Lecture 10

Last time we discussed some iterative methods to find roots

1. Bisection method

a_n, b_n s.t $f(a_n) f(b_n) < 0$

$$w = \frac{a_n + b_n}{2}$$

If $f(a_n) f(w) \leq 0$ set $a_{n+1} = a_n$, $b_{n+1} = w$

otherwise set $a_{n+1} = w$, $b_{n+1} = b_n$

root lies in $[a_{n+1}, b_{n+1}]$

Bisection method always converges to the root convergence is "slow"!

2. Regula Falsi-method

3. Modified Regula falsi-method

4. Secant method

5. Newtons Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n \geq 0$$

Newtons Method converges if x_0 is "close" to the root.

fixed pt iteration

Newton's method is a form of fixed pt iteration

$g : I \to I$ $\quad I = [a, b]$

$x_0 \in I$ $\quad x_{n+1} = g(x_n)$

Condition for convergence for fixed pt iteration

1) $g : I \to I$ $\quad I = [a, b]$

2) g is cts

3) (1) and (2) ensure that g has a fixed pt but still iteration may not converge.

(3) $|g'(x)| \leq k$ for some $k < 1$

$\forall x \in I$

(3) ensures convergence of fixed point convergence.

It also ensures that there is a unique fixed point for g

Linear convergence for fixed point convergence

$g : I \to I$ $\quad I = [a, b]$

g cts $\quad |g'(x)| \le k < 1 \quad \forall x \in I$

ξ unique fixed pt of g

Assume $g'(\xi) \neq 0$

$x_0 \in I$ any $\quad x_n = g(x_{n-1})$

$x_1, x_2, \ldots, x_n, \ldots$

$e_n = \xi - x_n$

$e_{n+1} = \xi - x_{n+1}$

$= g(\xi) - g(x_n)$

$= g'(\eta_n)(\xi - x_n)$ $\quad \eta_n$ between ξ and x_{n+1}

~~$e_{n+1} = g'(\eta_n)(\xi - x_n)$~~

$e_{n+1} = g'(\eta_n) e_n$

$\lim_{n \to \infty} x_n = \xi \quad \Rightarrow \quad \lim_{n \to \infty} \eta_n = \xi$

So $\lim_{n \to \infty} g'(x_n) = g'(\xi)$

$g'(x_n) \approx g'(\xi)$ for n-large.

So $e_{n+1} \approx g'(\xi)\, e_n$ for n large

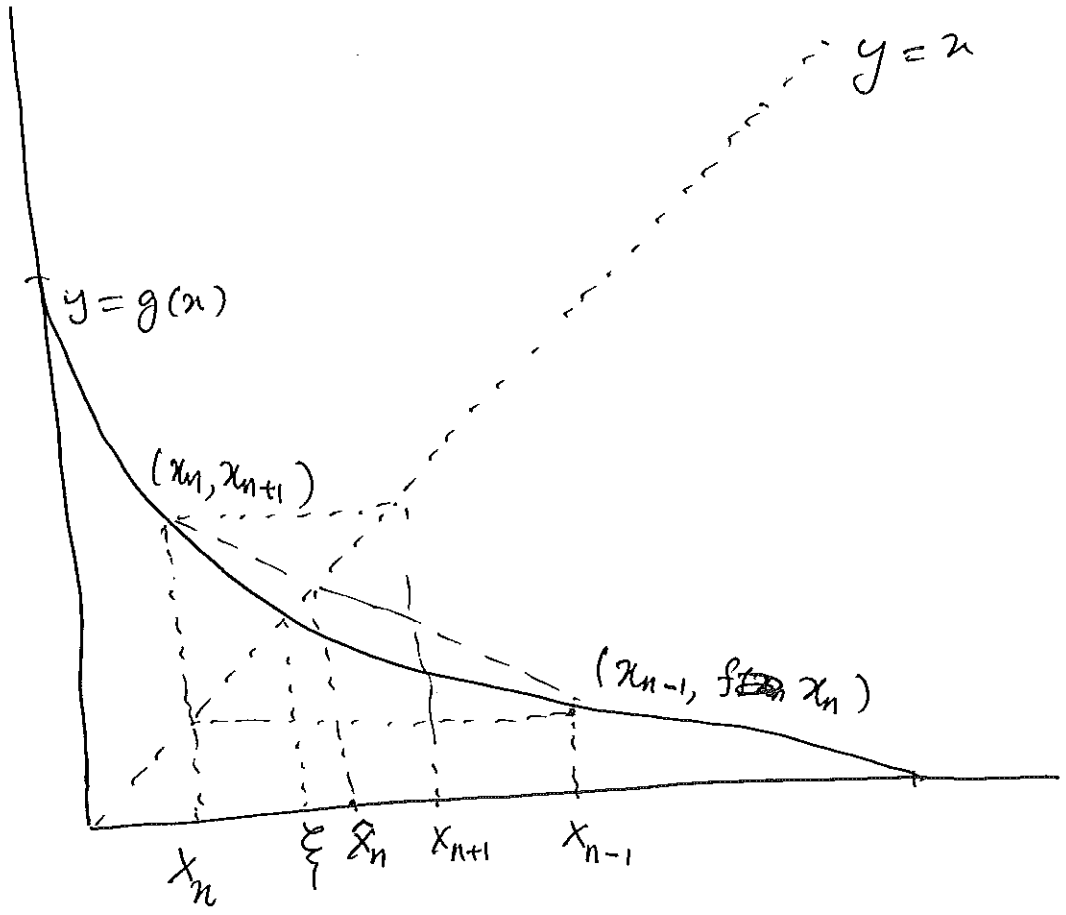
Thus the error in the $(n+1)^{st}$ iterate depends (more or less) linearly on the error e_n in the n^{th} iterate.

