Lecture 19

Last time we studied difference equations we studied linear difference equations with constant coefficients

(*) $y_{n+N} + a_{N-1} y_{n+N-1} + \cdots + a_0 y_n = 0$

The characteristic equation of (*) is

$$p(\beta) = \beta^N + a_{N-1}\beta^{N-1} + \cdots + a_0 = 0$$

If $p(\beta)$ has N distinct roots $\beta_1, \ldots, \beta_N$ then

$\beta_1^n, \beta_2^n, \ldots, \beta_N^n$ are all solution to (*)

So by linearity

(**) $y_n = c_1\beta_1^n + c_2\beta_2^n + \cdots + c_N\beta_N^n$ is also a solution (where $c_1, \ldots, c_N$ are arbitary constants)

(**) is called the general solution of (*)

We also analysed the case when $P(\beta)$ has conjugate complex zeros and also the case when $P(\beta)$ has a double root

We then studied non-homogeneous linear difference equation with constant coefficients

$$y_{n+N} + a_{N-1} y_{n+N-1} + \cdots + a_0 y_n = b_n$$

A general solution to above is of the form

$$y_n = y_n^G + y_n^P$$

where y_n^G is the general solution to the homogenous equation and y_n^P is a particular solution

Special case $b_n = b$ = constant for all n

a particular solution can be found by setting $y_n^P = \alpha$ and determining α by

$$\alpha = \frac{b}{1 + a_{N-1} + a_{N-2} + \cdots + a_0}$$

(provided denominator $\neq 0$)

We also studied

Stability of numerical methods to solve initial value differential equations

They arise because we replace a first order differential equation by a higher order difference equation.

If multistep method gives rise to a difference equation of order k, find the roots of the characteristic equation corresponding to homogeneous difference equation

call these β_i $(i=1,2,\ldots,k)$. The general soln of the homogeneous difference equation is $y_n = c_1\beta_1^n + \cdots + c_k\beta_k^n$

One of these solutions say β_1^n will tend to the exact soln of the diff. eqn as $h \to 0$.

All the other solutions are extraneous

A multistep method is defined to be strongly stable if the extraneous roots satisfy as $h \to 0$ the condition

$$|\beta_i| < 1 \quad \text{for } i = 2, 3, \ldots k.$$

If a method is strongly stable then errors introduced in the computation will decay as n - increases.

Adam-Bashforth order 4 method is strongly stable

Milne's method is not strongly stable.

Lecture 19

The Eigenvalue Problem

Eigenvalues of a matrix are of great importance in many physical problems. The stability of an aircraft, for example, is determined by the location in the complex plane of the eigenvalues of a certain matrix.

Recall that we say a (real or complex) number λ is an eigenvalue of the matrix B provided for some non-zero (real or complex) vector y

$$By = \lambda y.$$

The vector y is then called an ~~eigen~~ eigenvector of B belonging to the eigenvalue λ.

Lemma The number λ is an eigenvalue of B if and only if $B - \lambda I$ is not invertible.

Proof If λ is an eigen-value of B then there exists $y \neq 0$ such that $By = \lambda y$

So $(B - \lambda I)\, y = 0$ & $y \neq 0$

$\Rightarrow$ $B - \lambda I$ is not invertible.

Conversely if $B - \lambda I$ is not invertible.

then $\exists\ \bar{y} \neq 0$ such that

$$(B - \lambda I)\, \bar{y} = 0$$

$$\Rightarrow \quad B\bar{y} = \lambda \bar{y}.$$

Thus λ is an eigenvalue of B.

Characteristic polynomial

$$p(\lambda) = \det(\lambda I - B)$$

Corollary λ is an eigen-value of B if and only if λ is a root of the characteristic polynomial.

Proof λ is an eigen value of B

$\Leftrightarrow$ $B - \lambda I$ is not invertible

$\Leftrightarrow$ $\det(\lambda I - B) = 0$

$\Leftrightarrow$ λ is a root of $p(\lambda)$.

Remark $\deg p(\lambda) = n$ (n = order of B)

Corollary Let B be a $n \times n$ matrix. Then B has atmost n eigen values.

Example

1. $B = 0$ matrix

$$\lambda I - 0 = \begin{bmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{bmatrix}$$

$$p(\lambda) = \lambda^n$$

Thus 0 is the only eigenvalue of the zero-matrix.

