# Lecture 2

## Recall

1. Suppose $x^*$ is approximation to exact value $x$

Then

$|x - x^*|$ absolute error

$\frac{|x - x^*|}{|x|}$ relative error $(x \neq 0)$

2. $x^*$ is said to approximate $x$ to $t$ significant digits if

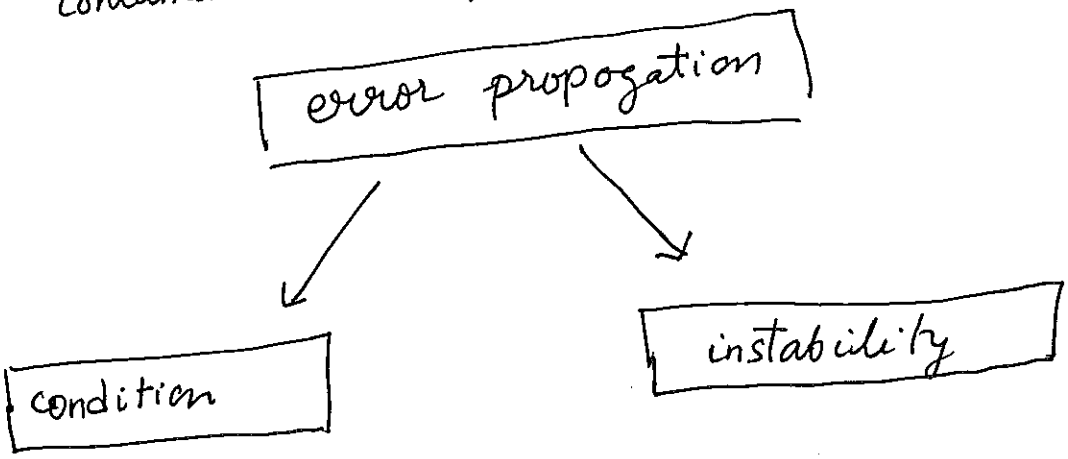
$$\frac{|x - x^*|}{|x|} \leq 5 \times 10^{-t}$$

3. Things which create loss in significant digits

a) Subtracting nearly equal quantities

b) Dividing by a very small number.

4. Once an error has been committed, it contaminates subsequent results




5. Condition → sensitivity of $f(x)$ to changes in $x$.

$$= \max \left\{ \frac{\left| \frac{f(x) - f(x^*)}{f(x)} \right|}{\left| \frac{x - x^*}{x} \right|} : |x - x^*| \text{ is small} \right\}$$

$$\approx \left| \frac{f'(x)\, x}{f(x)} \right|$$

Examples

(a). $f(x) = \sqrt{x}$ is well conditioned

(b) $f(x) = \frac{10}{1-x^2}$ is ill-conditioned near 1.

(c) 5. I gave an example

where $f = f_1 \circ f_2 \cdot f_3 \circ f_4$

with f well conditioned

but $f_3$ ill-conditioned

This also creates lot of error.

6. Last time I gave a spectacular example of instability.

Today we first discuss

Polynomials

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

This is called power form.

This may lead to loss of significant digits

Example (we work with 4 sig digits)

Suppose $p$ is a straight line such that

$p(6000) = \frac{1}{3}$ $\quad p(6001) = \frac{-2}{3}$

$p(x) = 6000 - x$ in 4 sig digits

This gives $p(6000) = 0$ $\quad p(6001) = -1$

Remedy (shift of centre)

$$p(x) = 3.333\,E{-}1 - (x - 6000)$$

$$p(6000) = 3.333\,E{-}1$$

$$p(6001) = -6.667\,E{-}1$$

# Nested form of a polynomial

$$p(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n$$

Computing directly requires more multiplication and additions.

Better to compute it as

$$p(x) = a_0 + (x-c)\left\{ a_1 + a_2(x-c) + a_3(x-c)^2 + \cdots + a_n(x-c)^{n-1} \right\}$$

$$= a_0 + (x-c)\left\{ a_1 + (x-c)\left\{ a_2 + a_3(x-c) + \cdots \right\} \right\}$$

$$= a_0 + (x-c)\left\{ a_1 + (x-c)\left\{ a_2 + (x-c)\left\{ a_3 + \cdots \right\} \right\} \right\}$$