Accelerating fixed point convergence

Fixed pt iteration at times requires many iterations.

There is a standard method to accelerate linearly converging sequences. It is called Aitken's Δ^2-process.

Graphical justification



~~Determin~~ $\hat{x}_n$ is closer to ξ, than x_n, x_{n-1}, x_{n+1}

Determination of $\hat{x}_n$

parametric form of line segment joining (x_n, x_{n+1}) & (x_{n-1}, x_n)

$$r(t) = (x_n, x_{n+1}) + t\,(x_{n-1} - x_n,\ x_n - x_{n+1})$$

$$t \in [0, 1]$$

determine t when $r(t) = (\hat{x}_n, \hat{x}_n)$

$$x_n + t(x_{n-1} - x_n) = x_{n+1} + t(x_n - x_{n+1})$$

$$x_n - x_{n+1} = t(2x_n - x_{n+1} - x_{n-1})$$

$$t = \frac{x_{n+1} - x_n}{x_{n+1} - 2x_n + x_{n-1}}$$

So pt

$$\hat{x}_n = x_{n+1} - \frac{(x_{n+1} - x_n)^2}{x_{n+1} - 2x_n + x_{n-1}}$$

For any sequence $\{y_n\}$

$$\Delta y_n = y_{n+1} - y_n$$

$$\Delta^2 y_n = \Delta y_{n+1} - \Delta y_n$$
$$= y_{n+2} - 2y_{n+1} + y_n$$

Thus $$\hat{x}_n = x_{n+1} - \frac{(\Delta x_n)^2}{\Delta^2 x_{n-1}}$$

This is called Aitken's Δ^2-process.

This can be applied to any linearly convergent sequence and not just coming from fixed pt iteration

Aitken's Δ^2-process

Algorithm

Given a sequence $x_0, x_1, x_2, \ldots$ converging to ξ calculate the sequence $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n, \ldots$ by

$$\hat{x}_n = x_{n+1} - \frac{(\Delta x_n)^2}{\Delta^2 x_{n-1}}$$

If the sequence $x_0, x_1, \ldots, x_n, \ldots$ converges linearly to ξ that is if

$$\xi - x_{n+1} = K(\xi - x_n) + o(\xi - x_n) \quad \text{for some } K \neq 0$$

then $\hat{x}_n = \xi + o(\xi - x_n)$

$$\left[\text{i.e } \frac{\hat{x}_n - \xi}{x_n - \xi} \longrightarrow 0 \right]$$

