end{aligned}$$

For $q = 1$, we recover the usual binomial coefficient.

18.4 q-binomial theorem

The q-binomial theorem says

$$\prod_{i=1}^{n}(x+q^{i-1}y)=\sum_{k=0}^{n}\binom{n}{k}_q q^{\binom{k}{2}}x^{n-k}y^k. \tag{18.4}$$

Let us work out the case $n=2$.

$$\begin{aligned} lhs=\prod_{i=1}^{2}(x+q^{i-1}y)&=(x+y)(x+qy)\\ &=x^2+(1+q)xy+qy^2. \end{aligned}$$

$$\begin{aligned} rhs&=\sum_{k=0}^{2}\binom{2}{k}_q q^{\binom{k}{2}}x^{2-k}y^k\\ &=\binom{2}{0}_q q^0x^2y^0+\binom{2}{1}_q q^0x^1y^1+\binom{2}{2}_q q^1x^0y^2\\ &=x^2+(1+q)xy+qy^2. \end{aligned}$$

For $q=1$, we recover binomial theorem

$$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k. \tag{18.5}$$

18.5 q-binomial identities

The q-binomial identities say:

For integers $0 \leq k \leq n$,

$$\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q. \tag{18.6a}$$

$$\binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-k} \binom{n-1}{k-1}_q. \tag{18.6b}$$

For $q = 1$, we recover binomial identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}. \tag{18.7}$$

18.6 q-Chu–Vandermonde identity

For integers $m, n, k \geq 0$,

$$\binom{m+n}{k}_q = \sum_{i+j=k} \binom{m}{i}_q \binom{n}{j}_q q^{(m-i)j}. \tag{18.8}$$

This is the q-Chu–Vandermonde identity.

For $m = 2, n = 1, k = 1$,

$$\begin{aligned} lhs &= \binom{2+1}{1}_q = \binom{3}{1}_q = 1 + q + q^2. \\ rhs &= \sum_{i+j=1} \binom{2}{i}_q \binom{1}{j}_q q^{(2-i)j} \\ &= \binom{2}{1}_q \binom{1}{0}_q q^{(2-1)\cdot 0} + \binom{2}{0}_q \binom{1}{1}_q q^{(2-0)\cdot 1} \\ &= 1 + q + q^2. \end{aligned}$$

For $q = 1$, we recover Chu–Vandermonde identity

$$\binom{m+n}{k} = \sum_{i+j=k} \binom{m}{i}\binom{n}{j}. \tag{18.9}$$

19 Enumeration in graded posets

19.1 Rank vector and Whitney numbers

Let P be a finite graded poset of rank r (with the rank function of P taking all values from 0 to r).

For $0 \leq i \leq r$, let $f_i(P)$ denote the number of elements in P of rank i.

The tuple $(f_0, f_1, \ldots, f_r)$ is the rank vector of P.

The f_i are Whitney numbers of the second kind.

19.2 Rank-generating polynomial

Define rank-generating polynomial of P by

$$f(P,x) := \sum_{i=0}^{r} f_i(P)x^i \tag{19.1}$$
$$= f_0 + f_1 x + \cdots + f_r x^r.$$

Fact 19.1. *For finite graded posets P and Q,*

$$f(P \times Q, x) = f(P, x)\, f(Q, x). \tag{19.2}$$

Proof. This follows from the observation that

$$f_k(P \times Q) = \sum_{i=0}^{k} f_i(P)\, f_{k-i}(Q).$$

This is because $\mathrm{rk}(x, y) = \mathrm{rk}(x) + \mathrm{rk}(y)$. See (17.2). □

Formula (19.2) extends to a finite number of posets:

$$f(P_1 \times \cdots \times P_k, x) = f(P_1, x) \ldots f(P_k, x). \tag{19.3}$$

19.3 Properties of a rank vector

A rank vector is symmetric if

$$f_i(P) = f_{r-i}(P) \tag{19.4}$$

for all $0 \leq i \leq r$.

Observe: The rank vector of opposite poset P^{op} is obtained by reversing the rank vector of P.

Hence: The rank vector of a finite self-dual poset is symmetric.

A rank vector is unimodal if

$$f_0(P) \leq \cdots \leq f_k(P) \geq f_{k+1}(P) \geq \cdots \geq f_r(P) \tag{19.5}$$

for some $0 \leq k \leq r$.

A rank vector is log-concave if

$$f_i(P)^2 \geq f_{i-1}(P)\, f_{i+1}(P) \tag{19.6}$$

for all $1 \leq i \leq r-1$.

Remark. Log-concavity is a stronger property than unimodality; that is, any log-concave tuple of positive numbers is unimodal.

19.4 Examples

Example 19.2. Consider Boolean poset $P(X)$ on a finite set X with $|X| = n$.

Elements of rank i are precisely the subsets of size i. Thus,

$$f_i(P(X)) = \binom{n}{i}. \tag{19.7}$$

Further,

$$f(P(X), x) = \sum_{i=0}^{n} \binom{n}{i} x^i = (1+x)^n. \tag{19.8}$$

This can also be deduced from formula (19.3) and poset isomorphism (15.2), noting that $f(C_2, x) = 1 + x$.

The rank vector is symmetric, unimodal, log-concave. (Log-concavity is addressed in the exercises. Symmetry says $\binom{n}{i} = \binom{n}{n-i}$; it also follows from self-duality.)

Example 19.3. Consider poset of divisors D_N. Write $N = p_1^{a_1} \dots p_k^{a_k}$. Then

$$\begin{aligned} f(D_N, x) &= \prod_{j=1}^{k} (1 + x + \cdots + x^{a_j}) \\ &= \prod_{j=1}^{k} \frac{1 - x^{a_j+1}}{1 - x}. \end{aligned} \tag{19.9}$$

This can be deduced from formula (19.3) and poset isomorphism (15.3).

The rank vector is symmetric, unimodal, log-concave. (Log-concavity is addressed in the exercises. Symmetry follows from self-duality.)

Let $N = 72$. Then $72 = 2^3 \cdot 3^2$. So rank of D_{72} is $r = 3 + 2 = 5$. Further,

$$\begin{aligned} f(D_{72}, x) &= (1 + x + x^2 + x^3)(1 + x + x^2) \\ &= 1 + 2x + 3x^2 + 3x^3 + 2x^4 + x^5. \end{aligned}$$

So the rank vector is $(1, 2, 3, 3, 2, 1)$.

Example 19.4. Let W be an n-dimensional vector space over a finite field $\mathbb{F}_q$. Consider poset of subspaces $\mathrm{Sub}(W)$ from Example 4.6.

Elements of rank i are i-dimensional subspaces of W.

$$f_i(\mathrm{Sub}(W)) = \binom{n}{i}_q. \tag{19.10}$$

The rank vector is symmetric, unimodal, log-concave. (We are not proving the latter two. Symmetry follows from self-duality.)

Note:

$$|\mathrm{Sub}(W)| = \sum_{i=0}^{n} \binom{n}{i}_q.$$

20 Problem set VI

20.1 Test your understanding

Say whether each of the following statements is true or false.

1. For any $m, n \geq 1$, we have $(mn)_q = (m)_q (n)_q$ as polynomials in q.
2. In $\prod_{i=1}^{n} (x + q^{i-1} y)$, coefficient of $x^{n-k} y^k$ is $q^{\binom{k}{2}} \binom{n}{k}_q$.
3. For a finite graded poset P, evaluating its rank-generating polynomial $f(P, x)$ at $x = 1$ yields $|P|$.
4. Let P and Q be finite graded posets. If P and Q have symmetric rank vectors, then $P \times Q$ has a symmetric rank vector.
5. Every log-concave tuple of nonnegative numbers is unimodal.

20.2 Problems related to theory

1. Binomial coefficient $\binom{n}{i}$ also makes sense when n is a fraction like $1/2$ or a negative number like -1. The same is true for a q-binomial coefficient. Write down these definitions.
2. Use (18.3) to prove identities (18.6a) and (18.6b). These give recursive formulas to compute q-binomial coefficients. For instance, you can try computing $\binom{4}{2}_q$ this way. Use induction on n to show that $\binom{n}{k}_q$ is a polynomial in q. Moreover, it has degree $k(n-k)$, and the coefficient of both q^0 and $q^{k(n-k)}$ is exactly 1.
3. Prove formula (19.10).
4. Use (18.3) to show $\binom{n}{i}_q = \binom{n}{n-i}_q$. Alternatively, deduce this identity from self-duality of subspace poset $\mathrm{Sub}(W)$ and formula (19.10). Strictly speaking we only get the identity when q is a prime power. However, since both sides are polynomials in q and agree on infinitely many values, they must coincide.

   Similarly, deduce identity $\binom{n}{i} = \binom{n}{n-i}$ from self-duality of Boolean poset $P(X)$ and formula (19.7).

5. Prove q-binomial theorem (18.4).

6. Prove q-Chu–Vandermonde identity (18.8). You may first try to prove Chu–Vandermonde identity (18.9).

7. Consider the result below.

   Theorem 20.1. *If rank vectors of finite graded posets P and Q are both log-concave, then so is rank vector of $P \times Q$.*

   Prove the following two special cases of this theorem.

   (a) Let $P = C_m$ and $Q = C_n$ be two chains. Use formula (20.3) to show that the rank vector of $C_m \times C_n$ is log-concave.

   (b) Let P be a finite graded poset of rank r with log-concave rank vector $(f_0, \ldots, f_r)$. Show that rank vector $(g_0, \ldots, g_{r+1})$ of $P \times C_2$ is also log-concave. (Use $g_i = f_i + f_{i-1}$ for all $0 \leq i \leq r+1$ where we set $f_{-1} = f_{r+1} = 0$.)

   Deduce log-concavity of the rank vectors of $P(X)$ and D_N from the above theorem using (15.2) and (15.3).

20.3 Problems related to examples

1. Show: $f_i(D_N)$ is number of weak compositions of i with k parts which are componentwise smaller than $(a_1, \ldots, a_k)$. More precisely, it is cardinality of the set $\{(m_1, \ldots, m_k) \in \mathbb{N}^k \mid \sum_{j=1}^k m_j = i \quad \text{and} \quad m_j \leq a_j \text{ for all } j\}$. Alternatively,

$$f_i(D_N) = \sum_{I \subseteq [k]} (-1)^{|I|} \binom{i - \sum_{j \in I}(a_j + 1) + k - 1}{k - 1}. \tag{20.1}$$

   (Extract coefficient of x^i in the two expressions in (19.9). Also use binomial theorem

$$(1 - x)^{-k} = \sum_{m=0}^{\infty} \binom{m + k - 1}{k - 1} x^m \tag{20.2}$$

   for negative exponents.)

   We highlight two special cases.

(a) Let $k = 2$ and assume $a_1 \leq a_2$. Then

$$f_i(D_N) = \begin{cases} i+1 & \text{if } 0 \leq i \leq a_1, \\ a_1+1 & \text{if } a_1 \leq i \leq a_2, \\ a_1+a_2+1-i & \text{if } a_2 \leq i \leq a_1+a_2. \end{cases} \tag{20.3}$$

(b) Put $(a_1, \ldots, a_k) = (1, \ldots, 1)$ to deduce the hypergeometric identity

$$\binom{n}{i} = \sum_{m=0}^{n} (-1)^m \binom{n}{m} \binom{i-2m+n-1}{n-1}. \tag{20.4}$$

2. Give examples to show that a symmetric tuple does not have to be log-concave, and a log-concave tuple does not have to be symmetric.

