Lecture 11

Last time we discussed order of convergence

$x_n \to \xi$

$e_n = \xi - x_n$

If $\exists\ p \geq 0$ and a constant $c \neq 0$ such that

$$\lim_{n \to \infty} \frac{|e_{n+1}|}{|e_n|^p} = c$$

Then p is called "order of convergence" and c is called the asymptotic error constant

Examples 1) Fixed pt iterating

ξ fixed pt of $g : I \to I$

$g'(\xi) \neq 0$

Then $p = 1$ and $c = |g'(\xi)|$

2. For Newtons method

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^2} = \frac{1}{2}\left|\frac{f''(\xi)}{f'(\xi)}\right|$$

provided $f'(\xi) \neq 0$

So order of convergence is 2 and error constant is $\frac{|f''(\xi)|}{2|f'(\xi)|}$

(if ξ is a double root then $p=1$)

3. For secant method

$$|e_{n+1}| = c_n |e_n| |e_{n+1}|$$

$$p = \frac{1+\sqrt{5}}{2} = 1.618\ldots$$

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^p} = \left|\frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\right|^{1/p}$$

(provided $f'(\xi) \neq 0$)

here $r \geq 1$

4. $\left\{\frac{1}{n^r}\right\} \to 0$

$p = 1$

In theory if order of convergence $p > 1$ then it does converge quickly to ξ.

Numerical methods to solve Linear system of equations

Suppose we have a system of equations

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}$$

We want to find a solution

In applications n is large (at least 1000). So doing it by hand is out of question. We have to use computers

It is convenient to use matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \bar{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Thus we have to solve

$$A\bar{x} = \bar{b}$$

This system either has

1) a unique solution
2) no solution
3) infinitely many solutions

Example

1) $x_1 + x_2 = 2$

$x_1 - x_2 = 0$

has a unique solution $x_1 = x_2 = 1$

2) $x_1 + x_2 = 1$

$2x_1 + 2x_2 = 3$

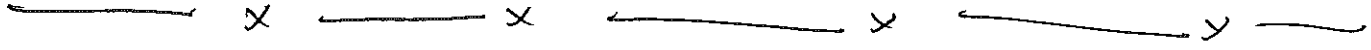
} has no solution

3) $2x_1 - x_2 = 3$

$4x_1 - 2x_2 = 6$

has infinitely many solutions

$\{(x_1, x_2) \mid 2x_1 - x_2 = 3\}$



For most applications the system

$A\bar{x} = \bar{b}$ has a unique solution.

Theorem

$A\bar{x} = \bar{b}$ has a unique solution iff A is an invertible matrix i.e., there exists a matrix B such that

$$BA = AB = I_n$$

I_n = $n \times n$ identity matrix

$$= \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$$

B is usually denoted by A^{-1}

$\therefore Ax = b$

$A^{-1}(Ax) = A^{-1}b$

So $x = A^{-1}\bar{b}$ is the unique solution to $A\bar{x} = \bar{b}$

In practise

Computation of A^{-1} takes too many computations.

Usually we don't need it.

In practise there is a two step procedure to find solution of $A\bar{x} = \bar{b}$

Step 1 (Gaussian Elimination)

The system $A\bar{x} = \bar{b}$ is transformed to an equivalent system $U\bar{x} = \tilde{b}$

where $U = (u_{ij})$ is an upper ~~diag~~ triangular matrix

i.e., $u_{ij} = 0$ for $i > j$

$$U = \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ 0 & & & * \end{bmatrix}$$

$A\bar{x} = \bar{b}$ is equivalent to $Ux = \tilde{b}$

means that $\bar{x}_0$ is a solution of $Ax = b$

iff $\bar{x}_0$ is a solution of $Ux = \tilde{b}$

Step 2 Solving

$$U\bar{x} = \tilde{b}.$$

We do Step 2 first

Note we are assuming $A\bar{x} = \bar{b}$ has a unique solution.

So $U\bar{x} = \bar{b}$ has a unique solution

$\Rightarrow$ U is an invertible matrix.

Exercise

Show that an upper triangular matrix $U = (u_{ij})$ is invertible iff all diagonal entries (i.e. u_{ii}) is non-zero!