Example

$$g(x) = \frac{1}{2} e^{x/2} \qquad x \in [0, 1]$$

$x_0 = 0$

$x_1 = 0.5$

$x_2 = 0.642013$

$x_3 = 0.689257$

$x_4 = 0.705733$

$x_5 = 0.711570$

$x_6 = 0.713651$

$x_7 = 0.714393$

$x_8 = 0.714658$

$x_9 = 0.714753$

$x_{10} = 0.714787$

$x_{11} = 0.714799$

$x_{12} = 0.714804$

$x_{13} = 0.714805$

$x_{14} = 0.714806$

$x_n = x_{14}$ for $n \geq 14$

$\hat{x}_1 = 0.698349$

$\hat{x}_2 = 0.712809$

$\hat{x}_3 = 0.714556$

$\hat{x}_4 = 0.714772$

$\hat{x}_5 = 0.714804$

$\hat{x}_6 = 0.714806$

For fixed point iteration

If $\hat{x}_n$ is a better approximation to ξ than x_n then it is certainly wasteful to continue generating x_{n+2}, x_{n+3}.

It seems more reasonable to start fixed point iteration afresh with $\hat{x}_n$ as the initial guess.

Steffensen iteration

Given the iteration function $g(x)$ and a point y_0

For $n = 0, 1, 2, \ldots$ until satisfied do

$x_0 = y_n$

calculate $x_1 = g(x_0)$ $\quad x_2 = g(x_1)$

calculate Δx_1 and $\Delta^2 x_0$

calculate $y_{n+1} = x_2 - \dfrac{(\Delta x_1)^2}{\Delta^2 x_0}$

Example

$$g(x) = \frac{1}{2} e^{x/2}$$

$y_0 = 0$

$x_0 = 0$

$x_1 = 0.5$

$x_2 = 0.642013$

$y_1 = 0.698349$

$x_0 = 0.698349$

$x_1 = 0.708948$

$x_2 = 0.712715$

$y_2 = 0.714792$

$x_0 = 0.714792$

$x_1 = 0.714801$

$x_2 = 0.714804$

$y_3 = 0.714806$

(we have done 6 function calculations and 3 Aitken Δ^2 computations.)

Convergence of Newton and Secant Method

Suppose $f(x)$ is continuously twice differentiable and $f(\xi) = 0$, $f'(\xi) \neq 0$

Iteration function of Newton's Method

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$g'(x) = \frac{f(x)\, f''(x)}{(f'(x))^2}$$

$$g'(\xi) = 0$$

Note $\exists\, \varepsilon > 0$ s.t on $|x - \xi| \leq \varepsilon$

$$|g'(x)| \leq k < 1$$

$$I = [\xi - \varepsilon, \xi + \varepsilon]$$

$$|x - \xi| \leq \varepsilon$$

$$|g(x) - \xi| = |g(x) - g(\xi)| \qquad \eta \text{ between } x \text{ and } \xi$$

$$= |g'(\eta)|\, |x - \xi|$$

$$\leq k |x - \xi|$$

$$\leq \varepsilon$$

$$g(x) \in I.$$

$g : I \to I$ is cts

$I = [\xi - \varepsilon, \xi + \varepsilon]$

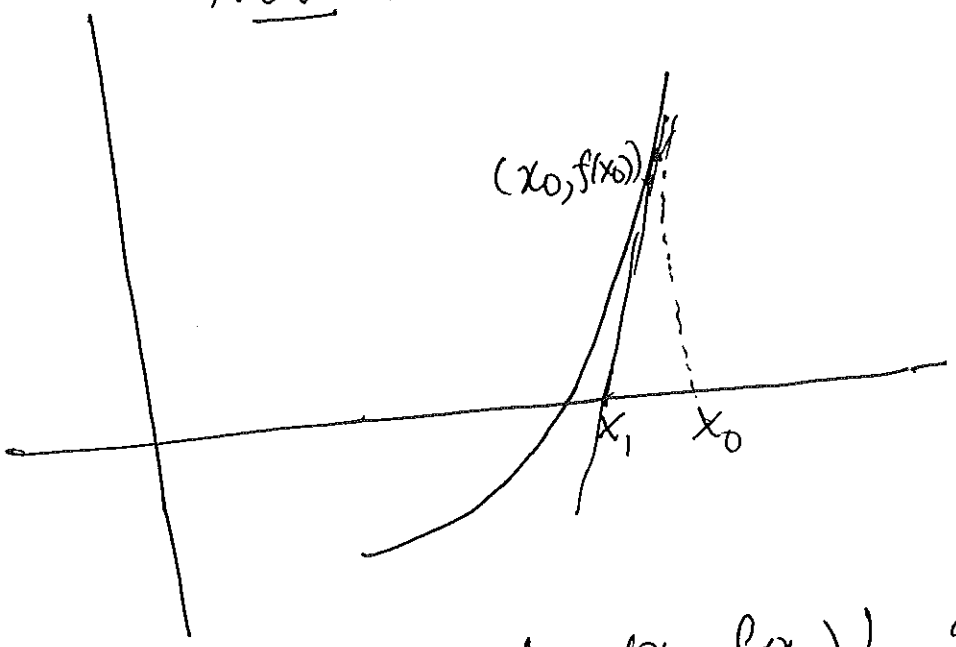
$|g'(x)| \leq k < 1 \quad \forall x \in I$