3. A network of roads forms a triangular grid (as in a Meruprastar or Pascal's triangle). From the top node (row 0), 2^{1000} people set out. The group splits into two: half go left-below and half go right-below. At each of the two nodes of row 1, the group splits into two as above. The same procedure continues. Question: How many people arrive at each node of the 1000-th row?

20.4 Explorations

1. The q-Chu–Vandermonde identity is related to matrix compositions. Even further, it is related to the existence of the graded q-bialgebra of polynomials $\mathbb{k}_q[x]$. See Section 2.11 and Section 10.2 of the notes "Braid arrangements and graded bialgebras" from my homepage.

2. Explore hypergeometric identities in connection to (18.9) and (20.4).

3. Look at Stanley's paper titled "Log-Concave and Unimodal Sequences in Algebra, Combinatorics, and Geometry".

4. There are entire books written on q-calculus. For instance, see those by Kac and Cheung, Ernst, Hirschhorn, Johnson.

21 Adjunction between posets

21.1 Galois connection

Let P and Q be two posets. Suppose

$$\lambda : P \to Q \quad \text{and} \quad \rho : Q \to P$$

are order-preserving maps.

We say (λ, ρ) is a Galois connection or an adjunction if

$$\lambda(x) \leq y \iff x \leq \rho(y) \tag{21.1}$$

for $x \in P$ and $y \in Q$.

We also say that ρ is the right adjoint of λ, and λ is the left adjoint of ρ.

21.2 Examples

Example 21.1. Consider posets $\mathbb{R}$ and $\mathbb{Z}$.

Consider the inclusion, floor, ceiling maps:

- $i : \mathbb{Z} \to \mathbb{R}$ where $i(n) = n$
- $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ where $\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leq x\}$
- $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ where $\lceil x \rceil = \min\{n \in \mathbb{Z} \mid x \leq n\}$.

They are order-preserving.

For any $x \in \mathbb{R}$ and $n \in \mathbb{Z}$,

$$\begin{aligned} i(n) \leq x &\iff n \leq \lfloor x \rfloor \\ \lceil x \rceil \leq n &\iff x \leq i(n). \end{aligned} \tag{21.2}$$

Thus, the floor function is the right adjoint of the inclusion map, while the ceiling function is the left adjoint of the inclusion map.

Example 21.2. Let W be a vector space over field $\Bbbk$.

For any subset $A \subseteq W$, let $L(A)$ denote the smallest subspace of W which contains A. It is obtained by taking linear spans of elements in A.

Observe:

$$A \subseteq U \iff L(A) \subseteq U. \tag{21.3}$$

for any subset A and subspace U of W.

This says that the inclusion map

$$i : \mathrm{Sub}(W) \to P(W), \qquad U \mapsto U \tag{21.4}$$

is the right adjoint of the map

$$L : P(W) \to \mathrm{Sub}(W), \qquad A \mapsto L(A). \tag{21.5}$$

Example 21.3. Fix a subset B of a set X.

Define order-preserving map

$$\lambda : P(X) \to P(X), \qquad A \mapsto A \cap B. \tag{21.6}$$

Define order-preserving map

$$\rho : P(X) \to P(X), \qquad C \mapsto B^c \cup C, \tag{21.7}$$

where $B^c := X \setminus B$.

Observe:

$$A \cap B \subseteq C \iff A \subseteq B^c \cup C. \tag{21.8}$$

Thus, (λ, ρ) is an adjunction between $P(X)$ and itself.

Example 21.4. Let $f : X \to Y$ be a function.

Define order-preserving map

$$f : P(X) \to P(Y), \quad A \mapsto f(A), \tag{21.9}$$

where $f(A) = \{f(a) \mid a \in A\}$.

Define order-preserving map

$$f^{-1} : P(Y) \to P(X), \quad B \mapsto f^{-1}(B), \tag{21.10}$$

where $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$.

Observe:

$$f(A) \subseteq B \iff A \subseteq f^{-1}(B) \tag{21.11}$$

for any subsets $A \subseteq X$ and $B \subseteq Y$.

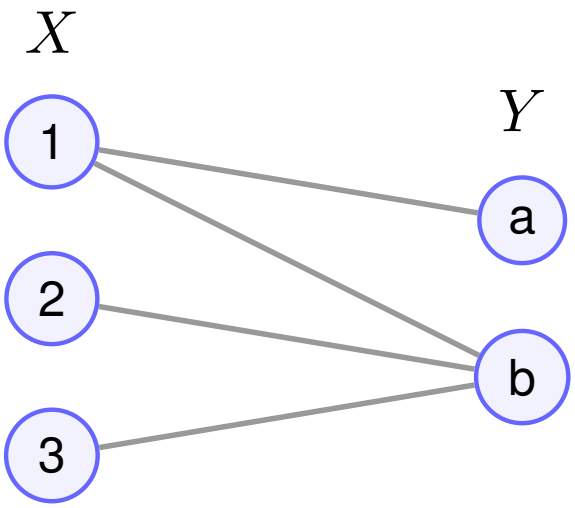
Thus, $\left(f, f^{-1}\right)$ is a Galois connection.

21.3 Bipartite graph

Let X and Y be any sets, and let $R \subseteq X \times Y$.

We refer to this data as a bipartite graph, and denote it by (X, R, Y).

Example 21.5. Let $X = \{1, 2, 3\}$ and $Y = \{a, b\}$. Let $R = \{(1, a), (1, b), (2, b), (3, b)\}$. This is illustrated below.



Example 21.6.

- X is a set of students, Y is a set of courses, $(x, y) \in R$ if student x takes course y.
- X is a set of words, Y is a set of poems, $(x, y) \in R$ if word x appears in poem y.
- X is a set of workers, Y is a set of tasks, $(x, y) \in R$ if worker x performs task y.

Example 21.7. Let (X, R, Y) be a bipartite graph.

We will now use R to connect Boolean posets $P(X)$ and $P(Y)$. For technical purposes, we will work with opposite poset $P(Y)^{op}$.

Define order-preserving map

$$\lambda_R : P(X) \to P(Y)^{op}, \quad A \mapsto \lambda_R(A), \tag{21.12}$$

where $\lambda_R(A) = \{y \in Y \mid \forall a \in A,\ (a, y) \in R\}$.

Define order-preserving map

$$\rho_R : P(Y)^{op} \to P(X), \quad B \mapsto \rho_R(B), \tag{21.13}$$

where $\rho_R(B) = \{x \in X \mid \forall b \in B,\ (x, b) \in R\}$.

What are $\lambda_R(\emptyset)$ and $\rho_R(\emptyset)$?

In Example 21.6:

- $\lambda_R(A)$ is the set of courses taken by every student in set A, and $\rho_R(B)$ is the set of students who take every course in set B.
- $\lambda_R(A)$ is the set of poems which contain every word in set A, and $\rho_R(B)$ is the set of words which appear in every poem in set B.
- $\lambda_R(A)$ is the set of tasks which are performed by every worker in set A, and $\rho_R(B)$ is the set of workers who perform every task in set B.

Observe:

(21.14) $$\lambda_R(A) \supseteq B \iff A \subseteq \rho_R(B)$$

for any subsets $A \subseteq X$ and $B \subseteq Y$.

Either side expresses the fact that
$\forall a \in A,\ \forall b \in B,\ (a, b) \in R$.

Thus, (λ_R, ρ_R) is a Galois connection.

In Example 21.6:

Either side of (21.14) holds iff

- every student in set A takes every course in set B.
- every word in set A appears in every poem in set B.
- every worker in set A performs every task in set B.

21.4 Properties of adjunctions

Fact 21.8. *Let (λ, ρ) be an adjunction between P and Q. Then*

$$\lambda(\rho(y)) \leq y \quad \textit{and} \quad x \leq \rho(\lambda(x)) \tag{21.15}$$

for $x \in P$ and $y \in Q$.

Proof. For the first inequality, put $x = \rho(y)$ in (21.1). For the second inequality, put $y = \lambda(x)$ in (21.1). □

It is of interest to know for which $x \in P$ and $y \in Q$ equalities hold in (21.15).

Fact 21.9. *Let (λ, ρ) be an adjunction between P and Q. Then*

(21.16) $\lambda(\rho(\lambda(x))) = \lambda(x)$ *and* $\rho(y) = \rho(\lambda(\rho(y)))$

for $x \in P$ and $y \in Q$.

Proof. For the first equality:

Apply map λ to $x \leq \rho(\lambda(x))$ to obtain $\lambda(x) \leq \lambda(\rho(\lambda(x)))$.

Substitute $y = \lambda(x)$ in $\lambda(\rho(y)) \leq y$ to obtain $\lambda(\rho(\lambda(x))) \leq \lambda(x)$.

Combining the two yields $\lambda(\rho(\lambda(x))) = \lambda(x)$.

The second equality is obtained in a similar manner. □

21.5 Preservations of joins and meets

Fact 21.10. *Let (λ, ρ) be an adjunction between P and Q. Then:*

- *If $x, y \in P$ such that their join exists in P, then join of $\lambda(x)$ and $\lambda(y)$ exists in Q, and is given by*

$$\lambda(x \vee y) = \lambda(x) \vee \lambda(y). \tag{21.17}$$

- *If P has bottom element $\bot_P$, then Q also has bottom element $\bot_Q$, and is given by*

$$\lambda(\bot_P) = \bot_Q. \tag{21.18}$$

Proof. For (21.17), we show that $\lambda(x \vee y)$ is the least upper bound of $\lambda(x)$ and $\lambda(y)$.

- **UB:** $x, y \leq x \vee y \implies \lambda(x), \lambda(y) \leq \lambda(x \vee y)$ since λ is order-preserving.
- **LUB:** Let $z \in Q$ be an upper bound of $\lambda(x)$ and $\lambda(y)$. Then:

$$\lambda(x) \leq z \text{ and } \lambda(y) \leq z \overset{(21.1)}{\iff} x \leq \rho(z) \text{ and } y \leq \rho(z)$$

$$\iff x \vee y \leq \rho(z) \overset{(21.1)}{\iff} \lambda(x \vee y) \leq z.$$

For (21.18):

For any $z \in Q$, $\bot_P \leq \rho(z)$ always holds in P.

By (21.1), $\bot_P \leq \rho(z) \iff \lambda(\bot_P) \leq z$. Thus, $\lambda(\bot_P) = \bot_Q$. □

Fact 21.11. *Let (λ, ρ) be an adjunction between posets P and Q.*

If join $\bigvee S$ exists for subset $S \subseteq P$, then $\bigvee \lambda(S)$ exists in Q, and is given by

$$\lambda\left(\bigvee S\right) = \bigvee \lambda(S), \tag{21.19}$$

where $\lambda(S) = \{\lambda(x) \mid x \in S\}$.

For $S = \{x, y\}$, we get (21.17).

For $S = \emptyset$, we get (21.18).

Thus Fact 21.11 generalizes Fact 21.10.

Let us now prove (21.19) which unifies the proofs of (21.17) and (21.18). Pay attention to the latter since it is the triviality of $\emptyset$ which can cause confusion.

Proof. Let $z \in Q$.

$$
\begin{aligned}
z \text{ is an UB of } \lambda(S) &\iff \forall x \in S,\ \lambda(x) \leq z \\
&\overset{(21.1)}{\iff} \forall x \in S,\ x \leq \rho(z) \\
&\iff \rho(z) \text{ is an UB of } S \text{ in } P \\
&\iff \bigvee S \leq \rho(z) \\
&\overset{(21.1)}{\iff} \lambda\left(\bigvee S\right) \leq z.
\end{aligned}
$$

This shows that $\lambda\left(\bigvee S\right)$ is the unique least upper bound of $\lambda(S)$.