Bonus of nesting

preservation of significant digits

Example (4 sig digits)

$$p(x) = x^3 - 6.1x^2 + 3.2x + 1.5$$

$$p(4.71) = -14.26 \quad \text{(correct upto 4 sig digits)}$$

However if you directly compute

$$p(4.71) = (4.71)^3 - 6.1(4.71)^2 + 3.2(4.71) + 1.5$$

$$= -14.23 \quad \text{(so correct upto 3 sig digits)}$$

In nested form

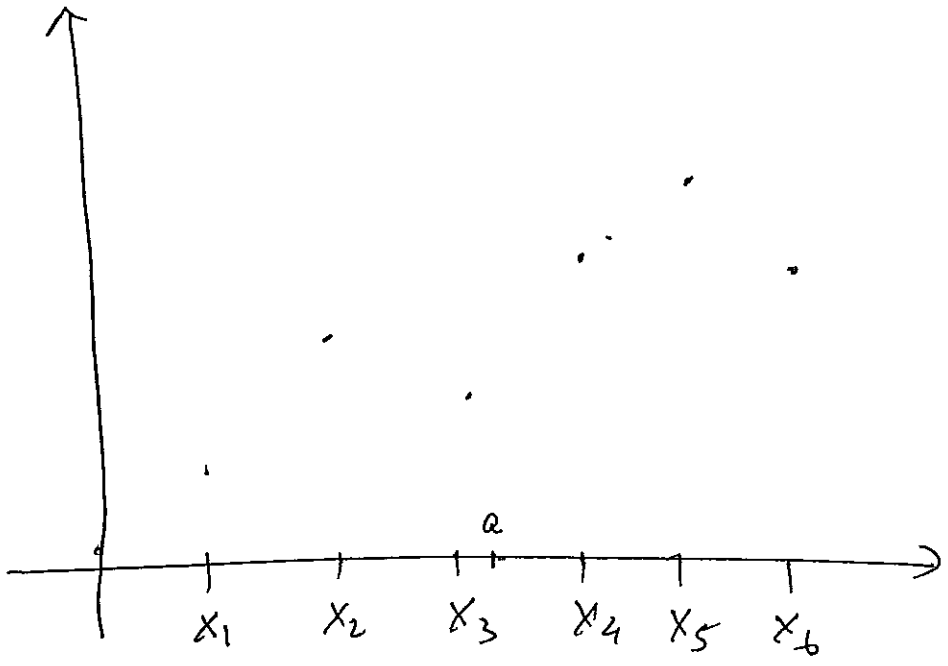
$$p(x) = x(x^2 - 6.1x + 3.2) + 1.5$$

$$= x(x(x - 6.1) + 3.2) + 1.5$$

$$p(4.71) \approx -14.26$$

We now discuss

"Interpolation".




We have $f(x_1), \ldots f(x_6)$.

We need to approximate $f(a)$

Idea :- fit a curve "passing through"

$(x_1, f(x_1)), (x_2, f(x_2)), \ldots, (x_n, f(x_n))$

and then approximate $f(a)$

Question :- Which curve to fit ?

One classical and non-trivial result is the following.

Theorem (Weierstrass Approximation theorem)

Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function. For each $\epsilon > 0$, there exists a polynomial $P(x)$ such that

$$|f(t) - P(t)| < \epsilon \quad \text{for all } t \text{ in } [a, b]$$

Remark :- The polynomial constructed for proving this theorem has slow convergence. So it is ineffective in practise

We might be tempted to use Taylor polynomials

Example $f(x) = \frac{1}{x}$ $\quad x \in [1, 4]$

$$f'(x) = -\frac{1}{x^2}$$

$$f''(x) = (-1)^2 \frac{2}{x^3}$$

$$f^{(k)}(x) = (-1)^k \frac{k!}{x^{k+1}}$$

So $n^{th}$ Taylor polynomial

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!} (x-1)^k$$

$$= \sum_{k=0}^{n} (-1)^k (x-1)^k$$

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| $T_n(3)$ | 3 | -5 | 11 | -21 | 43 | -85 |

## Problem

Given $n+1$ distinct points $x_0, x_1, \ldots, x_n$ in $[a,b]$ and a function $f: [a,b] \to \mathbb{R}$, does there exists a polynomial $p(x)$ of degree $\leq n$ which interpolates $f(x)$ at the points $x_0, x_1, \ldots, x_n$ i.e $p(x)$ satisfies

$$p(x_i) = f(x_i) \quad \text{for } i = 0, 1, \ldots, n$$

We prove that there exists a unique polynomial which does the job.