So fixed pt iteration converges.

Thus Newton's method converges for pts sufficiently close to the root.

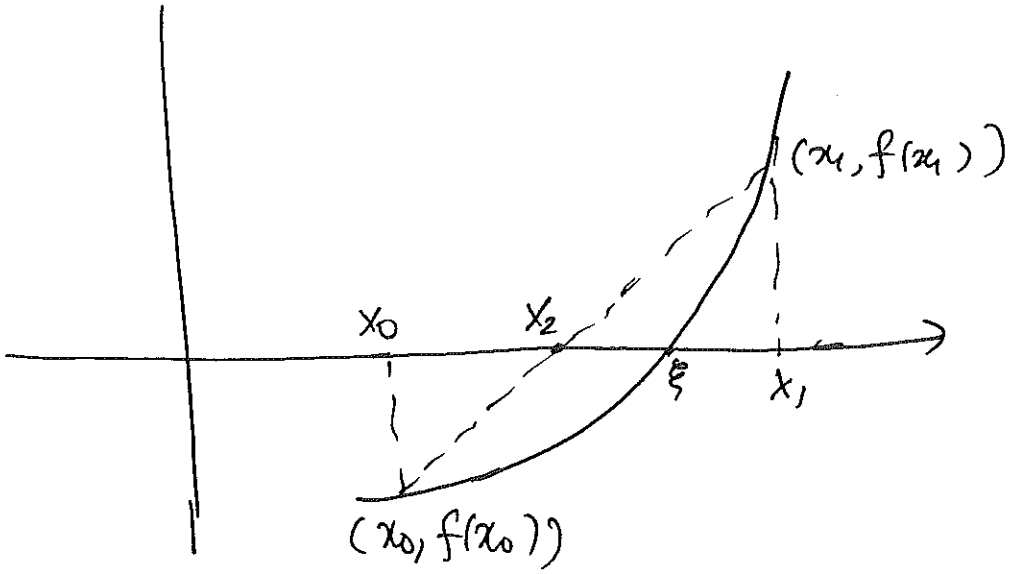
Error in Newton Method and Secant Method

Newtons method



We draw tgt at $(x_0, f(x_0))$ and see where it hits x-axis.

Secant Method



Both methods interpolates $f(x)$ at two points α and β by a st. line

$$p(x) = f(\alpha) + f[\alpha, \beta](x - \alpha)$$

whose zero $\hat{\xi} = \alpha - \dfrac{f(\alpha)}{f[\alpha, \beta]}$

is taken as the next approximation to the actual zero of $f(x)$

In secant method we take $\alpha = x_n$ $\beta = x_{n-1}$ and then produce

$$\hat{\xi} = x_{n+1}$$

In Newton's method we take

$\alpha = \beta = x_n$

and so $\hat{\xi} = x_{n+1}$

We know that

$$f(x) = f(\alpha) + f[\alpha, \beta](x-\alpha) + f[\alpha, \beta, x](x-\alpha)(x-\beta)$$