Thus, $\lambda\left(\bigvee S\right) = \bigvee \lambda(S)$. □

Fact 21.12. *Let (λ, ρ) be an adjunction between P and Q. Then:*

- *If $x, y \in Q$ such that their meet exists in Q, then meet of $\rho(x)$ and $\rho(y)$ exists in P, and is given by*

$$\rho(x \wedge y) = \rho(x) \wedge \rho(y). \tag{21.20}$$

- *If Q has top element $\top_Q$, then P also has top element $\top_P$, and is given by*

$$\rho(\top_Q) = \top_P. \tag{21.21}$$

Fact 21.13. *If meet $\bigwedge S$ exists for subset $S \subseteq Q$, then $\bigwedge \rho(S)$ exists in P, and is given by*

$$\rho\left(\bigwedge S\right) = \bigwedge \rho(S), \tag{21.22}$$

where $\rho(S) = \{\rho(y) \mid y \in S\}$.

Example 21.14. Consider adjunction (L, i) between $P(W)$ and $\mathrm{Sub}(W)$ in Example 21.2.

Since i is a right adjoint, it preserves meets. Thus, for any subspaces U and V,

$$U \wedge V = i(U \wedge V) = i(U) \wedge i(V) = U \cap V.$$

Since L is a left adjoint, it preserves joins. Thus, for any subsets A and B,

$$L(A \cup B) = L(A \vee B) = L(A) \vee L(B) = L(A) + L(B).$$

The above also works for arbitrary meets and joins.

21.6 Closed and coclosed elements

Let (λ, ρ) be an adjunction between P and Q. Put

$$P_c = \{x \in P \mid x = \rho(\lambda(x))\} \tag{21.23}$$

$$Q_o = \{y \in Q \mid \lambda(\rho(y)) = y\}. \tag{21.24}$$

We view P_c and Q_o as full subposets of P and Q, respectively.

An element of P is closed if it belongs to P_c.

An element of Q is coclosed if it belongs to Q_o.

Fact 21.15. *Let (λ, ρ) be an adjunction between P and Q. We have*

(21.25) $\quad P_c = \mathrm{image}(\rho) \quad$ *and* $\quad Q_o = \mathrm{image}(\lambda).$

Proof. For the first equality:

$\underline{P_c \subseteq \mathrm{image}(\rho)}.$

Take $x \in P_c$. Then $x = \rho(y)$ for $y = \lambda(x)$. So $x \in \mathrm{image}(\rho)$.

$\underline{\mathrm{image}(\rho) \subseteq P_c}.$

Take $\rho(y) \in \mathrm{image}(\rho)$. Then $\rho(y) = \rho(\lambda(\rho(y)))$ by (21.16). So $\rho(y) \in P_c$.

The second equality is proved similarly. □

Fact 21.16. *Let (λ, ρ) be an adjunction between P and Q. There are inverse poset isomorphisms*

$$P_c \rightleftarrows Q_o \tag{21.26}$$

obtained by restricting λ and ρ.

Thus we have a commutative diagram

$$\begin{array}{ccc} P & \underset{\rho}{\overset{\lambda}{\rightleftarrows}} & Q \\ \uparrow & & \uparrow \\ P_c & \underset{\rho}{\overset{\lambda}{\rightleftarrows}} & Q_o \end{array} \tag{21.27}$$

of posets.

Proof. It follows from (21.25) that λ restricts to a map from P_c to Q_o, and ρ restricts to a map from Q_o to P_c.

The fact that they are inverses follows from (21.23) and (21.24). □

21.7 Uniqueness of adjoints

Fact 21.17. *Adjoints (left or right) of an order-preserving map between posets may not exist, but they are unique whenever they exist.*

In more detail: Suppose we are given an order-preserving map $\lambda : P \to Q$. Then we can ask whether it has a right adjoint. That is, whether there exists an order-preserving map $\rho : Q \to P$ such that (21.1) holds. Such a ρ may not exist, but if it exists, then it is unique.

Similarly, we can ask whether λ has a left adjoint. Similar comments apply.

21.8 Distribution and quantile functions

This is only a sketch, for full details see Section 3.1 and Appendix A of probability notes on my homepage.

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

Here $\mathcal{F}$ is a σ-algebra on Ω.

Elements of $\mathcal{F}$ are called events.

For an event A, $P(A)$ denotes the probability that A occurs.

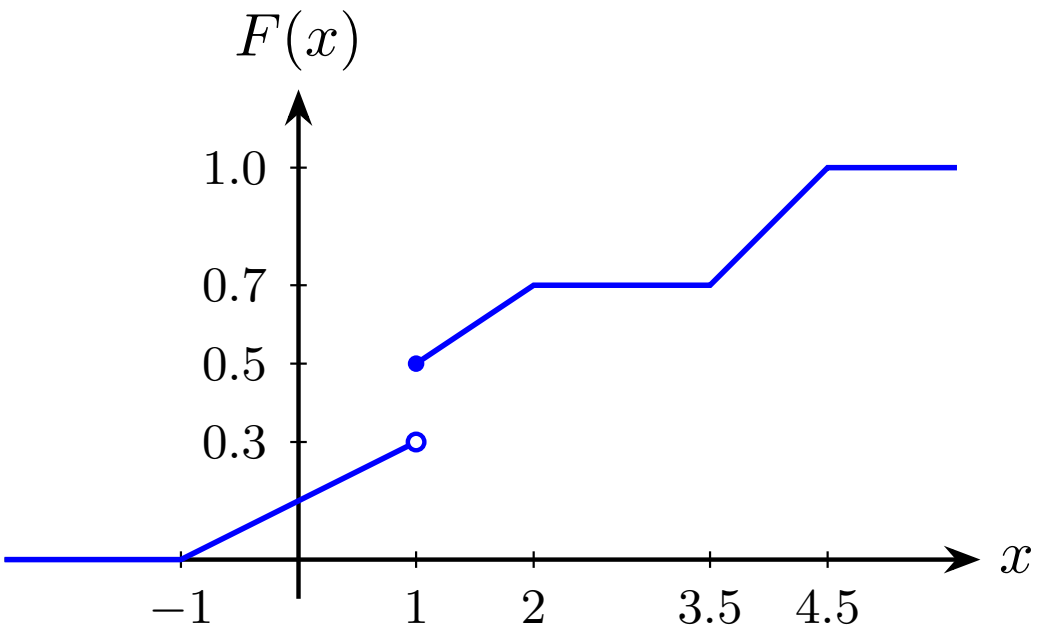
Let $X : \Omega \to \mathbb{R}$ be a random variable.

Define $F : \mathbb{R} \to [0, 1]$ by

$$F(x) := P[X \leq x]. \tag{21.28}$$

This is the distribution function of X. It is increasing and right continuous.

An illustrative example is shown below.



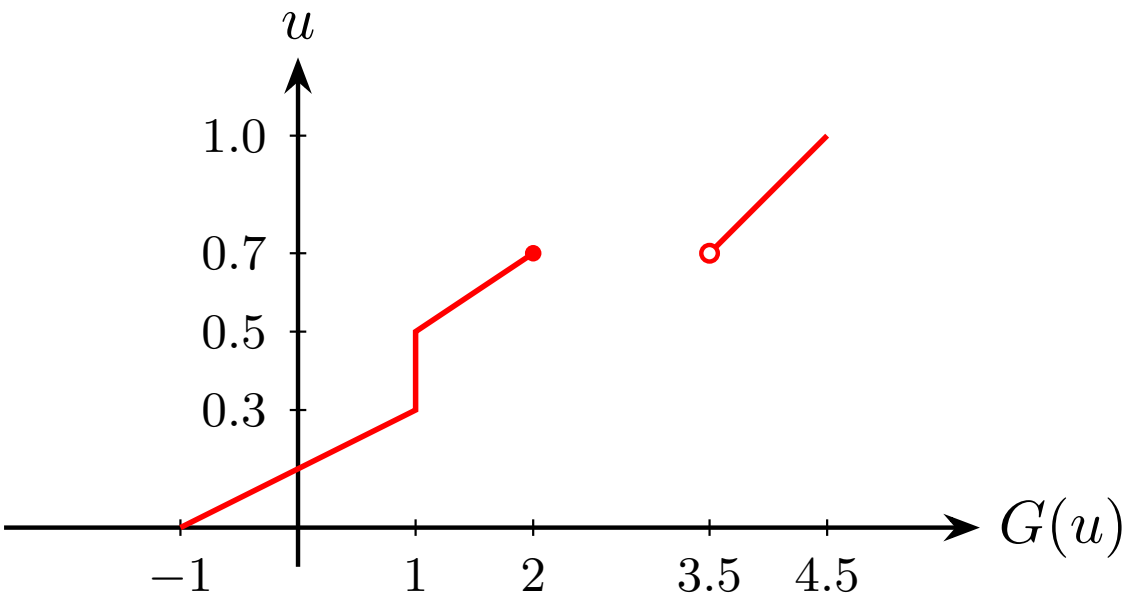
Define $G : (0,1) \to \mathbb{R}$ by

$$
\begin{aligned}
G(u) &= \inf\{x \mid u \le F(x)\} \\
&= \inf\{x \mid u \le P[X \le x]\}.
\end{aligned}
\tag{21.29}
$$

This is the quantile function of X. It is increasing and left continuous.

$G(1/2)$ is the median, $G(1/100), G(2/100), \ldots,$ $G(99/100)$ are percentiles.

On our illustrative example, we get the following.



We have

$$G(u) \leq x \iff u \leq F(x) \tag{21.30}$$

for $x \in \mathbb{R}$ and $u \in (0, 1)$.

Up to a technical point, G is the left adjoint of F.

22 Closure operator

22.1 Closure operator

A closure operator on poset P is a map

$$c : P \to P \tag{22.1}$$

such that for every $x, y \in P$,

- $x \leq c(x)$,
- if $x \leq c(y)$, then $c(x) \leq c(y)$.

We refer to $c(x)$ as the closure of x.

Fact 22.1. *Let $c : P \to P$ be a closure operator. Then:*

- $c(c(x)) = c(x)$,
- *if $x \leq y$, then $c(x) \leq c(y)$.*

In particular, closure operator is an order-preserving map from P to itself.

22.2 Closed elements

Let c be a closure operator on P.

An element $z \in P$ is closed if $c(z) = z$, that is, the closure of z is itself.

Let P_c denote the full subposet of all closed elements.

The map c induces an order-preserving map $c : P \to P_c$.

Let $i : P_c \to P$ be the inclusion map.

22.3 Examples

Example 22.2. Let S be a subset of $\mathbb{R}^2$.

We say S is convex if, for any two points in S, the straight line segment joining them lies entirely in S.

Some examples of convex sets are:

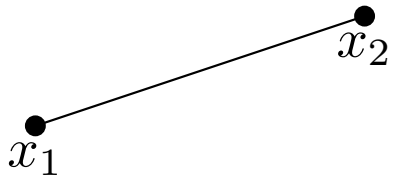
- empty set $\emptyset$, a single point,
- a line segment (with or without its endpoints), lines,
- an open disc, a closed disc,
- the interior of a convex polygon, convex polygon with its boundary included,
- all of $\mathbb{R}^2$,

and so on.