# Lagrange polynomials

Given $x_0, x_1, \cdots, x_n$ distinct points

$$l_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}$$

$k = 0, 1, \cdots, n$

## Example

$n = 2$ $\quad x_0, x_1, x_2$ distinct points

$$l_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

$$l_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

$$l_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

$x_0, x_1, \ldots, x_n$ $(n+1)$ distinct points

$$l_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i} \qquad k = 0, 1, \ldots, n$$

Properties.

1. $l_k(x)$ is a degree $n$-polynomial $\forall k = 0, 1, \ldots, n$

2. $l_k(x_k) = 1$

3. $l_k(x_i) = 0$ for $i \neq k$

$$P_n(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x)$$

Then . $P_n(x)$ is a polynomial of degree $\leq n$

. $P_n(x_i) = f(x_i)$ $\forall i = 0, 1, \ldots, n$.

Thus $P_n(x)$ is an interpolating polynomial.

**Proposition**

Let $p(x)$, $q(x)$ interpolate $f(x)$ in $x_0, \ldots, x_n$. Then $p(x) = q(x)$

**Proof** degree $p(x) \le n$

degree $q(x) \le n$

So $h(x) = p(x) - q(x)$ is a polynomial of degree $\le n$.

$$\left.\begin{aligned} h(x_i) &= p(x_i) - q(x_i) \\ &= f(x_i) - f(x_i) \\ &= 0 \end{aligned}\right\} \text{ for } i = 0, 1, \ldots, n$$

Thus $h(x)$ has $n+1$ zeros $x_0, \ldots, x_n$

Therefore $h(x) = 0$

Thus $p(x) = q(x)$.

Example

| $x$ | $f(x)$ |
|---|---|
| 2 | $6.931 \; E-1$ |
| 3 | $1.099$ |
| 4 | $1.386$ |

approximate $f(3.2)$

Ans $P_2(x) = f(2)\,l_0(x) + f(3)\,l_1(x) + f(4)\,l_2(x)$

interpolates $f(x)$

Recall

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

Thus $$l_0(3.2) = \frac{(3.2-3)(3.2-4)}{(2-3)(2-4)}$$

$$= -8 \; E-2$$

Similarly

$$l_1(3.2) = \frac{(3.2-2)(3.2-4)}{(3-2)(3-4)}$$

$$= 9.6 \; E-1$$

Also

$$l_2(3.2) = \frac{(3.2-2)(3.2-3)}{(4-2)(4-3)}$$

$$= 1.2\ E-1$$

$$f(3.2) \approx (6.931\ E-1)(-8\ E-2)$$

$$+ (9.6\ E-1)(1.099)$$

$$+ (1.386)(1.2\ E-1)$$

$$= 1.166$$

The function $f(x) = \log(x)$

$$f(3.2) = \log(3.2)$$

$$= 1.163$$

Thus our approximation is correct upto 3 significant digits

**Definition** $$P_n(x) = \sum_{k=0}^{n} f(x_k)\, \ell_k(x)$$

is called "Lagrange" form of interpolating polynomial

**Problem with Lagrange form of interpolating polynomial**

Suppose we have found $P_n(x)$ by interpolating $f(x)$ at points $x_0, x_1, \ldots, x_n$.

Suppose we also know $f(x_{n+1})$

Then we can form $P_{n+1}(x)$ by interpolating $f(x)$ at $x_0, x_1, \ldots, x_n, x_{n+1}$

There is no obvious relation between the Lagrange form of $P_n(x)$ and $P_{n+1}(x)$.

## Newton form of interpolating polynomial

We write

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1)$$
$$+ \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

Important remark $x - x_n$ does not appear in the last term above.

Set

$$q(x) = a_0 + a_1(x-x_0) + \cdots + a_{n-1}(x-x_0)\cdots(x-x_{n-2})$$

$$P_n(x) = q(x) + a_n(x-x_0)\cdots(x-x_{n-1})$$

Note $q(x_i) = P_n(x_i) = f(x_i)$

$i = 0, 1, \ldots, n-1$

Also $\deg q(x) \leq n-1$

By uniqueness of interpolating polynomial

$$q(x) = P_{n-1}(x)$$

So

$$P_n(x) = P_{n-1}(x) + a_n(x-x_0)\cdots(x-x_{n-1})$$

$$P_n(x) = P_{n-1}(x) + a_n (x-x_0) \cdots (x-x_{n-1}) \qquad (18)$$

Note $a_n$ = coefficient of $x^n$ in $P_n(x)$.

definition

$$f[x_0, x_1, \ldots, x_n] =: a_n$$

$f[x_0, x_1, \ldots, x_n]$ is called the $n^{th}$ divided difference of $f(x)$ at the points

$x_0, x_1, \ldots, x_n$.