This holds for all x

So for $x = \xi$ $\quad f(\xi) = 0$

$$0 = f(\xi) = f(\alpha) + f[\alpha, \beta](\xi - \alpha) + f[\alpha, \beta, \xi](\xi-\alpha)(\xi-\beta)$$

$$f[\alpha, \beta](\xi - \alpha) = -f(\alpha) - f[\alpha, \beta, \xi](\xi-\alpha)(\xi-\beta)$$

$$\xi = \alpha - \frac{f(\alpha)}{f[\alpha, \beta]} - \frac{f[\alpha, \beta, \xi]}{f[\alpha, \beta]}(\xi-\alpha)(\xi-\beta)$$

$$\hat{\xi} = \alpha - \frac{f(\alpha)}{f[\alpha, \beta]}$$

$$\text{So } \xi = \hat{\xi} - \frac{f[\alpha, \beta, \xi]}{f[\alpha, \beta]}(\xi-\alpha)(\xi-\beta)$$

This equation can now be used to obtain error bounds for Newton and Secant Method

for Newton's Method

$\alpha = \beta = x_n$ $\qquad \hat{\xi} = x_{n+1}$

$e_j = \xi - x_j$ error at the j^{th} iterate

We obtain

$$e_{n+1} = -\frac{f[x_n, x_n, \xi]}{f[x_n, x_n]} e_n^2$$

Recall $f[x_n, x_n] = f'(x_n)$

and $f[x_n, x_n, \xi] = \frac{1}{2} f''(\eta_n)$ for some η_n between x_n and ξ

Thus $$\boxed{e_{n+1} = -\frac{1}{2} \frac{f''(\eta_n)}{f'(x_n)} e_n^2}$$

This shows that Newton's Method converges quadratically.

Error bound for secant Method

Set $\alpha = x_n$, $\beta = x_{n-1}$ $\quad$ [$e_j = \xi - x_j$ error at the j^{th} stage

$$e_{n+1} = -\frac{f[x_{n-1}, x_n, \xi]}{f[x_{n-1}, x_n]} e_n e_{n-1}$$

Error in the $(n+1)^{st}$ stage is proportional to the product of the errors in the n^{th} and $(n-1)^{st}$ stage

$$f[x_{n-1}, x_n, \xi] = \frac{1}{2} f''(\delta_n)$$

$$f[x_{n-1}, x_n] = f'(\eta_n)$$

for some η_n, δ_n between x_n, x_{n-1} and ξ.

$$e_{n+1} \approx -\frac{1}{2} \frac{f''(\xi)}{f'(\xi)} e_n e_{n-1}$$

for $n >> 0$.

Def^n (Order of convergence)

Let $x_0, x_1, \ldots x_n, \ldots$ be a sequence which converges to a number ξ and set $e_n = \xi - x_n$. If there exists a number p and a constant $C \neq 0$ such that

$$\lim_{n \to \infty} \frac{|e_{n+1}|}{|e_n|^p} = C$$

then p is called order of convergence and C is called the asymptotic error constant

Examples

1) Fixed pt iteration

Let ξ be fixed pt of g

Assume $g'(\xi) \neq 0$

$$e_{n+1} \approx e_n g'(\xi)$$

$$\lim_{n \to \infty} \frac{|e_{n+1}|}{|e_n|} = |g'(\xi)|$$

So order of convergeca is 1
and the asymptotic error constant is

$|g'(\xi)|$

(2) For Newtons Method

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^2} = \frac{1}{2} \frac{|f''(\xi)|}{|f'(\xi)|}$$

Provided $f'(\xi) \neq 0$

So order of convergence is 2 and the asymptotic error constant is

$$\frac{|f''(\xi)|}{2|f'(\xi)|}$$

(3) Secant Method

We get from our previous calculation

$$(*) \quad |e_{n+1}| = C_n |e_n| |e_{n+1}|$$

with

with

$$\lim_{n\to\infty} c_n = c_\infty = \frac{1}{2}\frac{|f''(\xi)|}{|f'(\xi)|}$$

(we are assuming $f'(\xi) \neq 0$)