Let $y \neq 0$. Note $By = 0 = 0y$

Thus y is an eigenvalue corresponding to 0.

2. $B = I_n$ ($n \times n$ identity matrix)

$$\lambda I - I = \begin{bmatrix} \lambda-1 & & \\ & \ddots & \\ & & \lambda-1 \end{bmatrix}$$

$$p(\lambda) = (\lambda-1)^n$$

Thus 1 is the only eigenvalue of I.

Let $y \neq 0$. Then $By = y = 1 \cdot y$

Thus y is an eigenvector corresponding to 1.

3) Let B be an upper-triangular matrix

$$\lambda I - B = \begin{bmatrix} \lambda - b_{11} & * & \cdots & * \\ & \lambda - b_{22} & & * \\ & & \ddots & \\ & & & \lambda - b_{nn} \end{bmatrix}$$

$$p(\lambda) = \det(\lambda I - B)$$

$$= (\lambda - b_{11})(\lambda - b_{22}) \cdots (\lambda - b_{nn}).$$

Thus $b_{11}, \ldots, b_{nn}$ are eigenvalues of B.

It requires some work to find eigenvectors of B.

(6)

(4) $B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$|\lambda I - B| = \begin{vmatrix} \lambda-1 & -2 & 0 \\ -2 & \lambda-1 & 0 \\ 0 & 0 & \lambda+1 \end{vmatrix}$$

$$= (\lambda-1)(\lambda-1)(\lambda+1) + 2(-2(\lambda+1))$$

$$= (\lambda+1)((\lambda-1)^2 - 4)$$

$$= (\lambda+1)(\lambda^2 - 2\lambda - 3)$$

$$= (\lambda+1)^2(\lambda-3)$$

Thus $-1, 3$ are eigenvalues of B.

To find eigen-vector corresponding to 3

we solve $(3I - B)y = 0$

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} 2y_1 - 2y_2 \\ -2y_1 + 2y_2 \\ 4y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So $y_3 = 0$ $\quad y_1 = y_2$

$y = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is eigen-vector corresponding to $\lambda = 3$.

To find eigenvector corresponding to $\lambda = -1$

we solve $(-I - B)y = 0$

$$\begin{pmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-2y_1 - 2y_2 = 0 \quad \Rightarrow y_1 = -y_2$

No condition on y_3

$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are two eigenvectors corresponding to $\lambda = -1$

Prop" Let x, y be two eigen-vectors for the matrix A corresponding to the eigenvalues λ and μ of A respectively. If $\lambda \neq \mu$ then x and y are linearly independent.

Proof Suppose we have a relation

$$aX + bY = 0 \quad \text{①}$$

$$\Rightarrow A(ax + by) = A0 = 0$$

$$aAx + bAy = 0$$

$$a\lambda x + b\mu y = 0 \quad \text{②}$$

λ① − ② gives

$$b(\lambda - \mu)y = 0$$

note $y \neq 0$

$\lambda \neq \mu$ so $b = 0$

$aX = 0 \quad X \neq 0 \Rightarrow a = 0$

Thus x and y are linearly independent

Def^n A basis of n-vectors which consists entirely of eigen-vectors of B is called a complete set of eigenvectors for B.

Corollary :- Let A be a $n \times n$ matrix. Suppose A has n-distinct eigen-values then it has a complete set of eigenvectors.

Proof Let $\lambda_1, \ldots, \lambda_n$ be n-distinct eigenvalues of B.
Let $x_1, \ldots x_n$ be corresponding eigenvectors.

Claim $\{x_1, \ldots, x_n\}$ is a basis

Suffices to show it is linearly independent
We show by induction on k that-

$x_1, \ldots x_k$ is linearly independent.