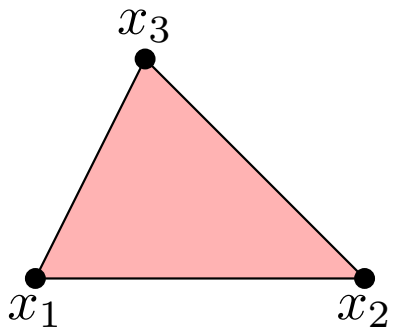
Observe: Intersection of two convex sets is convex, but union of two convex sets is not convex in general. For example, the set consisting of two disjoint closed discs is not convex.

For subset $S \subseteq \mathbb{R}^2$, let $\mathrm{conv}(S)$ denote convex hull of S (smallest convex set containing S).

The convex hull of x_1 and x_2 is the line segment joining them.

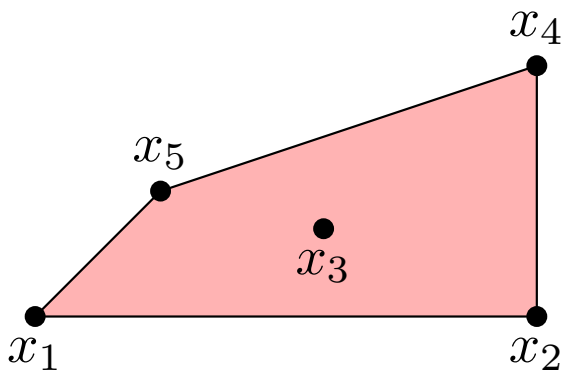


The convex hull of x_1, x_2, x_3 is the triangle formed by these points.



It is possible that x_1, x_2, x_3 lie on a straight line, in which case the convex hull will be a line segment joining the two extreme points.

A convex hull of five points in the plane is shown below.



Note very carefully that x_3 lies in the interior of this convex set. Thus, the above convex set is also the convex hull of x_1, x_2, x_4, x_5.

Consider Boolean poset $P(\mathbb{R}^2)$.

Define map

(22.2) $$\mathrm{conv} : P(\mathbb{R}^2) \to P(\mathbb{R}^2)$$

which sends subset S to its convex hull $\mathrm{conv}(S)$.

Clearly, $S \subseteq \mathrm{conv}(S)$, and if $S \subseteq \mathrm{conv}(T)$, then $\mathrm{conv}(S) \subseteq \mathrm{conv}(T)$.

Thus, conv is a closure operator.

Its closed elements are precisely the convex sets in $\mathbb{R}^2$.

Example 22.3. Consider Boolean poset $P(\mathbb{R}^2)$.

Define map $c : P(\mathbb{R}^2) \to P(\mathbb{R}^2)$ where $c(S)$ is the smallest axis-aligned rectangular bounding box for S. By definition, we include the boundary of the rectangle.

This is a closure operator.

The full subposet P_c of closed elements consists precisely of all axis-aligned rectangles (including empty sets, points, and line segments).

Example 22.4. Let X be a topological space, and consider Boolean poset $P(X)$.

Define map $c : P(X) \to P(X)$ by $c(A) = \bar{A}$, where $\bar{A}$ is the smallest closed set in X containing A.

The full subposet $P(X)_c$ of closed elements consists precisely of all closed sets in X.

The terminology of closure operators and closed sets is motivated by this example.

Example 22.5. Let M be a monoid, and consider Boolean poset $P(M)$.

Define map $c : P(M) \to P(M)$ by $c(A) = \langle A \rangle_M$ is the submonoid of M generated by A.

This is a closure operator.

The full subposet $P(M)_c$ of closed elements consists precisely of all submonoids of M.

We have a similar example with monoid M replaced by group G. Now $c(A) = \langle A \rangle_G$ is the subgroup of G generated by A. Poset of closed elements P_c is the poset of all subgroups of G.

22.4 Adjunction of a closure operator

Let c be a closure operator on poset P.

Fact 22.6. *We have:* (c, i) *is an adjunction between* P *and* P_c*. That is,*

(22.3) $$c(x) \leq y \iff x \leq y$$

for $x \in P$ *and* $y \in P_c$*.*

Proof. Forward implication. Suppose $c(x) \leq y$. Then $x \leq c(x) \leq y$.

Backward implication. Suppose $x \leq y$. Since $y \in P_c$, $c(y) = y$. Hence $c(x) \leq y$. □

Example 22.7. Let W be a vector space over a field $\mathbb{k}$, and consider Boolean poset $P(W)$.

Define map $L : P(W) \to P(W)$ where $L(A)$ is the linear span of subset A.

This is a closure operator.

The full subposet $P(W)_L$ of closed elements consists precisely of all subspaces of W.

The resulting adjunction (L, i) was discussed in Example 21.2.

Fact 22.8. *If P is a complete lattice, then so is P_c.*

The meet in P_c coincides with the meet in P, while the join in P_c is the closure of the join in P. That is,

$$\text{(22.4)} \qquad \bigwedge_{P_c} S = \bigwedge_{P} S \quad \textit{and} \quad \bigvee_{P_c} S = c\big(\bigvee_{P} S\big)$$

for all subsets $S \subseteq P_c$.

Proof. We use adjunction (c, i) from (22.3).

Since i is a right adjoint, it preserves meets by (21.22). This yields the first equality.

Since c is a left adjoint, it preserves joins by (21.19). This yields the second equality. Note: Since $S \subseteq P_c$, $c(S) = S$. □

22.5 Coclosure operator

A coclosure operator on poset Q is a map

$$k : Q \to Q \tag{22.5}$$

such that for every $x, y \in Q$,

- $k(x) \leq x$,
- if $k(x) \leq y$, then $k(x) \leq k(y)$.

We refer to $k(x)$ as the coclosure (or kernel) of x.

Fact 22.9. *Let $k : Q \to Q$ be a coclosure operator. Then:*

- $k(k(x)) = k(x)$,
- *if $x \leq y$, then $k(x) \leq k(y)$.*

In particular, a coclosure operator is an order-preserving map from Q to itself.

22.6 Coclosed (open) elements

Let k be a coclosure operator on Q.

An element $z \in Q$ is coclosed (or open) if $k(z) = z$, that is, the coclosure of z is itself.

Let Q_o denote the full subposet of all coclosed elements.

The map k induces an order-preserving map $k : Q \to Q_o$.

Let $i : Q_o \to Q$ be the inclusion map.

Example 22.10. Let X be a topological space, and consider Boolean poset $P(X)$.

Define map $k : P(X) \to P(X)$ by $k(A) = A^\circ$, where A° is the largest open set in X contained in A.

The full subposet $P(X)_o$ of coclosed elements consists precisely of all open sets in X.

22.7 Adjunction of a coclosure operator

Let k be a coclosure operator on poset Q.

Fact 22.11. *We have:* (i, k) *is an adjunction between* Q_o *and* Q*. That is,*

(22.6) $$x \leq y \iff x \leq k(y)$$

for $x \in Q_o$ *and* $y \in Q$*.*

Fact 22.12. *If Q is a complete lattice, then so is Q_o.*

The join in Q_o coincides with the join in Q, while the meet in Q_o is the coclosure of the meet in Q. That is,

$$\bigvee_{Q_o} S = \bigvee_{Q} S \quad \text{and} \quad \bigwedge_{Q_o} S = k\big(\bigwedge_{Q} S\big) \tag{22.7}$$

for all subsets $S \subseteq Q_o$.

22.8 (Co)closure operators of an adjunction

Fact 22.13. *Let (λ, ρ) be an adjunction between P and Q. Then:*

1. *Composite $\rho\lambda$ is a closure operator on P. Closed sets P_c of (λ, ρ) coincide with closed sets P_c of $\rho\lambda$.*

2. *Composite $\lambda\rho$ is a coclosure operator on Q. Coclosed sets Q_o of (λ, ρ) coincide with coclosed sets Q_o of $\lambda\rho$.*

Proof. For the first part, we verify the two axioms of a closure operator:

- $x \leq \rho\lambda(x)$ from (21.15).
- If $x \leq \rho\lambda(y)$, then $\lambda(x) \leq \lambda(y)$ from (21.1), and so $\rho\lambda(x) \leq \rho\lambda(y)$.

The second part is similar. □

Theorem 22.14. *Let P and Q be complete lattices. Let (λ, ρ) be an adjunction between P and Q.*

Then (21.26) *are inverse isomorphisms between complete lattices P_c and Q_o which preserve both meets and joins.*

23 Problem set VII

23.1 Test your understanding

Say whether each of the following statements is true or false.

1. Let $c : P \to P$ be a closure operator on a lattice P. If x and y are closed sets of P, then so is $x \wedge y$.
2. The circle S^1 is a convex set of $\mathbb{R}^2$.
3. For poset P, identity map $P \to P$ is both a closure operator and a coclosure operator.

23.2 Problems related to theory

1. Prove Fact 21.13 by following the proof pattern of Fact 21.11. An alternative way is given in the next exercise.
2. Check: The following two statements are equivalent.
   - (λ, ρ) is an adjunction between P and Q.
   - (ρ, λ) is an adjunction between P^{op} and Q^{op}.

   In other words, left and right adjoints swap places when we go to opposite posets.

   Use this to deduce Fact 21.11 and Fact 21.13 from each other.
3. Prove the following result using the outline provided.

   Theorem 23.1. *Let P be a complete lattice, and let Q be a poset.*

   An order-preserving map $\lambda : P \to Q$ has a right adjoint $\rho : Q \to P$ iff λ preserves all joins, that is, (21.19) *holds for all $S \subseteq P$.*

   The forward implication follows from Fact 21.11. For the backward implication: We are only given λ. Define ρ by the formula

   $$\rho(y) := \bigvee \{x \in P \mid \lambda(x) \leq y\}. \tag{23.1}$$

   Rhs exists since P is complete. Check that ρ is indeed the right adjoint of λ, that is, (21.1) holds.

   Deduce the following dual result.

Theorem 23.2. *Let Q be a complete lattice, and let P be a poset.*

An order-preserving map $\rho : Q \to P$ has a left adjoint $\lambda : P \to Q$ iff ρ preserves all meets, that is, (21.22) holds for all $S \subseteq Q$.

Given ρ, the left adjoint λ is defined by

$$\lambda(x) := \bigwedge \{y \in Q \mid x \leq \rho(y)\}. \tag{23.2}$$

4. Prove Fact 21.17 that adjoints are unique whenever they exist.

5. Prove the following fact.

   Fact 23.3. *Adjunctions can be composed:*

   If (λ, ρ) is an adjunction between P and Q, and (μ, δ) is an adjunction between Q and R, then $(\mu\lambda, \rho\delta)$ is an adjunction between P and R.

6. Prove the following fact.

   Fact 23.4. *Let $f : P \to Q$ be a poset isomorphism. Then f clearly preserves all meets and joins which exist in P. So it has both a left and a right adjoint. Both adjoints are given by $f^{-1} : Q \to P$. In this case, $P_c = P_o = P$ and $Q_o = Q_c = Q$.*

7. Consider the following statement.

   Fact 23.5. *Having an adjunction between $P(X)$ and $P(Y)$ is equivalent to having an adjunction between $P(X)$ and $P(Y)^{op}$.*

   Recall from (4.2) that $P(X)$ is isomorphic to $P(X)^{op}$ under taking complements in X. Thus by Fact 23.4, we have an adjunction between any Boolean poset and its opposite. Now use Fact 23.3 and the diagrams below to deduce Fact 23.5.

$$P(X) \underset{\rho}{\overset{\lambda}{\rightleftarrows}} P(Y) \rightleftarrows P(Y)^{op}$$

$$P(X) \underset{\rho'}{\overset{\lambda'}{\rightleftarrows}} P(Y)^{op} \rightleftarrows P(Y)$$

8. Let P be a bounded poset. Let $f : P \to \{*\}$ be the unique map to a singleton set. It is clearly order-preserving, and preserves all existing meets and joins in P. (Note: We are not assuming P to be a complete lattice.) Show that the left and right adjoints of f exist. Describe them.

