# Lecture 12



Last time we did Gauss Elimination (GE)

We have a system of equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

We can write it as

$$Ax = b$$

$$A = (a_{ij}) \qquad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

After doing Row transformations we convert it into an equivalent system

$$Ux = \tilde{b} \quad \text{where}$$

$U$ is upper triangular.

Then $\underline{x}$ can be solved by back substitution.

If ~~row~~ GE can be done without row interchanges then A can be factored as $A = LU$

where L is lower triangular and U is upper triangular

Then to solve $Ax = b$

$LUx = b$

set $Ux = y$

solve $Ly = b$ by forward substitution

then solve $Ux = y$ by backward substitution

Advantage of LU factorization is when we have to solve $Ax = b$ for many different values of $b$.

Question :- which classes of matrices admit LU decomposition

We have to find classes of matrices for which Gauss Elimination can be performed effectively without row interchanges.

The classes of matrices are

(1) strictly diagonally dominant matrices

(2) +ve definite matrices

Recall

An $n \times n$ matrix $A = (a_{ij})$ is said to be strictly diagonally dominant if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \quad \text{for each } i = 1, 2, \cdots, n$$

Example

$$A = \begin{bmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{bmatrix}$$

A is strictly diagonally dominant

Note $A^t = \begin{bmatrix} 7 & 3 & 0 \\ 2 & 5 & 5 \\ 0 & -1 & -6 \end{bmatrix}$

is NOT strictly diagonally dominant

Theorem :- A strictly diagonally dominant matrix A is non-singular.

Proof We prove by contradiction

Suppose A is singular

$\Rightarrow \exists\, x \neq 0$ such that $Ax = 0$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Let $k$ be index for which

$$0 < |x_k| = \max_{1 \le j \le n} |x_j|$$

Since $Ax = 0$

we have $\sum_{j=1}^{n} a_{ij} x_j = 0$ for each $i = 1, 2, \ldots, n$

When $i = k$

$$\sum_{j=1}^{n} a_{kj} x_j = 0$$

So $a_{kk} x_k = - \sum_{j \neq k} a_{kj} x_j$

$$|a_{kk}| |x_k| \le \sum_{j \neq k} |a_{kj}| |x_j|$$

So $|a_{kk}| \leq \sum_{j \neq k} |a_{kj}| \frac{|x_j|}{|x_k|} \leq \sum_{j \neq k} |a_{kj}|$

This inequality contradicts the strict diagonal dominance of A.

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## Positive definite matrix

A matrix A is +ve definite if

1) it is symmetric i.e, $A^t = A$

2) $x^t Ax > 0$ for every n-dim vector $x \neq 0$.

**Remarks** A +ve definite matrix is non-singular

**Prob**

$$Ax = 0$$
$$\Rightarrow x^t Ax = 0$$
$$\Rightarrow x = 0.$$

# Cholesky Algorithm

Given a positive definite $n \times n$ matrix $A$ it factors into $LL^t$ where $L$ is lower triangular

**Example** $A = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix}$

$$A = LL^t$$

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$l_{11}^2 = 4 \Rightarrow l_{11} = 2$$

$$l_{21} l_{11} = 2 \Rightarrow l_{21} = 1$$

$$l_{31} l_{11} = 14 \Rightarrow l_{31} = 7$$

$$l_{21}^2 + l_{22}^2 = 17$$

$$l_{22}^2 = 16$$

$$\Rightarrow l_{22} = 4$$

$$l_{31} l_{21} + l_{32} l_{22} = -5$$

$$7 + l_{32}\, 4 = -5$$

$$\Rightarrow l_{32} = -3$$

$$l_{31}^2 + l_{32}^2 + l_{33}^2 = 83$$

$$49 + 9 + l_{33}^2 = 83$$

$$l_{33}^2 = 25$$

$$l_{33} = 5$$

So $$L = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{pmatrix}$$

# Cholesky's Algorithm

To factor A into $LL^t$ where L is lower-triangular (Here A is +ve definite)