$k=1$, Nothing to show as $x_1 \neq 0$

$k=2$ follows from previous proposition

We assume for m and prove for $m+1$

Suppose $c_1 x_1 + \cdots + c_m x_m + c_{m+1} x_{m+1} = 0$ ①

$$A(c_1 x_1 + \cdots + c_{m+1} x_{m+1}) = 0$$

$$c_1 A x_1 + \cdots + c_{m+1} A x_{m+1} = 0$$

$$c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \cdots + c_m \lambda_m x_m + c_{m+1} \lambda_{m+1} x_{m+1} = 0 \quad ②$$

λ_{m+1} ① − ② gives

$$c_1(\lambda_{m+1} - \lambda_1) x_1 + \cdots + c_m(\lambda_{m+1} - \lambda_m) x_m = 0$$

By induction hypothesis $x_1, \ldots x_m$ is L.I

$$c_i(\lambda_{m+1} - \lambda_i) = 0 \qquad i = 1, 2, \ldots, m$$

$\lambda_{m+1} \neq \lambda_i$ so $c_i = 0$ for $i = 1, \ldots m$

Thus $c_{m+1} x_{m+1} = 0$

$x_{m+1} \neq 0 \Rightarrow c_{m+1} = 0$

Thus $\{x_1, \ldots x_m, x_{m+1}\}$ is L.I.

It follows from induction that

$\{x_1, \ldots x_n\}$ is L.I. So it is a basis

If the eigenvalues are not-distinct then B may or may not have a complete set of eigen-vectors.

Example (previous)

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Eigenvalues were $3, -1$

eigen vector corresponding to 3 is $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

eigen vector corresponding to -1 are

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Note $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis of $\mathbb{R}^3$.

So B has a complete set of eigenvectors in this case

Example $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & -1 \\ 0 & \lambda-1 \end{vmatrix} = (\lambda-1)^2$$

Thus 1 is the only eigenvalue of A.

To find eigenvector corresponding to 1

$$(I - A)y = 0$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$y_2 = 0$$

Thus $y = \begin{pmatrix} y_1 \\ 0 \end{pmatrix}$ $y_1 \neq 0$ is eigen-vector corresponding to 1

$$y = y_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Thus A does not have complete set of eigen-vectors.

Question :- When does a matrix A has a complete set of eigen-vector

Example (difficult) If A is symmetric then A has a complete set of eigen-vectors

_____ × _____ × _____

One says that two matrices A and B are similar if

$$A = C^{-1}BC$$

for some invertible matrix C

→ Similar matrices have same eigenvalues

> eigenvalue of A

$$Ay = \lambda y \quad \text{with } y \neq 0$$

$Cy \neq 0$ ∵ C is invertible

$$BCy = BCy = CAy = CAy$$

$$B(Cy) = \lambda(Cy)$$

Thus for eigenvalue - eigenvector pair (λ, x) of A there corresponds eigenvalue - eigenvector pair $(\lambda, (x)$ of B.

In numerical methods we have algorithm which converts a matrix to another similar matrix for which eigenvalue and eigenvectors are easy to find

For example a symmetric matrix A can be converted into a symmetric tridigonal matrix (by similarity).

Theorem Let A be a $n \times n$ matrix.

TFAE

1) A has a complete set of eigenvectors

2) A is similar to a diagonal matrix.

Proof 1) $\Rightarrow$ 2)

Let $\mathcal{B} = \{x_1, \ldots, x_n\}$ be a complete set of eigenvectors of A. Also let $Ax_i = \lambda_i x_i$ $i = 1, 2, \ldots, n$

Then $\mathcal{B}$ is a basis of $\mathbb{R}^n$.

Let $C = [x_1, \ldots, x_n]$ be matrix with i^{th} column to be x_i. So $Ce_i = x_i$

Note C is invertible

Consider $D = C^{-1}AC$

$De_i = C^{-1}ACe_i$

$= C^{-1}Ax_i$

$= C^{-1}\lambda_i x_i$ $(\because C^{-1}x_i = e_i)$

$= \lambda_i e_i$

Thus $De_i = \lambda_i e_i$

So $D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$ is a diagonal matrix

Also A is similar to D.

Conversely let A be similar to a diagonal matrix D

Let $D = C^{-1} A C$

Let $C = [x_1, x_2, \ldots, x_n]$ i.e $Ce_i = x_i$

Note $De_i = \lambda_i e_i$

$$A = CDC^{-1}$$

$$Ax_i = CDC^{-1}x_i \qquad \left(\begin{array}{l} Ce_i = x \\ C^{-1}x_i = e \end{array} \right.$$

$$= CDe_i$$

$$= C\lambda_i e_i$$

$$= \lambda_i Ce_i$$

$$= \lambda_i x_i$$

So $\{x_1, x_2, \ldots, x_n\}$ are eigen-vectors of A

Since C is invertible $\{x_1, x_2, \ldots, x_n\}$ is L.I

Thus A has a complete set of eigenvectors

Localization

At times one is only interested in a rough estimate of some or all eigen-values of a matrix B. Even if one eventually intends to calculate the eigenvalues one may have to start with some information about their approximate location.