We seek a number p such that

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^p} = C \quad \text{for some}$$

non-zero constant C

From (*)

$$\frac{|e_{n+1}|}{|e_n|^p} = c_n |e_n|^{1-p} |e_{n-1}| = c_n \left(\frac{|e_n|}{|e_{n-1}|^p}\right)^\alpha \tag{**}$$

provided $\alpha = 1-p$ and $\alpha p = -1$

i.e provided

$$p - p^2 = -1$$

The equation $p^2 - p - 1 = 0$ has the simple +ve root

$$p = \frac{1+\sqrt{5}}{2} = 1.618$$

With this choice of p and of $\alpha = 1-p$ (**) defines a "fixed-point like iteration"

$$y_{n+1} = c_n \, y_n^{-\frac{1}{p}}$$

$$y_{n+1} = \frac{|e_{n+1}|}{|e_n|^p} \qquad \text{and} \lim_{n\to\infty} c_n = c_\infty$$

It follows that y_n converges to the fixed point of the equation

$$x = c_\infty x^{-1/p}$$

$$x^{1+\frac{1}{p}} = c_\infty \qquad 1+\tfrac{1}{p} = p$$

$$\text{so } x = c_\infty^{1/p}$$

Thus $\lim_{n\to\infty} \dfrac{|e_{n+1}|}{|e_n|^p} = \left| \dfrac{1}{2} \dfrac{f''(\xi)}{f'(\xi)} \right|^{1/p}$

Thus order of convergence of secant method
$= p =$ +ve root of $p^2 - p - 1$
$= 1.618$

Asymptotic error constant $\left| \dfrac{f''(\xi)}{2f'(\xi)} \right|^{1/p}$

Convergence of Newton's Method at double roots

Recall that when we said that Newton's method converge quadratically we assumed $f'(\xi) \neq 0$ (i.e, ξ is a simple root of $f(x)$).

What happens if f is a double root?

i.e., $f(\xi) = 0 = f'(\xi)$, $f''(\xi) \neq 0$

Let $g(x) = x - \dfrac{f(x)}{f'(x)}$ be the iteration function of Newton's method

$$g'(x) = \frac{f(x)\, f''(x)}{(f'(x))^2}$$

$$\lim_{x \to \xi} g'(\xi) = \lim_{x \to \xi} \frac{f(x)}{f'(x)^2} \lim_{x \to \xi} f''(x)$$

$$= \left(\lim_{x \to \xi} \frac{f(x)}{f'(x)^2} \right) f''(\xi)$$

↑ Apply L'Hopitals rule

$$= \left(\lim_{x \to \xi} \frac{f'(x)}{2 f'(x) f''(x)} \right) f''(\xi)$$

$$= \frac{1}{2 f''(\xi)} f''(\xi)$$

$$= \frac{1}{2}.$$

So $g'(\xi) = \frac{1}{2} \neq 0$

Thus the fixed point iteration is <u>linear</u>.

Recall that Newton's method will converge if the initial approximation x_0 is "close-enough" to the root ξ.

The phrase "close enough" is vague and many times Newton's iteration will not converge or it will converge to another zero than the one being sought-

It would be desirable to establish conditions which guarantee convergence of Newton's method for any choice of the initial iterate in a given interval.

One such set of condition is contained in the following theorem

Theorem :- Let $f:[a,b] \longrightarrow \mathbb{R}$ be twice continuously differentiable on the interval $[a,b]$ and let the following conditions be satisfied

(i) $f(a)f(b) < 0$

(ii) $f'(x) \neq 0 \quad \forall x \in [a,b]$

(iii) $f''(x)$ is either ≥ 0 or ≤ 0 for all $x \in [a,b]$

(iv) At the end point

$$\frac{|f(a)|}{|f'(a)|} \leq b-a, \qquad \frac{|f(b)|}{|f'(b)|} \leq b-a$$

Then the Newtons' method converges to the unique solution ξ of $f(x)=0$ in $[a,b]$ for any choice $x_0 \in [a,b]$

Comments on these conditions

(i) ensures that f has a zero in $[a,b]$

(ii) ensures that f has a unique zero in $[a,b]$

(iii) $\Rightarrow$ graph of $f(x)$ is either concave from above or concave from below. Furthermore together with (ii) implies that $f'(x)$ is monotone

(iv) states that tgt to the curve at either end point intersects the x-axis within the interval $[a,b]$.