Step 2 Solution of $U\bar{x} = \tilde{b}$

$$u_{11}x_1 + u_{12}x_2 + \cdots + u_{1n}x_n = \tilde{b}_1$$
$$u_{22}x_2 + \cdots + u_{2n}x_n = \tilde{b}_2$$

$$u_{n-1,n-1}x_{n-1} + u_{n-1,n}x_n = \tilde{b}_{n-1}$$
$$u_{nn}x_n = \tilde{b}_n$$

$$x_n = \frac{\tilde{b}_n}{u_{nn}}$$

$$x_{n-1} = \frac{\tilde{b}_{n-1} - u_{n-1,n}x_n}{u_{n-1,n-1}}$$

$$x_i = \frac{\tilde{b}_i - \sum_{j>i} u_{ij}x_j}{u_{ii}}$$

for $i = n-1, n-2, \ldots, 2, 1$

This process is called back-substitution

Example

$$3x_1 + x_2 + 2x_3 = 6$$
$$4x_2 + 2x_3 = 7$$
$$3x_3 = 9$$

$$x_3 = 9$$
$$4x_2 + 6 = 7$$
$$x_2 = 1/4$$
$$3x_1 + 1/4 + 6 = 6$$
$$x_1 = -\frac{1}{12}.$$

Gaussian Elimination

Recall two linear system $Ax = b$ and $\tilde{A}x = \tilde{b}$ are equivalent if any solution of one is a solution of the other

Theorem Let $A\bar{x} = \bar{b}$ be a linear system and suppose we subject this system to a sequence of operation of the following kind

(i) Multiplication of one equation by a non-zero constant

(ii) Addition of a multiple of one equation to another equation

(iii) Interchange of two equations

If this system produces a new system $\tilde{A}x = \tilde{b}$ then the systems $Ax = b$ and $\tilde{A}x = \tilde{b}$ are equivalent.

In particular A is invertible iff $\tilde{A}$ is invertible

Gaussian Elimination

It is possible to convert $Ax = b$ to equivalent system $Ux = \tilde{b}$ (U upper triagular) by using the above 3 operation

Example

$$x_1 - x_2 + 2x_3 = -6$$
$$2x_1 - 2x_2 + 3x_3 = -14$$
$$x_1 + x_2 + x_3 = -2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -6 \\ 2 & -2 & 3 & -14 \\ 1 & 1 & 1 & -2 \end{array}\right]$$

$R_2 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -6 \\ 0 & 0 & -1 & -2 \\ 1 & 1 & 1 & -2 \end{array}\right]$$

$R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -6 \\ 0 & 0 & -1 & -2 \\ 0 & 2 & -1 & 4 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -6 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & -1 & -2 \end{array}\right]$$

$-x_3 = -2$ so $x_3 = 2$

$2x_2 - x_3 = 4$ $\quad x_2 = 3$

$x_1 - x_2 + 2x_3 = -6$ $\quad x_1 - 3 + 4 = -6$

$x_1 = -7.$

Algorithim for Gaussian Elimination

To solve $Ax = b$

$W = [A : b]$ "augumented matrix"

Step 1 for $i = 1, 2, \ldots, n-1$ do steps 2, 3, 4

Step 2 det p be the smallest integer with $i \le p \le n$ and $a_{pi} \neq 0$

If no integer p can be found then output "no unique solution exists" & stop

Step 3 If $p \neq i$ then interchange Row R_i $\leftrightarrow$ Row R_j

Step 4 for $j = i+1, \ldots, n$ do steps 5, 6

Step 5 Set $m_{ji} = \frac{a_{ji}}{a_{ii}}$

Step 6 perform $R_j - m_{ji} R_i$

Step 7 If $a_{nn} = 0$ then
"no unique solution exist"
& stop

Step 8 U = first n columns of ω
$\tilde{b}$ = last column of ω

then $Ax = b$ is equivalent to $Ux = \tilde{b}$

where U is an upper triangular matrix

Operation count

We count the number of multiplication / division and addition / subtraction to do GE

In general the amount of time required to perform a multiplication or division on a computer is approximately the same and is considerably greater than that required to perform addition or subtraction

No arithmetic operation is performed until step 5 in the algorithm.