$$L = (l_{ij})$$

**Step 1** set $l_{11} = \sqrt{a_{11}}$

**Step 2** for $j = 2, 3, \ldots, n$ set

$$l_{j1} = \frac{a_{j1}}{l_{11}}$$

**Step 3** For $i = 2, \ldots, n-1$ do steps a, b.

**Step a** Set $l_{ii} = \sqrt{a_{ii} - \sum_{K=1}^{i-1} l_{iK}^2}$

**step b** for $j = i+1, \ldots, n$

set $l_{ji} = \dfrac{a_{ji} - \sum_{K=1}^{i-1} l_{jK} l_{iK}}{l_{ii}}$

Step 7

Set $l_{nn} = \sqrt{a_{nn} - \sum_{k=1}^{n-1} l_{nk}^2}$

$_____ \times _____ \times _____$

Q Why do Cholesky factorization ?

LU factorization requires $O(n^3/3)$ multiplication / division and $O(n^3/3)$ addition / subtraction

The $LL^t$ Cholesky factorization requires $O(n^3/6)$ multiplications / divisions and $O(n^3/6)$ addition / subtraction

Thus it requires only 50% of calculations

Disadvantage of Cholesky's algorithm

It is valid only for +ve definite matrices

Note that LU decomposition is possible if GE can be done without row changes.

## What to do when GE has row changes?

A $n \times n$ permutation matrix $P = (P_{ij})$ is obtained by rearranging the rows of $I$

Example $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$R_2 \leftrightarrow R_3$

$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ is $3 \times 3$ permutation matrix

$A = (a_{ij})_{3\times 3}$

$$PA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

$R_2 \leftrightarrow R_3$ in $A$

## Two useful properties of permutation matrices

Suppose $k_1, \ldots, k_n$ is a permutation of $1, 2, \ldots, n$ and the permutation matrix $P = (P_{ij})$ is defined by

$$P_{ij} = \begin{cases} 1 & \text{if } j = k_i \\ 0 & \text{otherwise} \end{cases}$$

Then (i) $PA$ permutes the rows of $A$

(ii) $P^{-1}$ exists and $P^{-1} = P^t$

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## PLU factorization of a matrix

Let $A$ be a matrix

Suppose if possible we have done some row changes while doing ~~Gauss~~ Gauss Elimination on $A$.

This implies that there exists a permutation matrix $P$ such that GE can be done on $PA$ without any row change

Thus $PA = LU$

Solving $Ax = b$

$$PAx = Pb = b'$$

$$LUx = b'$$

$$y = Ux$$

first solve $Ly = b'$

Then solve $Ux = y$

## Example

$$A = \begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ -1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 2 \end{bmatrix}$$

$R_1 \longleftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ -1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 2 \end{bmatrix}$$

$R_3 + R_1$ $\quad$ $R_4 - R_1$ giving

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$R_3 \longleftrightarrow R_4$

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix} = U$$

$P =$ $R_1 \leftrightarrow R_2$, $R_3 \leftrightarrow R_4$ done on identity matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$PA = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 1 & 2 & 0 & 2 \\ -1 & -1 & 1 & 0 \end{bmatrix}$$

$R_3 - R_1$ $\quad$ $R_4 + R_1$ $\quad$ gru

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$R_3 - R_2$ $\quad$ gru

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

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# Errors associated with Gauss Elimination

## Example

$$0.0003x_1 + 1.566x_2 = 1.569$$
$$0.3454x_1 - 2.436x_2 = 1.018$$

Exact answer $x_1 = 10, \quad x_2 = 1$

Comp in 4 sig digits

$$\left[\begin{array}{cc|c} 0.0003 & 1.566 & 1.569 \\ 0.3454 & -2.436 & 1.018 \end{array}\right]$$

$$m_{21} = \frac{0.3454}{0.0003} = 1,151$$

$$a_{22}^{(2)} = -2.436 - (1,151)(1.566)$$
$$= -1804$$

$$b_2^{(2)} = 1.018 - (1.451)(1.569)$$

$$= -1805$$

$$\left[\begin{array}{cc|c} 0.0003 & 1.566 & 1.569 \\ 0 & -1804 & -1805 \end{array}\right]$$

$$x_2 = \frac{-1805}{-1804} = 1.001$$

Hence from the first equation

$$x_1 = \frac{1.569 - (1.566)(1.001)}{0.0003}$$

$$x_1 = 3.333$$

Exact is 10

So $x_1$ has lot of error

## Plausible explanation

$a_{11} = 0.0003$ is very small

So the algorithm performs badly for $a_{11}$ is "near zero".