Sketch of proof of theorem

We assume without loss of generality $f(a) < 0$

We then distinguish two cases

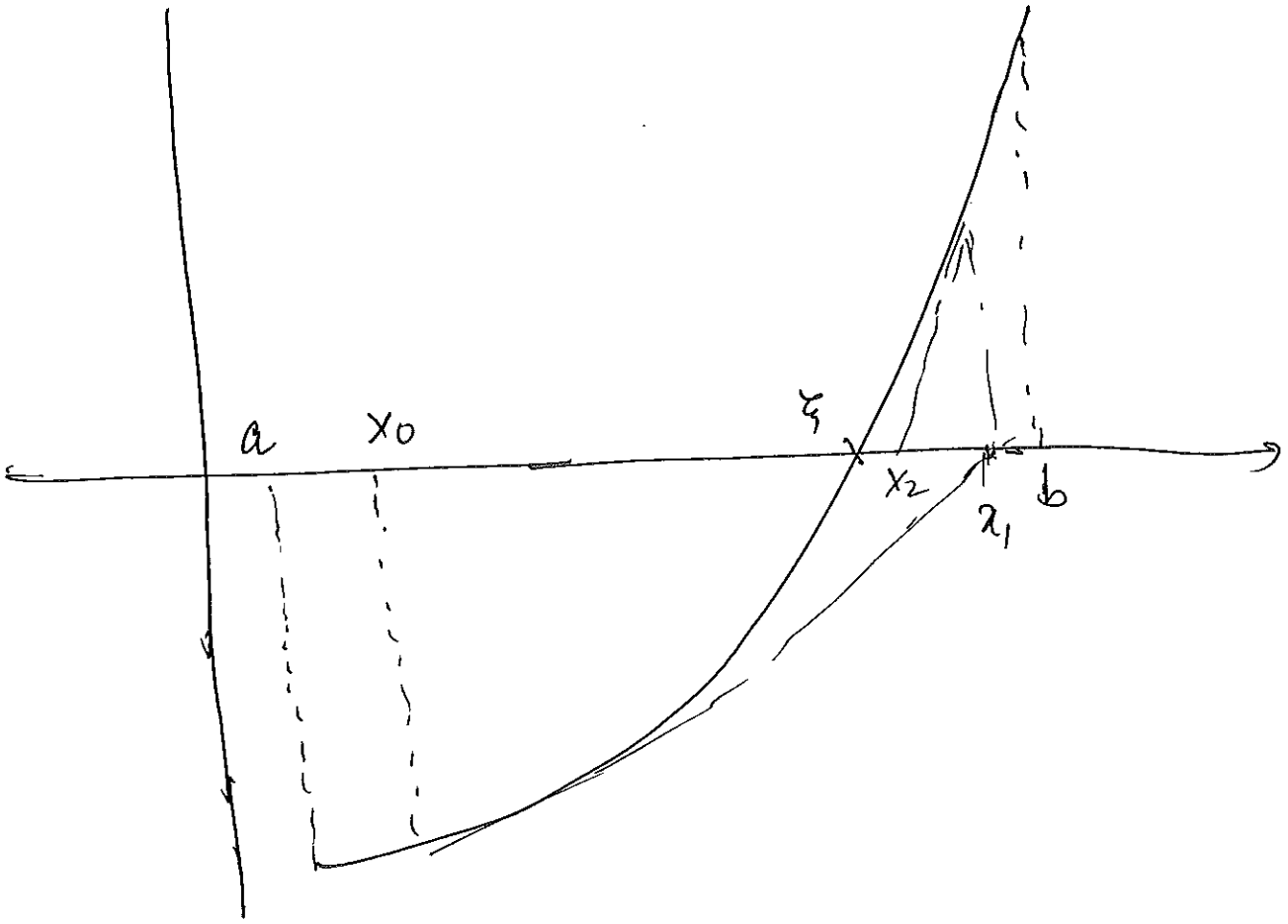
Case (i) $f''(x) \geq 0$

Case (ii) $f''(x) \leq 0$.

Case (ii) reduces to case (i) if we replace f by $-f$.

It therefore suffices to consider case (i)

Graph of $f(x)$ looks like



From the graph it is evident that for $x_0 > \xi$ the resulting iterates decrease monotonously to ξ while for $a \leq x_0 < \xi$ x_1 falls between ξ and b. Then the subsequent iterates converge monotonously to ξ.