step 5 requires about $n-i$ divisions be performed

In step 6 we replace row R_j by

$$R_j - m_{ji} R_i$$

This requires m_{ji} be multiplied to each term in R_i

This requires $(n-i)(n-i+1)$ multiplications

Afterwards each term of the resulting equation is subtracted from the corresponding term in R_j. This requires $(n-i)(n-i+1)$ subtractions

Then for each $i = 1, 2, \ldots, n-1$ the operations required are

multiplications/divisions	
$n-i + (n-i)(n-i+1)$	$= (n-i)(n-i+2)$

Addition / Subtraction	
$(n-i)(n-i+1)$	

Total multiplication / division

$$\sum_{i=1}^{n-1} (n-i)(n-(i+2)) = \frac{2n^3+3n^2-5n}{6}$$

Total addition / subtraction

$$\sum_{i=1}^{n-1} (n-i)(n-(i+1)) = \frac{n^3-n}{3}$$

For back substitution (i.e. step 2)

One can show are required

$\frac{n^2+n}{2}$ multiplication / division

$\frac{n^2-n}{2}$ addition / subtraction

Note for large n

n^3 is considerably larger than n^2.

for example when $n=100$

100^2 is 1% of 100^3

Thus GE is $\theta(n^3/3)$ operations

Tridigonal matrix

$$\begin{bmatrix} a_1 & b_1 & & & & 0 \\ c_1 & a_2 & b_2 & & & \\ & c_3 & a_3 & b_3 & \cdots & \\ & & & & & \\ & & & & & b_{n-1} \\ 0 & & & & c_{n-1} & a_n \end{bmatrix}$$

$A = [a_{ij}]$ is said to be tridigonal

if $a_{ij} = 0$ for $|i-j| > 1$

Exercise

$Ax = b$ for tridigonal ~~matrix~~ systems can be solved in $O(n)$ steps.

LU - factorization

Steps used to solve a system $Ax = b$ can be used to factor the matrix A. The factorization is particularly useful if it has the form $A = LU$ where L is a lower triangular matrix and U is upper triangular matrix.

Not all matrices have this type of representation. However many matrices that occur in ~~the study of~~ practise have this property.

Application :- We would want to solve $Ax = b$ for many different values of b

~~Lecture 1~~

If we do GE each time then we would need $O(n^3/3)$ operation each time we solve $Ax = b$.

On the other hand once $A = LU$

Then we can solve $Ax = b$ as follow

Set $y = Ux$

we solve $LUx = b$

$Ly = b$

L is lower triangular. So determining y requires $O(n^2)$ operation.

Then solving $Ux = y$ U upper triangular requires only $O(n^2)$ operations

Thus number of operation to solve the system $Ax = b$ is reduced from $O(n^3/3)$ to $O(2n^2)$ when $n \geq 1000$ this reduces number of computation by more than 99%

~~Gen~~ Construction of LU factorization

Assumption $Ax = b$ can be solved without row interchanges.

First step in GE process consists of performing for each $j = 2, 3, \ldots, n$ the operation

$$R_i - m_{j1} R_1 \quad \text{where } m_{j1} = \frac{a_{j1}^{(1)}}{a_{11}^{(1)}}$$

Equivalently we can multiply the original matrix A on the left by the matrix

$$M^{(1)} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -m_{21} & 1 & 0 & & 0 \\ -m_{31} & 0 & 1 & & \\ \vdots & & & & 0 \\ -m_{n1} & 0 & \cdots & & 1 \end{bmatrix}$$

Set $A = A^{(1)} \quad b = b^{(1)}$

$$A^{(1)} x = b^{(1)}$$

$$M^{(1)} A^{(1)} x = M^{(1)} b^{(1)}$$

Set $A^{(2)} = M^{(1)} A^{(1)}$

$b^{(2)} = M^{(1)} b^{(1)}$

So we have system

$$A^{(2)} x = b^{(2)}$$

$A^{(2)}$ has $a_{i1}^{(2)} = 0$ for $i \geq 2$.