However consider the system in example but with first equation multiplied by $10^m$ where $m$ is some integer

$$(0.0003.)\,10^m x_1 + 1.566 \times 10^m x_2 = 1.569 \times 10^m$$

$$0.3454\, x_1 - 2.436\, x_2 = 1.018$$

$$m_{21} = \frac{0.3454}{0.0003\ 10^m} = 1,151 \cdot 10^{-m}$$

$$a_{22}^{(2)} = -2.436 - (1,151\ 10^{-m})(1.566\ 10^m)$$

$$= -1804$$

Ily $b_{22}^{(2)} = -1805$

So we get $x_2 = 1.001$

and finally $x_4 = 3.333$

## Explanation of the error

$|a_{11}|$ is small compared with $|a_{12}|$.

Thus a small error in computed value of $x_2$ leads to a large error in $x_1$.

$$\left|\frac{a_{12}}{a_{11}}\right| \approx 5220$$

$$\left|\frac{a_{22}}{a_{21}}\right| \approx 6$$

So we do $R_1 \leftrightarrow R_2$

We get $m_{11} = \frac{0.0003}{0.3454} = 0.0008681$

So now new ~~sec~~ second equation becomes

$$1.568 x_2 = 1.568$$

$$x_2 = 1$$

and from "new" first equation

we get $x_1 = 10$

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## Scaled partial pivoting

Step 1 $\quad s_i = \max_{1 \le j \le n} |a_{ij}|$

scale factor for row $i$

$s_i \neq 0$ since otherwise all entries in row $i$ is zero

$\Rightarrow$ A is singular. $\times$

$$\frac{|a_{p1}|}{s_p} = \max_{1 \le k \le n} \frac{|a_{k1}|}{s_k}$$

perform $R_1 \longleftrightarrow R_p$ if $p \neq 1$

In a similar manner before eliminating variable $x_i$ from rows $i+1, i+2, \cdots, n$

we select the smallest integer $p \geq i$ with

$$\frac{|a_{pi}|}{s_p} = \max_{i \le k \le n} \frac{|a_{ki}|}{s_k}$$

and perform $R_i \longleftrightarrow R_p$ if $p \neq i$

Example (calculation in 3 sig digits)

$$2.11 x_1 - 4.21 x_2 + 0.921 x_3 = -2.01$$

$$4.01 x_1 + 10.2 x_2 - 1.12 x_3 = -3.09$$

$$1.09 x_1 + 0.987 x_2 + 0.832 x_2 = 4.21$$

$$s_1 = 4.21 \qquad s_2 = 10.2 \qquad s_3 = 1.09$$

$$\frac{|a_{11}|}{s_1} = \frac{2.11}{4.21} = 0.501$$

$$\frac{|a_{21}|}{s_2} = \frac{4.01}{10.2} = 0.393$$

$$\frac{|a_{31}|}{s_3} = \frac{1.09}{1.09} = 1$$

So we do $R_1 \longleftrightarrow R_3$.

$$\left[\begin{array}{ccc|c} 1.09 & 0.987 & 0.832 & 4.21 \\ 4.01 & 10.2 & -1.12 & -3.09 \\ 2.11 & -4.21 & 0.921 & 2.01 \end{array}\right]$$

$R_2 - \frac{4.09}{1.09} R_1 \qquad R_3 - \frac{2.11}{1.09} R_1$ gives

$$\left[\begin{array}{ccc|c} 1.09 & 0.987 & 0.832 & 4.21 \\ 0 & 6.57 & -4.18 & -18.6 \\ 0 & -6.12 & -0.689 & -6.16 \end{array}\right]$$

Note $s_2 = 10.2$ $\quad s_3 = 4.21$ ↑ since we did $R_1 \leftrightarrow R_3$

$$\frac{|a_{22}|}{s_2} = \frac{6.57}{10.2} = 0.644$$

$$\frac{|a_{32}|}{s_3} = \frac{6.12}{4.21} = 1.45$$

So we do $R_3 \leftrightarrow R_2$ and do further computations