Such information is provided by localization theorems which describe regions in the complex plane in which the eigenvalues are known to lie.

Recall if $\|\ \|$ is a norm in $\mathbb{R}^n$ and $\|A\|$ corresponding matrix norm of a matrix A

then $\|Ax\| \leq \|A\| \|x\|$.

Prop^n:- Let A be a $n \times n$-matrix. Let $\|\ \|$ be a norm in $\mathbb{R}^n$ and consider the associated matrix norm

Then $|\lambda| \leq \|A\|$

Proof λ is an eigenvalue of A

So $\exists\ x \neq 0$ s.t- $Ax = \lambda x$

Then

$$|\lambda| \|x\| = \|\lambda x\|$$
$$= \|Ax\|$$
$$\leq \|A\| \|x\|$$

As $\|x\| \neq 0$

we get- $|\lambda| \leq \|A\|$.

Examples

1. $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$\|A\|_\infty = 3$$

So if λ is an any eigenvalue of A then $|\lambda| \leq 3$

(Infact eigenvalues of A are $3, -1$)

2) $A = \begin{bmatrix} 0.6 & 0.7 \\ 0.3 & 0.2 \end{bmatrix}$

Note $\|A\|_\infty = \max\{0.6+0.7, 0.3+0.2\} = 1.3$

$\|A\|_1 = \max\{0.6+0.3, 0.7+0.2\} = 0.9$

Note that in prop^n any matrix norm can be used.

So $|\lambda| \leq 0.9$ for any eigenvalue of A.

Gershgorin's discs

Let $A = (a_{ij})$ be a $n \times n$-matrix

Consider

$$R_i = \left\{ z \in \mathbb{C} \;\middle|\; |z - a_{ii}| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \right\}$$

Theorem: Let λ be an eigen-value of A then $\lambda \in R_i$ for some $i = 1, 2, \ldots, n$.

Proof Suppose $z \notin R_i$ for each i

Consider the matrix $B = zI - A$

Note that

$$|b_{ii}| = |z - a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| = \sum_{\substack{j=1 \\ j \neq i}}^{n} |b_{ij}|$$

$\Rightarrow$ B is strictly diagonally dominant matrix.

Recall that strictly diagonally dominant

matrices are invertible.

$\Rightarrow$ z is not an eigen-value of A $\quad *$

Thus if λ is an eigenvalue of A then

$\lambda \in R_i$ for some i.

Remark

The disc's $R_1, R_2, \ldots, R_n$ in complex plane are called Gershgorin's disk's corresponding to the matrix A.

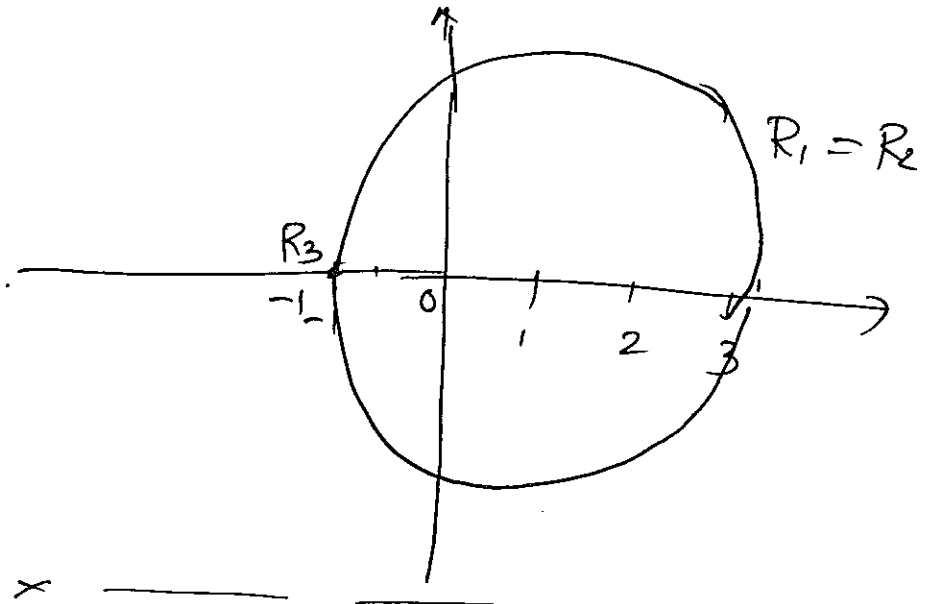
Example $\quad A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$R_1 = \{ z \mid |z-1| \leq 2 \}$$