In a similar manner we construct $M^{(2)}$, the identity matrix with entries below the diagonal in the second column replaced by the negative of the multiple

$$m_{j2} = \frac{a_{j2}^{(2)}}{a_{22}^{(2)}}$$

$$A^{(3)} = M^{(2)} A^{(2)}$$

$$b^{(3)} = M^{(2)} b^{(2)}$$

$A^{(3)}$ has zeros below the diagonal in first 2 columns

So we have

$$A^{(3)}x = b^{(3)}$$

In general with $A^{(k)}x = b^{(k)}$ already formed, multiply both sides by

$$M^{(k)} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & 0 & & \\ & & -m_{k+1,k} & \ddots & & \\ & 0 & & & 1 & \\ & & -m_{n,k} & & & \end{bmatrix} \quad (k)$$

$$A^{(k+1)} = M^{(k)}A^{(k)} = M^{(k)} \cdots M^{(1)}A$$

$$b^{(k+1)} = M^{(k)}b^{(k)} = M^{(k)}M^{(k-1)} \cdots M^{(1)}b$$

So we have $A^{(k+1)}x = b^{(k+1)}$

The process ends with

$$A^{(n)}x = b^{(n)}$$

where $A^{(n)} = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} \\ & & & \\ 0 & 0 & \cdots & a_{nn}^{(n)} \end{pmatrix}$

is upper-triangular

Set $U = A^{(n)}$.

Therefore $U = M^{(n-1)} M^{(n-2)} \cdots M^{(1)} A$

$$L = \left[M^{(n-1)} M^{(n-2)} \cdots M^{(1)} \right]^{-1}$$

$$= M^{(1)-1} M^{(2)-1} \cdots M^{(n-1)-1}$$

Note each $M^{(k)}$ is lower triangular

So $M^{(k)-1}$ is " "

$\Rightarrow$ L is lower triangular

Also $A = LU$

$$M^{(1)} = \begin{bmatrix} 1 & & & 0 \\ -m_{21} & 1 & & \\ -m_{31} & & & \\ \vdots & & & \\ -m_{n1} & & & 1 \end{bmatrix}$$

$$M^{(1)-1} = \begin{bmatrix} 1 & & & 0 \\ m_{21} & 1 & & \\ m_{31} & & & \\ \vdots & & & \\ m_{n1} & & & 1 \end{bmatrix}$$

and so on

One can prove that

$$L = \begin{pmatrix} 1 & 0 & & \\ m_{21} & 1 & & \\ m_{31} & m_{32} & & \\ \vdots & \vdots & & \\ m_{n1} & m_{n2} & & 1 \end{pmatrix}$$

Example

$$x_1 + x_2 + 0x_3 + 3x_4 = 4$$
$$2x_1 + x_2 - x_3 + x_4 = 1$$
$$3x_1 - x_2 - x_3 + 2x_4 = -3$$
$$-x_1 + 2x_2 + 3x_3 - x_4 = 4$$

Step 1 $R_2 - 2R_1$ $R_3 - 3R_1$ $R_4 + R_1$

gives $A^{(2)} = \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & -4 & -1 & -7 \\ 0 & 3 & 3 & 2 \end{pmatrix}$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & & & 0 \\ -1 & & & 1 \end{pmatrix}$$

Step (2) $R_3 - 4R_2$ $R_4 + 3R_2$

$$A^{(3)} = \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{pmatrix}$$

Solve $Ax = b = \begin{pmatrix} 8 \\ 7 \\ 14 \\ -7 \end{pmatrix}$

Set $y = Ux$

$$Ly = b$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{pmatrix} 8 \\ 7 \\ 14 \\ -7 \end{pmatrix}$$

$y_1 = 8$

$2y_1 + y_2 = 7$ $\quad y_2 = -9$

$3y_1 + 4y_2 + y_3 = 14$ $\quad y_3 = 26$

$-y_1 - 3y_2 + y_4 = -7$ $\quad$ so $y_4 = -26$

We then solve

$$Ux = y$$

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 8 \\ -9 \\ 26 \\ -26 \end{pmatrix}$$

We use "back substitution"

$$-13x_4 = -26 \implies x_4 = 2$$

$$3x_3 + 13x_4 = 26 \implies x_3 = 0$$

$$-x_2 - x_3 - 5x_4 = -9 \implies x_2 = -1$$

$$x_1 + x_2 + 3x_4 = 8 \implies x_1 = 3$$