9. Let P be a poset. Let Q be the full subposet of $P \times P$ consisting of those (x, y) such

that $x \leq y$. Consider order-preserving maps

$$\begin{array}{lll} i : P \to Q & p_1 : Q \to P & p_2 : Q \to P \\ x \mapsto (x, x) & (y, z) \mapsto y & (y, z) \mapsto z. \end{array} \tag{23.3}$$

Check that (p_2, i) and (i, p_1) are adjunctions.

10. For P a lattice, define order-preserving map $i : P \to P \times P$ by $i(x) = (x, x)$. Determine its left and right adjoints.

11. Let $f : P \to Q$ be an order-preserving bijection. Show that the following statements are equivalent.

    (a) f is a poset isomorphism.

    (b) f has a left adjoint.

    (c) f has a right adjoint.

    In this situation, left and right adjoints of f coincide, and are given by the inverse of f.

12. Prove Fact 22.1 on closure operators, and Fact 22.9 on coclosure operators.

13. A closure operator on poset P is the same as a coclosure operator on poset P^{op}.

23.3 Problems related to examples

1. Consider poset $(\mathbb{Z}, \leq)$. Fix positive integer c. Define order-preserving map $f : \mathbb{Z} \to \mathbb{Z}$ by $f(m) = c \cdot m$. Determine its left and right adjoints.

2. This is continuation of Example 21.4. Let $f : X \to Y$ be a function. Define order-preserving map

$$f_! : P(X) \to P(Y)$$

where $f_!(A) = \{b \in B \mid f^{-1}(\{b\}) \subseteq A\}$. Check: $(f^{-1}, f_!)$ is a Galois connection.

Thus, f^{-1} has both left and right adjoints. Deduce that f^{-1} preserves arbitrary meets and joins:

$$f^{-1}(\bigcup_j B_j) = \bigcup_j f^{-1}(B_j) \quad \text{and} \quad f^{-1}(\bigcap_j B_j) = \bigcap_j f^{-1}(B_j).$$

In contrast, for f we only have

$$f(\bigcup_i A_i) = \bigcup_i f(A_i) \quad \text{and} \quad f(\bigcap_i A_i) \subseteq \bigcap_i f(A_i).$$

3. Recall adjunction (λ_R, ρ_R) from Example 21.7. Use (21.18) and (21.21) to deduce that $\lambda_R(\emptyset) = Y$ and $\rho_R(\emptyset) = X$.

4. Examples 21.4 and 21.3, up to Fact 23.5, are special case of Example 21.7 as follows.
   - For a function $f : X \to Y$, define $R \subseteq X \times Y$ by $(x, y) \in R$ if $f(x) \neq y$. This yields an adjunction (λ_R, ρ_R) between $P(X)$ and $P(Y)^{op}$. Now use Fact 23.5 to turn it into an adjunction between $P(X)$ and $P(Y)$. Check that this is precisely (f, f^{-1}) in Example 21.4.
   - Do something similar for Example 21.3.

5. Let $P = (\mathbb{N}_+, |)$, that is, $m \leq_P n$ if $m|n$, and $Q = (\mathbb{N}_+, \leq)$, that is, $m \leq_Q n$ if $m \leq n$. Define map $f : P \to Q$ by $f(n) = n$. It is order-preserving since $m|n$ implies $m \leq n$. Show that f does not have either a left adjoint or a right adjoint.

6. Consider poset $\hat{\mathbb{R}}$ of extended real numbers. For any real number c, define order-preserving map $T_c : \hat{\mathbb{R}} \to \hat{\mathbb{R}}$ by $T_c(x) = x + c$. By convention, $T_c(\infty) = \infty$ and $T_c(-\infty) = -\infty$. Note: T_c is a poset isomorphism with inverse given by T_{-c}.

   Use Fact 23.4 to deduce that (T_c, T_{-c}) is an adjunction between $\hat{\mathbb{R}}$ and itself. It expresses the simple fact that $x + c \leq y \iff x \leq y - c$.

7. Consider poset $\hat{\mathbb{N}}$ of extended natural numbers. For any positive integer c, define order-preserving map $T_c : \hat{\mathbb{N}} \to \hat{\mathbb{N}}$ by $T_c(x) = x + c$. By convention, $T_c(\infty) = \infty$.
   - Show that T_c preserves arbitrary meets. So it has a left adjoint f_c given by formula (23.2). Explicitly, $f_c(n) = \max(n - c, 0)$. Use (21.1) to check directly that (f_c, T_c) is an adjunction.
   - Show that T_c does not preserve arbitrary joins. So it cannot have a right adjoint. Prove this directly using (21.1).

   What happens for $T_c : \mathbb{N} \to \mathbb{N}$ defined by the same formula $T_c(x) = x + c$? Recall that $\mathbb{N}$ is not a complete lattice. So Theorem 23.1 and Theorem 23.2 do not apply.

8. Determine the closure and coclosure operators associated to the adjunctions in Examples 21.1, 21.3, 21.4, 21.7.

23.4 Explorations

1. Let $\mathcal{S}$ be the set of all sentences in a formal language, and let $\mathcal{M}$ be the class of all possible models (structures). Let $\models$ be the satisfaction relation, where $M \models \sigma$ denotes that model M satisfies sentence σ. Understand the above terms precisely. The construction of Example 21.7 then yields a Galois connection between $P(\mathcal{S})$ and $P(\mathcal{M})^{op}$ (syntax and semantics).

2. Let K be a field, let $A = K[x_1, \ldots, x_n]$ be the polynomial ring over K, and let $\mathbb{A}^n = K^n$ be affine n-space. Define relation $R \subseteq A \times \mathbb{A}^n$ by

$$(f, p) \in R \iff f(p) = 0.$$

   The construction of Example 21.7 then yields a Galois connection between $P(A)$ and $P(\mathbb{A})^{op}$. The left adjoint V maps a set of polynomials $S \subseteq A$ to their zero-set:

$$V(S) = \{p \in \mathbb{A}^n \mid \forall f \in S,\ f(p) = 0\}.$$

   The right adjoint I maps a set of points $Y \subseteq \mathbb{A}^n$ to its vanishing ideal:

$$I(Y) = \{f \in A \mid \forall p \in Y,\ f(p) = 0\}.$$

   We have

$$V(S) \supseteq Y \iff S \subseteq I(Y)$$

   for all $S \subseteq A$ and $Y \subseteq \mathbb{A}^n$.

   The closed sets in $P(\mathbb{A})^{op}$ are called affine varieties/algebraic sets. When K is algebraically closed, closure operator $IV : P(A) \to P(A)$ maps an ideal J to its radical ideal $\sqrt{J}$. This is Hilbert's Nullstellensatz.

3. Let L/K be a field extension, and let $G = \mathrm{Aut}(L/K)$ be the automorphism group of L over K. Define relation $R \subseteq L \times G$ by

$$(x, \sigma) \in R \iff \sigma(x) = x.$$

   Understand the above terms precisely. The construction of Example 21.7 then yields a Galois connection between $P(L)$ and $P(G)^{op}$. Applying (21.26) yields a bijection between the set of Galois-closed intermediate subfields of L and the set of Galois-closed subgroups of

G. The fundamental theorem of Galois theory says that under suitable assumption on L/K (finiteness, normality, separability), we get a bijection between the set of all intermediate subfields of L and the set of all subgroups of G. In this case, we write $G = \mathrm{Gal}(L/K)$ and call it the Galois group of L over K.

4. Explore Heyting algebras and cartesian closed categories in connection to Example 21.3.

5. Explore how matroids and convex geometries are defined as closure operators.

23.5 Problems related to category theory

1. Recall that each poset P has an associated category C_P. Further, an order-preserving map of posets yields a functor between the associated categories. Moreover, a Galois connection between posets yields an adjunction between the associated categories.

24 Incidence algebra

24.1 Incidence algebra of a poset

Fix a field $\mathbb{k}$.

Let P be a locally finite poset.

Let $I(P)$ denote the set of all $\mathbb{k}$-valued functions on the set

$$\{(x, y) \in P \times P \mid x \leq y\},$$

the set of multichains of length 1. We say multichain since we are allowing $x = y$.

$I(P)$ is a vector space with addition and scalar multiplication defined pointwise.

That is,

$$(f+g)(x,y) := f(x,y) + g(x,y)$$
$$(cf)(x,y) := cf(x,y) \tag{24.1}$$

for $c \in \Bbbk$.

Note: We get a vector space in this manner whenever we consider $\Bbbk$-valued functions on any set X. There is nothing special about X being the set of multichains of length 1 in P.

We can also multiply functions as follows.

For $f, g \in I(P)$, define a new function $fg \in I(P)$ by

$$(fg)(x,z) := \sum_{y:\, x \le y \le z} f(x,y)g(y,z). \tag{24.2}$$

The sum is over all y in interval $[x,z]$.

Note how we made use of the poset structure of P. We also used that P is locally finite.

This turns $I(P)$ into an algebra.

It is the incidence algebra of P.

The unit element δ is given by

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases} \tag{24.3}$$

24.2 Zeta and Möbius functions of a poset

The zeta function $\zeta \in I(P)$ is defined by

$$\zeta(x, y) = 1 \tag{24.4}$$

for all $x \leq y$.

It is an invertible element of $I(P)$.

The Möbius function $\mu \in I(P)$ is defined as the inverse of ζ.

Unlike ζ which is always 1, values of μ depend on poset P. For more clarity, we may sometimes write μ_P instead of μ.

Since ζ and μ are inverses, $\zeta\mu = \mu\zeta = \delta$.

Expanding this out, we get the following.

Fact 24.1. *For any x in P,*

$$\mu(x, x) = 1 \tag{24.5}$$

and for any $x < y$ in P,

$$\sum_{z:\, x \le z \le y} \mu(x, z) = \sum_{z:\, x \le z \le y} \mu(z, y) = 0. \tag{24.6}$$

The sums are over all z in interval $[x, y]$.

By rewriting (24.6), the Möbius function can be computed recursively as follows.

Observation 24.2. For any x in P,

$$\mu(x, x) = 1$$

and for $x < y$ in P,

$$\mu(x, y) = -\sum_{z:\, x \le z < y} \mu(x, z) = -\sum_{z:\, x < z \le y} \mu(z, y).$$

Note very carefully the strict inequality in the above sums.

Example 24.3. Consider Boolean poset $P(X)$ on a finite set X. For $A \subseteq B$,

$$\mu(A, B) = (-1)^{|B|-|A|}. \tag{24.7}$$

In particular,

$$\mu(\emptyset, X) = (-1)^{|X|}. \tag{24.8}$$

Proof. Let us first prove special case (24.8). We do an induction on the size of X.

- The formula is clearly valid for $X = \emptyset$. This is the induction base.
- For induction step:

$$\mu(\emptyset, X) = -\sum_{A\colon A<X} \mu(\emptyset, A) = -\sum_{A\colon A<X} (-1)^{|A|}$$
$$= -\sum_{i\colon 0\le i<n} (-1)^i \binom{n}{i} = (-1)^n,$$

  where $n = |X|$.

To prove the more general formula (24.7), note that interval $[A, B]$ is isomorphic to $[\emptyset, B \setminus A]$ as a poset.