$$R_2 = R_1$$

$$R_3 = \{ z \mid |z+1| \leq 0 \}$$

$$= \{-1\}.$$



Extension of Gerschgorin's theorem

If p Gerschgorin disk's form a set S that is disjoint from the n-p other discs of a given matrix A, then S contains precisely p eigenvalues of A (counted with multiplicity)

Proof Too advanced for this class

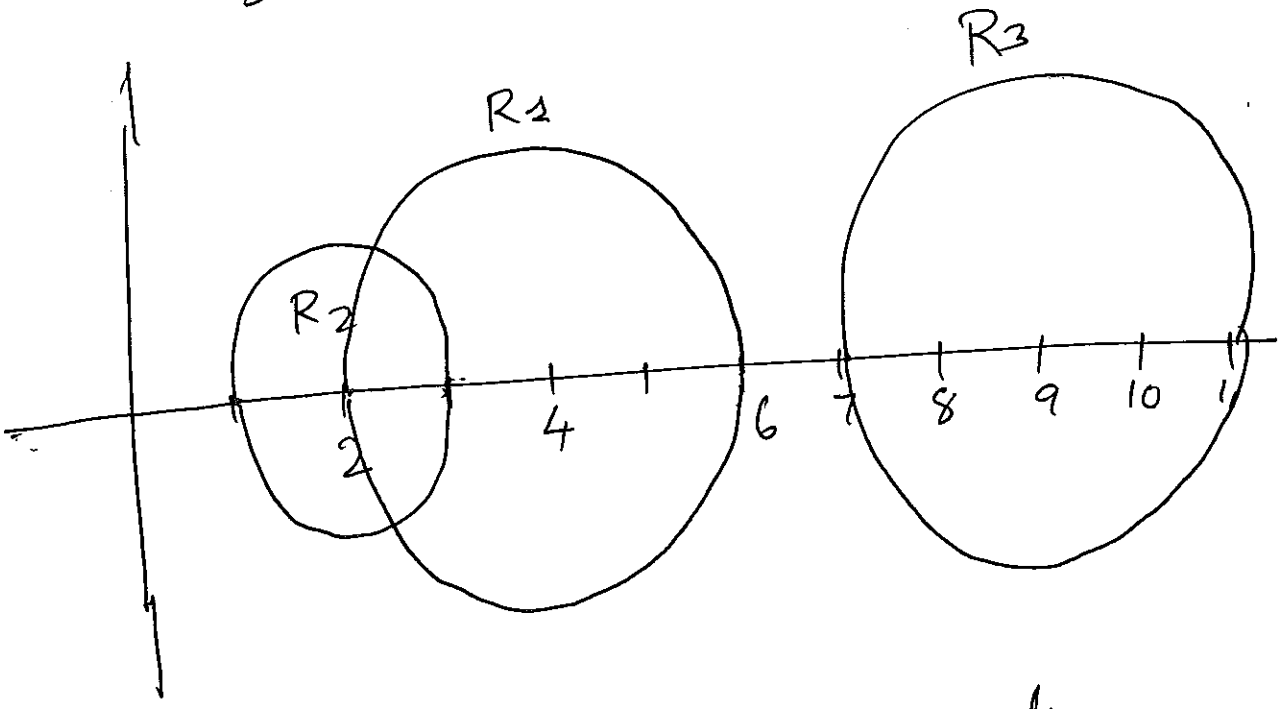
Example

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 0 & 2 & 1 \\ -2 & 0 & 9 \end{bmatrix}$$

$$R_1 = \{ z \in \mathbb{C} \mid |z-4| \leq 2 \}$$

$$R_2 = \{ z \in \mathbb{C} \mid |z-2| \leq 1 \}$$

$$R_3 = \{ z \in \mathbb{C} \mid |z-9| \leq 2 \}$$



$R_1 \cup R_2$ has 2 eigen-values

R_3 has 1 - eigen value