Applying formula (24.8) to $[\emptyset, B \setminus A]$ yields

$$\mu(A, B) = (-1)^{|B \setminus A|} = (-1)^{|B|-|A|}.$$

□

24.3 Algebra of upper triangular matrices

Example 24.4. Let $P = \{1, 2, \ldots, n\}$ with usual order. This is a linearly ordered set.

An element $f \in I(P)$ consists of scalars $f(i, j)$ for $1 \leq i \leq j \leq n$.

The product of f and g is given by

$$(fg)(i, k) = \sum_{j:\, i \leq j \leq k} f(i, j)\, g(j, k). \tag{24.9}$$

This is a specialization of (24.2).

The unit element δ is given by

$$\delta(i, j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{24.10}$$

This is a specialization of (24.3).

The zeta function is given by

$$\zeta(i,j) = 1 \tag{24.11}$$

and Möbius function is given by

$$\mu(i,j) = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } i+1 = j, \\ 0 & \text{otherwise.} \end{cases} \tag{24.12}$$

We can get this formula by using the Möbius recursion.

Observation 24.5. For linearly ordered set $P = \{1, 2, \ldots, n\}$,

incidence algebra $I(P)$ identifies with the algebra of upper triangular square matrices of size n:

A function f corresponds to the matrix whose ij-th entry is $f(i, j)$.

The unit element δ is the identity matrix.

ζ is the upper triangular matrix with all entries 1.

μ is the upper triangular matrix with 1's on the diagonal, -1's on the slanted line above the diagonal and 0 elsewhere.

For example, for $n = 3$,

$$\zeta = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mu = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

24.4 Powers of a function

Let P be a locally finite poset.

Fact 24.6. *For $k \geq 0$ and $f \in I(P)$ and $x \leq y$ in P,*

(24.13) $$f^k(x,y) = \sum_{x=x_0 \leq x_1 \leq \cdots \leq x_k = y} f(x_0,x_1) \ldots f(x_{k-1},x_k).$$

The sum is over all multichains of length k from x to y.

The first few cases are listed below.

- For $k = 0$, we have $f^0 = \delta$.
- For $k = 1$, we get tautology $f(x,y) = f(x,y)$.

- For $k = 2$,

$$f^2(x, y) = \sum_{x=x_0 \le x_1 \le x_2 = y} f(x_0, x_1) f(x_1, x_2).$$

This is a specialization of (24.2) with $f = g$.

- For $k = 3$,

$$f^3(x, y) = \sum_{x=x_0 \le x_1 \le x_2 \le x_3 = y} f(x_0, x_1) f(x_1, x_2) f(x_2, x_3).$$

24.5 Locally nilpotent and nilpotent elements

Suppose $h \in I(P)$ with $h(x,x) = 0$ for all $x \in P$.

Then: For $k \geq 0$ and $x \leq y$ in P,

(24.14) $$h^k(x,y) = \sum_{x=x_0<x_1<\cdots<x_k=y} h(x_0,x_1)\ldots h(x_{k-1},x_k).$$

This follows from formula (24.13).

Note very carefully the strict inequalities in the above sum. In other words, the sum is over chains from x to y of length k.

For $x \leq y$, we have $h^k(x, y) = 0$ for sufficiently large k. Hence, we say that h is locally nilpotent.

More precisely: If r is the maximum length of a chain from x to y, then $h^k(x, y) = 0$ for all $k > r$.

Suppose further that P is finite. Then $h^k = 0$ for sufficiently large k. In this case, we say that h is nilpotent.

24.6 Alternating sum formula for the inverse

Fact 24.7. *Let $f \in I(P)$ be such that $f(x,x) = 1$ for all $x \in P$. Then f is invertible in $I(P)$ and*

(24.15) $$f^{-1}(x,y) = \sum_{k \geq 0} (-1)^k \sum_{x = x_0 < x_1 < \cdots < x_k = y} f(x_0, x_1) \ldots f(x_{k-1}, x_k).$$

Summand corresponding to $k = 0$ is 0 unless $x = y$, in which case it is 1.

Proof. Expand

$$f^{-1} = (\delta - (\delta - f))^{-1} = \sum_{k \geq 0} (\delta - f)^k.$$

The sum is well-defined since $\delta - f$ is locally nilpotent. □

Example 24.8. Recall the algebra of upper triangular $n \times n$ matrices from Observation 24.5.

If A has all diagonal entries 1, then A is invertible with inverse

$$A^{-1} = \sum_{k \geq 0} (I - A)^k.$$

Explicitly, for $1 \leq i \leq j \leq n$,

$$A^{-1}_{ij} = \sum_{k \geq 0} (-1)^k \sum_{i = i_0 < i_1 < \cdots < i_k = j} A_{i_0 i_1} \ldots A_{i_{k-1} i_k}.$$

24.7 Hall formula

Proposition 24.9 (Philip Hall formula)**.** *For any $x \leq y$ in locally finite poset P,*

$$\mu(x,y) = \sum_{k \geq 0} (-1)^k c_k(x,y), \tag{24.16}$$

where $c_k(x,y)$ is the number of chains of length k from x to y.

Proof. This is a special case of (24.15). □

Can we give an illustration?

24.8 Weisner formula

Let P be a finite lattice with bottom element $\bot$ and top element $\top$.

Proposition 24.10. *For $y > \bot$ and $z \in P$,*

(24.17a) $$\sum_{x\colon y \vee x = z} \mu(\bot, x) = 0.$$

For $y < \top$ and $z \in P$,

(24.17b) $$\sum_{x\colon x \wedge y = z} \mu(x, \top) = 0.$$

The result above and proof below are both due to Weisner (1935).

Proof. We prove (24.17a) by induction on the partial order P.

The base case is clear.

We may assume $y \leq z$, otherwise lhs is clearly 0.

$$\sum_{x\colon y \vee x = z} \mu(\bot, x) = \sum_{x\colon x \leq z} \mu(\bot, x) - \sum_{x\colon y \vee x < z} \mu(\bot, x).$$

The first term is zero by definition (24.6), while the second term is zero by the induction hypothesis.

This completes the induction step.

The second identity (24.17b) follows by duality. □

Example 24.11. Let P be the lattice with four elements $\bot < a, b < \top$. It is isomorphic to Boolean poset $P(X)$ for $X = \{a, b\}$.

Weisner formula (24.17a) applied to $y = a$ and $z = \top$ yields

$$\mu(\bot, b) + \mu(\bot, \top) = 0.$$

Contrast this with the usual equation that we have, namely,

$$\mu(\bot, \bot) + \mu(\bot, a) + \mu(\bot, b) + \mu(\bot, \top) = 0.$$

The Weisner formula is theoretically significant.

24.9 Möbius function of a semimodular lattice

Proposition 24.12. *Let P be a finite upper or lower semimodular lattice of rank r. Then*

$$(-1)^r \mu(\bot, \top) = |\mu(\bot, \top)|. \tag{24.18}$$

In other words, sign of the Möbius function is the same as parity of the rank.

In addition, if P is interval-complemented (that is, every interval of P is complemented), then $\mu(\bot, \top) \neq 0$.

This result appears in Rota's paper (1964).

Proof. For definiteness, let us assume P is an upper semimodular lattice.

To prove (24.18), we induct on the rank of P.

The base case $r = 0$ is clear.

For induction step, fix an atom y (element of rank 1).

Then $x \vee y = \top$ implies that either $x = \top$ or $\mathrm{rk}(x) = r - 1$.

By Weisner formula (24.17a) applied to $z = \top$, the sum of $\mu(\bot, x)$ over all such x is 0.

So

$$\mu(\bot, \top) = -\sum_x \mu(\bot, x),$$

where the sum is over certain elements x of rank $r - 1$ (coatoms).

Applying induction hypothesis to each interval $[\bot, x]$ we obtain that sign of each summand is parity of $r - 1$.

Thus sign of $\mu(\bot, \top)$ is parity of r.

When P is interval-complemented, y possesses a complement. Thus the above sum is over a nonempty set. A similar inductive argument implies that $\mu(\bot, \top) \neq 0$. $\square$

25 Incidence module

Fix a field $\mathbb{k}$.

Let P be a locally finite poset.

Let $M(P)$ denote the vector space of $\mathbb{k}$-valued functions on a poset P with addition and scalar multiplication defined pointwise.

That is,

$$\begin{aligned} (f+g)(x) &:= f(x) + g(x) \\ (cf)(x) &:= cf(x) \end{aligned} \tag{25.1}$$

for $c \in \mathbb{k}$ and $x \in P$.

25.1 Left action of incidence algebra

Incidence algebra $I(P)$ acts on vector space $M(P)$ on the left as follows.

For $f \in I(P)$ and $g \in M(P)$, define $fg \in M(P)$ by

$$(fg)(x) := \sum_{y:\, x \leq y} f(x,y)g(y). \tag{25.2}$$

Thus, $M(P)$ is a left module over incidence algebra $I(P)$.

We call it the left incidence module of P.

25.2 Right action of incidence algebra

Similarly:

There is a right action of $I(P)$ on $M(P)$ given by

$$(fg)(y) := \sum_{x:\, x \leq y} f(x)g(x,y) \tag{25.3}$$

for $f \in M(P)$ and $g \in I(P)$.

We call $M(P)$ with the above action the right incidence module of P.

Example 25.1. Let $P = \{1, 2, \ldots, n\}$ as in Example 24.4.

An element $g \in M(P)$ consists of scalars $g(j)$ for $1 \leq j \leq n$.

Left action of $f \in I(P)$ on $g \in M(P)$ is given by

$$(fg)(i) = \sum_{j:\, i \leq j} f(i,j)\, g(j). \tag{25.4}$$

This is a specialization of (25.2).

Right action of $g \in I(P)$ on $f \in M(P)$ is given by

$$(fg)(j) := \sum_{i:\, i \leq j} f(i)\, g(i,j). \tag{25.5}$$

Observation 24.5 can be continued as follows.

Observation 25.2. For $P = \{1, 2, \ldots, n\}$, vector space $M(P)$ identifies with $\mathbb{k}^n$: A function g corresponds to the vector whose i-th coordinate is $g(i)$.

The left and right action of $I(P)$ on $M(P)$ correspond to the left and right action of matrices on column and row vectors, respectively.

25.3 Möbius inversion

Fact 25.3. *For functions f and g on P, we have*

$$g(x) = \sum_{y:\, x \le y} f(y) \tag{25.6}$$

$$\iff f(x) = \sum_{y:\, x \le y} \mu(x,y) g(y).$$

This is equivalent to $g = \zeta f \iff f = \mu g$.

The passage from the first equation to the second is Möbius inversion.

Similarly,

$$g(y) = \sum_{x:\, x \le y} f(x) \tag{25.7}$$

$$\iff f(y) = \sum_{x:\, x \le y} g(x) \mu(x,y).$$

This is equivalent to $g = f\zeta \iff f = g\mu$.

25.4 Inclusion-exclusion principle

Consider Boolean poset $P(X)$ on a finite set X.

Using formula (24.7), Möbius inversion takes the following form.

$$g(A) = \sum_{B\colon A\subseteq B} f(B) \tag{25.8}$$
$$\iff f(A) = \sum_{B\colon A\subseteq B} (-1)^{|B|-|A|} g(B).$$

$$g(B) = \sum_{A\colon A\subseteq B} f(A) \tag{25.9}$$
$$\iff f(B) = \sum_{A\colon A\subseteq B} (-1)^{|B|-|A|} g(A).$$

This is the inclusion-exclusion principle.

Example 25.4. Let $[n] = \{1, \ldots, n\}$. Let

- $g(A)$ = number of permutations of $[n]$ where at least all elements in A are fixed.
- $f(B)$ = number of permutations of $[n]$ whose exact set of fixed points is B.

Observe: Lhs of (25.8) holds.

Further: $g(A) = (n - |A|)!$.

So rhs of (25.8) yields

$$f(A) = \sum_{B: A \subseteq B} (-1)^{|B|-|A|}(n - |B|)!.$$

Setting $A = \emptyset$,

$$f(\emptyset) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n-k)! = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

A derangement of $[n]$ is a permutation of $[n]$ with no fixed points.

For example, for $n = 2$, the identity permutation of $[2]$ is not a derangement, while the transposition is a derangement.

Let d_n denote the number of derangements of $[n]$.

We showed that

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}. \tag{25.10}$$

We now provide a general context for ‘at least’ and ‘exact’.

Example 25.5. Let (X, R, Y) be a bipartite graph. Let B be any subset B of Y. Define

- $g(B)$ = number of $x \in X$ such that $(x, b) \in R$ for all $b \in B$. In other words, $g(B) = |\rho_R(B)|$, with ρ_R as defined in (21.13).
- $f(B)$ = number of $x \in X$ such that $(x, b) \in R$ for $b \in B$, and $(x, b) \notin R$ for $b \notin B$.

Observe: Lhs of (25.8) holds.

For example, let $Y = [n]$ and X be the set of permutations of $[n]$. Define R as follows.

- $(\sigma, i) \in R$ if permutation σ fixes i.

This leads to Example 25.4.

You can create more examples using Example 21.6:

- $g(B)$ = number of students who take all courses listed in B.
- $f(B)$ = number of students who take exactly the courses listed in B.

25.5 Rota formula

Let (λ, ρ) be an adjunction between finite posets P and Q.

Define

$$\begin{aligned} X_{\lambda,\rho} &= \{(x, w) \in P \times Q \mid \lambda(x) \leq w\} \\ &= \{(x, w) \in P \times Q \mid x \leq \rho(w)\}. \end{aligned}$$

Proposition 25.6. *For* $(x, w) \in X_{\lambda,\rho}$,

$$\sum_{y \in P:\, \lambda(y) = w} \mu_P(x, y) = \sum_{v \in Q:\, \rho(v) = x} \mu_Q(v, w). \tag{25.11}$$

This is the Rota formula.

See THA, Section C.1.10.

26 Characteristic polynomial

26.1 Characteristic polynomial

Let P be a finite graded lattice.

Define characteristic polynomial of P by

$$\chi(P,t) := \sum_{y \in P} \mu(y, \top)\, t^{\mathrm{rk}(y)}. \tag{26.1}$$

Its degree equals the rank of P.

If P has rank 0, then $\chi(P,t) = 1$, independent of t.

26.2 Special values

For $t = 0$, only summand for $y = \bot$ contributes to rhs of (26.1). Thus,

$$\chi(P, 0) = \mu(\bot, \top). \tag{26.2}$$

For $t = 1$, using (24.5) and (24.6),

$$\chi(P, 1) = \begin{cases} 1 & \text{if } P \text{ has rank } 0, \\ 0 & \text{otherwise.} \end{cases} \tag{26.3}$$

The value $t = -1$ is also of interest, and nontrivial. We will discuss it later.

26.3 Whitney numbers of the first kind

For $0 \leq k \leq \mathrm{rk}(P)$, define

$$\mathrm{wy}(P,k) := \sum_{x:\,\mathrm{rk}(x)=k} \mu(x,\top). \tag{26.4}$$

These are Whitney numbers of the first kind.

Observe: The number $\mathrm{wy}(P,k)$ is coefficient of t^k in characteristic polynomial $\chi(P,t)$.

26.4 Examples

Example 26.1. For Boolean poset $P(X)$ with $|X| = n$,

$$\chi(P(X), t) = (t-1)^n. \tag{26.5}$$

The calculation is shown below.

$$\begin{aligned}
\chi(P(X), t) &= \sum_{y \in P(X)} \mu(y, \top)\, t^{\mathrm{rk}(y)} \\
&= \sum_{y \in P(X)} (-1)^{n - \mathrm{rk}(y)}\, t^{\mathrm{rk}(y)} \\
&= \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i}\, t^i \\
&= (t-1)^n.
\end{aligned}$$

What happens when you put $t = -1$?

27 Zaslavsky formula

The Zaslavsky formula counts the number of chambers in an arrangement in terms of the absolute values of the Möbius function of the lattice of flats $\Pi[\mathcal{A}]$. It is given as follows.

Theorem 27.1. *For any linear arrangement $\mathcal{A}$,*

$$\sum_{\mathrm{X}\in\Pi[\mathcal{A}]} |\mu(\mathrm{X}, \top)| = c(\mathcal{A}), \tag{27.1}$$

where $c(\mathcal{A})$ is the number of chambers of $\mathcal{A}$.

This result appears in Zaslavsky's thesis (1975). For a proof, see THA, Section 1.13.

For lattice $\Pi[\mathcal{A}]$, there is a nice answer for the value of its characteristic polynomial at $t = -1$, namely,

$$\chi(\Pi[\mathcal{A}], -1) = (-1)^{\mathrm{rk}(\Pi[\mathcal{A}])} c(\mathcal{A}). \tag{27.2}$$

Proof.

$$\begin{aligned}
\chi(\Pi[\mathcal{A}], -1) &= \sum_{\mathrm{X} \in \Pi[\mathcal{A}]} \mu(\mathrm{X}, \top)\, (-1)^{\mathrm{rk}(\mathrm{X})} \\
&= (-1)^{\mathrm{rk}(\Pi[\mathcal{A}])} \sum_{\mathrm{X} \in \Pi[\mathcal{A}]} |\mu(\mathrm{X}, \top)| \\
&= (-1)^{\mathrm{rk}(\Pi[\mathcal{A}])} c(\mathcal{A}).
\end{aligned}$$

The first step used definition (26.1).

The second step used (24.18) for each interval $[\mathrm{X}, \top]$.

The third step used Zaslavsky formula (27.1). □

28 Partition lattice

Let I be a finite set. A partition X of I is a collection X of disjoint nonempty subsets of I such that

$$I = \bigsqcup_{B \in \mathrm{X}} B.$$

The subsets B are the blocks of the partition.

For example, for $I = \{p, r, i, y, e, s, h\}$,

$$\mathrm{X} = \{\{i, e\}, \{p, r, y, s, h\}\}$$

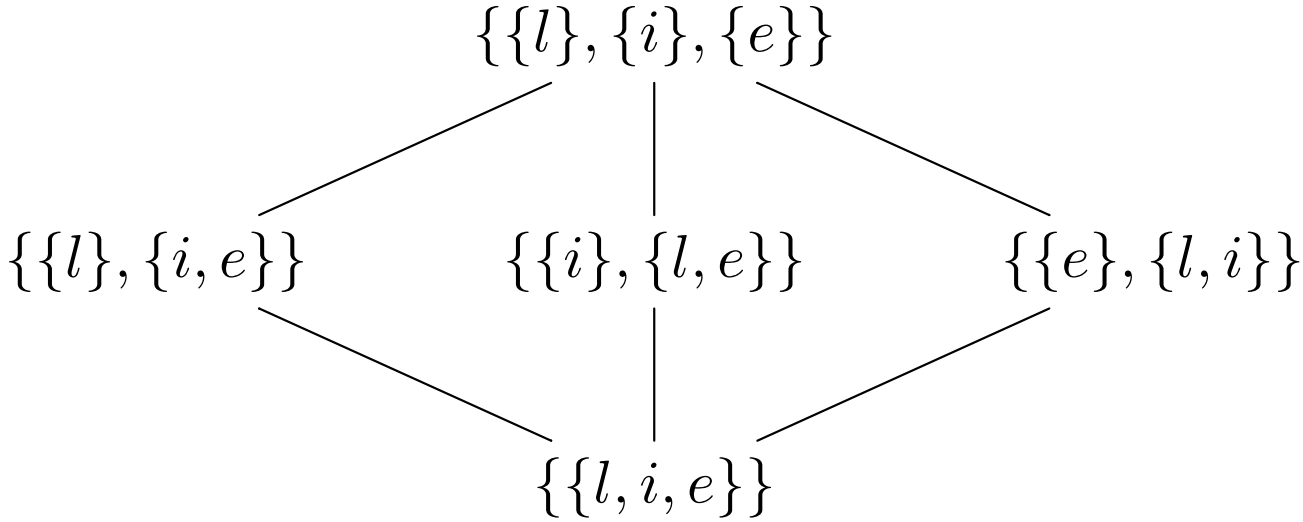
is a partition with two blocks, namely $\{i, e\}$ and $\{p, r, y, s, h\}$.

Let $\Pi[I]$ denote the set of partitions of I.

For partitions X and Y, we say that Y refines X if each block of X is a union of blocks of Y. In this case, we write $\mathrm{X} \leq \mathrm{Y}$.

This defines a partial order on $\Pi[I]$.

Example 28.1. The Hasse diagram of the poset of partitions of $\{l, i, e\}$ is shown below.



You may similarly draw the Hasse diagram of the poset of partitions of $\{r, a, v, i\}$. Apart from the bottom and top elements, it has 7 elements of rank 1, and 6 elements of rank 2.

Fact 28.2. *Poset of partitions $\Pi[I]$ is a finite graded lattice. Moreover, it is lower semimodular.*

The maximum element is the partition into singletons, and the minimum element is the partition whose only block is the whole set I.

The rank of a partition X of I is the number of blocks of I minus 1.

What is the meet and join of two partitions?

29 Problem set VIII

29.1 Test your understanding

Say whether each of the following statements is true or false.

1. Incidence algebra $I(P)$ of a poset P is commutative.
2. An element $f \in I(P)$ is invertible iff $f(x, x) = 1$ for all $x \in P$.
3. An upper triangular square matrix with diagonal entries zero is nilpotent.
4. If $x \lessdot y$ in P, then $\mu_P(x, y) = -1$.
5. For every $x \leq y$ in P, $\zeta^2(x, y)$ is the cardinality of interval $[x, y]$.
6. For $n \geq 1$, the number of permutations of $[n]$ having exactly $n - 1$ fixed points is n.
7. If P is a finite graded lattice of positive rank, then $t - 1$ divides $\chi(P, t)$.
8. For a field $\Bbbk$ and nonempty set X, the algebra of all $\Bbbk$-valued functions on X with pointwise addition and multiplication is an example of an incidence algebra.

29.2 Problems related to theory

1. Let P be a locally finite poset. Verify that:
   - For $f, g, h \in I(P)$, $f(gh) = (fg)h$.
   - For $f \in I(P)$, $\delta f = f\delta = f$.
   - For $f, g, h \in I(P)$, $f(g + h) = fg + fh$ and $(f + g)h = fh + gh$.
   - For $f, g \in I(P)$ and $c \in \Bbbk$, $c(fg) = (cf)g = f(cg)$.

   These properties ensure that $I(P)$ is an algebra.
2. Establish formula (24.13) by induction on k.
3. Show: The invertible elements of $I(P)$ are precisely those functions f for which $f(x, x) \neq 0$ for all $x \in P$. See THA, Proposition C.2.
4. Let P be a discrete poset. Describe its incidence algebra $I(P)$.

5. Let P be a locally finite poset. Verify that:

   - For $f, g \in I(P)$ and $h \in M(P)$, $f(gh) = (fg)h$.
   - For $f \in M(P)$, $\delta f = f$.
   - For $f \in I(P)$ and $g, h \in M(P)$, $f(g+h) = fg + fh$.
   - For $f, g \in I(P)$ and $h \in M(P)$, $(f+g)h = fh + gh$.
   - For $f \in I(P), g \in M(P), c \in \Bbbk$, $c(fg) = (cf)g = f(cg)$.

   These properties ensure that $M(P)$ is a left module over algebra $I(P)$. Similar for right module.

6. For a nonempty finite lattice P, consider linear system

$$\sum_{x:\ y \vee x = z} c_x = 0, \qquad \perp < y \leq z.$$

   We also impose $c_{\perp} = 1$. Show that $(c_x = \mu(\perp, x))$ is the unique solution of this system. This result highlights the theoretical significance of the Weisner formula.

   What happens if we take $c_{\perp}$ to be some other scalar?

7. For locally finite posets P and Q, show:

$$\mu_{P \times Q}((x,y),(x',y')) = \mu_P(x,x')\mu_Q(y,y'). \tag{29.1}$$

   Generalize to a finite number of posets.

   We mention a related fact: If P and Q are finite, then $I(P \times Q) = I(P) \otimes I(Q)$ as algebras. Rhs is tensor product of algebras $I(P)$ and $I(Q)$.

8. Recall that any algebra A has an opposite algebra A^{op}: The product of a and b in A^{op} is defined to be the product of b and a in A. Note: If A is commutative, then there is no distinction between A and A^{op}.

   Show: For locally finite poset P,

$$\varphi : I(P)^{op} \xrightarrow{\cong} I(P^{op}) \tag{29.2}$$

   defined by $\varphi(f)(y,x) = f(x,y)$ is an isomorphism of algebras.

   An algebra morphism preserves a pair of mutually inverse elements. Use this to deduce that $\mu_P(x,y) = \mu_{P^{op}}(y,x)$.

Use the above ideas to show how (24.17a) and (24.17b) imply each other.

Similarly, show how Proposition 24.12 for a lower semimodular lattice, and for an upper semimodular lattice imply each other.

9. Show: For finite graded lattices P and Q,

$$\chi(P \times Q, t) = \chi(P, t)\, \chi(Q, t). \tag{29.3}$$

Generalize to a finite number of lattices.

Use this and (15.2) to deduce formula (26.5).

29.3 Problems related to examples

1. Compute the Möbius function of the two posets in Example 8.6.

2. Consider poset of positive integers $\mathbb{N}_+$ where $a \leq b$ if a divides b. Observe that $\mu(a, b) = \mu(1, b/a)$ since interval $[a, b]$ is isomorphic to interval $[1, b/a]$. Let us shorten $\mu(1, n)$ to $\mu(n)$.

   Let p be a prime divisor of n. Apply Weisner formula (24.17a) with $\perp = 1$, $y = p$, $z = n$ to deduce

$$\mu(n) = \begin{cases} -\mu(n/p) & \text{if } p \text{ does not divide } n/p, \\ 0 & \text{otherwise.} \end{cases} \tag{29.4}$$

   Deduce that

$$\mu(n) = \begin{cases} (-1)^r & \text{if } n \text{ has } r \text{ distinct prime factors, all of multiplicity } 1, \\ 0 & \text{if } n \text{ has repeated prime factors.} \end{cases} \tag{29.5}$$

   This is the classical Möbius function.

   Alternatively: Use observations (29.1) and (15.3) to deduce formula (29.5) above.

3. Fix a parameter s. A Dirichlet series is an expression of the form

$$\hat{f}(s) = \sum_{n \geq 1} \frac{f(n)}{n^s},$$

with $f(n) \in \mathbb{k}$. The set of all Dirichlet series forms an algebra with componentwise sum and scalar multiplication. The product is defined by expanding and collecting like terms. Consider Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$

Show that this is an invertible element in the algebra of Dirichlet series and compute its inverse.

4. Use observations (29.1) and (15.2) to deduce formula (24.7) for the Möbius function of the Boolean poset.

   How about applying Weisner formula (24.17a) to Boolean poset? Use it to derive (24.8).

5. Verify Hall formula (24.16) for Boolean poset $P(X)$ for $|X| = 1, 2, 3, 4$.

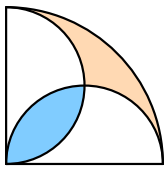
6. Use inclusion-exclusion principle (25.8) and (25.9) to deduce that:

   For subsets $A, B, C, \ldots$ of a finite set X,

$$|A \cup B| = |A| + |B| - |A \cap B|,$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|,$$

   and so on. Write down the precise alternating formula for the cardinality of the union of n sets. Also prove this formula directly by induction.

7. 

   In the above picture, show that the two shaded regions have the same area.

8. Verify derangement formula (25.10) directly for $n = 1, 2, 3, 4$. Show: The derangement numbers satisfy the recursion

$$d_n = (n-1)(d_{n-1} + d_{n-2}) \tag{29.6}$$

   for $n \geq 2$, with initial conditions $d_0 = 1$ and $d_1 = 0$. Similarly,

$$d_n = n d_{n-1} + (-1)^n \tag{29.7}$$

for $n \geq 1$, with initial condition $d_0 = 1$.

9. Give an example of a modular lattice whose Möbius function takes zero values. In particular, it is not interval-complemented.

10. Consider the Möbius function of partition lattice $\Pi[n]$. First show that

$$\mu(\bot, \top) = (-1)^{n-1}(n-1)!. \tag{29.8}$$

(Apply Weisner formula (24.17a) with $\bot = \{1 \ldots n\}$ having one block, $y = \{1, 2 \ldots n\}$ having two blocks, $z = \top = \{1, \ldots, n\}$ having n blocks, and use induction on n.)

For the general formula, use that any interval $[\mathrm{X}, \mathrm{Y}]$ in $\Pi[n]$ is the product of (smaller) partition lattices, one for each block of X. Show:

$$\mu(\mathrm{X}, \mathrm{Y}) = \prod_{B \in \mathrm{X}} (-1)^{k_B - 1}(k_B - 1)! \tag{29.9}$$

where each block B of X is partitioned into k_B blocks inside Y.

Show: The characteristic polynomial of $\Pi[n]$ is given by

$$\chi(\Pi[n], t) = \prod_{i=1}^{n-1} (t - i). \tag{29.10}$$

The falling factorial is the polynomial $t^{\underline{n}} = t(t-1)\cdots(t-n+1)$. For $n \geq k$, let $s(n, k)$ to be the coefficient of t^k in $t^{\underline{n}}$. These are the Stirling numbers of the first kind. Deduce that

$$\mathrm{wy}(\Pi[n], k) = s(n, k+1). \tag{29.11}$$

11. Consider subspace poset $\mathrm{Sub}(W)$ with $\dim W = n$ over a finite field $\mathbb{F}_q$. Show:

$$\mu(U, V) = (-1)^{j-i} q^{\binom{j-i}{2}}, \tag{29.12}$$

where $\dim U = i$ and $\dim V = j$.

The characteristic polynomial of subspace poset $\mathrm{Sub}(W)$ is given by

$$\chi(\mathrm{Sub}(W), t) = \prod_{i=0}^{n-1} (t - q^i). \tag{29.13}$$

29.4 Problems related to hyperplane arrangements

1. Fact: Boolean poset $P(X)$ with $|X| = n$ is isomorphic to lattice of flats of the coordinate arrangement $\mathcal{A}$ of rank n. The latter has $c(\mathcal{A}) = 2^n$ chambers. Verify Zaslavsky formula (27.1) and formula (27.2) explicitly.

2. Consider the linear rank-two arrangement $\mathcal{A}$ of n lines, with $n \geq 2$. It has $c(\mathcal{A}) = 2n$ chambers. Its lattice of flats has a bottom element $\bot$, a top element $\top$, and n elements of rank 1 between the two. Check:

$$\mu(\bot, \top) = n - 1 \quad \text{and} \quad \chi(\Pi[\mathcal{A}], t) = t^2 - nt + n - 1. \tag{29.14}$$

   Verify Zaslavsky formula (27.1) and formula (27.2) explicitly.

3. Fact: Partition lattice $\Pi[n]$ is isomorphic to lattice of flats of the braid arrangement $\mathcal{A}$ of rank $n - 1$. The latter has $c(\mathcal{A}) = n!$ chambers. Verify Zaslavsky formula (27.1) and formula (27.2) explicitly. You may use (29.8) and (29.10).

29.5 Explorations

1. A finite graded poset P is eulerian if for any $x \leq y$ in P,

$$\mu(x, y) = (-1)^{\mathrm{rk}(y) - \mathrm{rk}(x)}. \tag{29.15}$$

   The face poset of a convex polytope is eulerian. In particular, the face poset of the simplex which is the same as the Boolean poset is eulerian. We saw this in (24.7). The poset of faces of an oriented matroid and in particular of a real hyperplane arrangement is also eulerian.

2. What is the connection of Hall formula (24.16) to euler characteristic of a cell complex?

3. The poset of set partitions was (first) considered by Birkhoff (1935). For some context on the rich mathematics that it leads to, see Braid arrangements and graded bialgebras, Chapter 2 on my homepage.

4. There is a class of monoids called left regular bands. They may be viewed as noncommutative lattices. More precisely, the join operation of the lattice (which is commutative) is replaced by the product in the monoid. The motivating example of a LRB is the Tits monoid of a hyperplane arrangement.

   Just like a lattice, a left regular band (LRB) has an incidence algebra. Instead of just one zeta and one Möbius function, it contains a family of noncommutative zeta functions and a family of noncommutative Möbius functions which are mutually inverse. For the beginnings of this theory, see THA, Chapter 15.

   There is also a noncommutative Zaslavsky formula in which the usual Möbius function is replaced by a noncommutative Möbius function. For more on this, see Braid arrangements and graded bialgebras, Chapter 5 on my homepage.

5. What is Stanley's negative one color theorem on acyclic orientations of a graph? How is it a special case of the Zaslavsky theorem? See THA, Section 6.9.

6. Consider all partition lattices $\Pi[n]$ for $n \geq 1$. For each n, there is a monoid morphism from the monoid of formal power series in one variable under substitution to the incidence algebra $I(\Pi[n])$. It takes exponential power series $e^x - 1$ to the zeta function, and logarithmic power series $\log(1 + x)$ to the Möbius function of $I(\Pi[n])$. For precise details, see Braid arrangements and graded bialgebras, Chapters 3 and 4 on my homepage.

7. Algebra is a technical term. It is usually defined over a field $\mathbb{k}$. In particular, it is a vector space over $\mathbb{k}$. In more generality, one can take $\mathbb{k}$ to be a commutative ring with identity. In this situation, a $\mathbb{Z}$-algebra is the same as a ring (such as the ring of 2×2 matrices with integer coefficients).

   Module over an algebra/ring is another technical term. It is an object on which the algebra acts. It is equivalent to specifying a representation of the algebra. For more on algebras and modules, see for instance THA, Appendix D.

   We may linearize a monoid/group using coefficients in $\mathbb{k}$ to get an algebra over $\mathbb{k}$. It is called a monoid algebra/group algebra. A module over it is the same as a linear representation of the monoid/group. A basic example is the action of a monoid/group on a set. In this case, we can linearize the set to get a module.

30 References

- Stanley. Enumerative combinatorics. Volume 1. Chapter 3.
- Stanley. Enumerative combinatorics. Volume 2. Chapter 5.
- Bona. A walk through combinatorics. Chapter 16.
- Jukna. Extremal combinatorics. Chapter 8.
- Gratzer. General lattice theory.
- Davey and Priestley. Introduction to lattices and order. Chapter 7.
- Aguiar and Mahajan. Topics in hyperplane arrangements. Appendices B and C.