We write

$$\begin{aligned} P_n(x) = f(x_0) &+ f[x_0, x_1](x-x_0) + \cancel{f[x_0, x_1,} \\ &+ f[x_0, x_1, x_2](x-x_0)(x-x_1) \\ &+ \cdots \\ &\vdots \\ &+ f[x_0, x_1, \ldots, x_n](x-x_0)\cdots(x-x_{n-1}) \end{aligned}$$

Thus to determine $P_n(x)$ we only have to find $f[x_0, x_1], f[x_0, x_1, x_2], \ldots, f[x_0, x_1, \ldots, x_n]$.

## Determining $f[x_0, x_1, \ldots, x_n]$

$$f[x_0] = f(x_0)$$

$$P_1(x) = f(x_0) + f[x_0, x_1](x - x_0)$$

$$P_1(x_1) = f(x_1)$$

$$f(x_1) = f(x_0) f[x_0, x_1] (x_1 - x_0)$$

$$\text{So } f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

## Proposition

$$f[x_0, x_1, \ldots, x_k] = \frac{f[x_1, \ldots, x_k] - f[x_0, \ldots, x_{k-1}]}{x_k - x_0}$$

## Proof :-

Let $P_i(x)$ = polynomial of degree $\leq i$ which agrees with $f(x)$ at the points $x_0, x_1, \ldots, x_i$

Let $q_{k-1}(x)$ = polynomial of degree $\leq k-1$ which agrees with $f(x)$ at the points $x_1, \ldots, x_k$.

Set

$$p(x) = \frac{x - x_0}{x_k - x_0} q_{k-1}(x) + \frac{x_k - x}{x_k - x_0} p_{k-1}(x)$$

degree $p(x) \le k$.

$$p(x_0) = f(x_0) \qquad p(x_k) = f(x_k)$$

$1 \le i \le k-1$

$$p(x_i) = \frac{x_i - x_0}{x_k - x_0} f(x_i) + \frac{x_k - x_i}{x_k - x_0} f(x_i)$$

$$= f(x_i).$$

By uniqueness of interpolating polynomial

$$p(x) = P_k(x)$$

$f[x_0, x_1, \ldots, x_k]$ = coeff of $x^k$ in $P_k(x)$

$$= \frac{\text{coeff of } x^k \text{ in } q_{k-1}(x)}{x_k - x_0} - \frac{\text{coeff of } x^{k-1} \text{ in } p_{k-1}(x)}{x_k - x_0}$$

$$= \frac{f[x_1, \ldots, x_k] - f[x_0, \ldots, x_{k-1}]}{x_k - x_0}$$

Table of divided differences

| $x$ | $f(x)$ | $f[,]$ | $f[,,]$ | $f[,,,]$ |
|---|---|---|---|---|
| $x_0$ | $f[x_0]$ | $f[x_0, x_1]$ | $f[x_0, x_1, x_2]$ | $f[x_0, x_1, x_2, x_3]$ |
| $x_1$ | $f(x_1)$ | $f[x_1, x_2]$ | $f[x_1, x_2, x_3]$ | |
| $x_2$ | $f(x_2)$ | $f[x_2, x_3]$ | | |
| $x_3$ | $f(x_3)$ | | | |

Example fill the table and approximate $f(3.2)$

| $x$ | $f(x)$ | $f[,]$ | $f[,,]$ | $f[,,,]$ |
|---|---|---|---|---|
| 2 | $6.931\ E-1$ | $4.059\ E-1$ | $-5.945\ E-2$ | $9.15\ E-3$ |
| 3 | $1.099$ | $2.87\ E-1$ | $-3.2\ E-2$ | |
| 4 | $1.386$ | $2.23\ E-1$ | | |
| 5 | $1.609$ | | | |

$$P_2(x) = f(2) + f[2,3](x-2) + f[2,3,4](x-2)(x-3)$$

$$= 6.931\ E-1 + 4.059\ E-1\,(x-2) + (-5.945\ E-2)(x-2)(x-3)$$

$$P_2(3.2) = 1.166$$

Exact value

$$f(3.2) = \log(3.2) = 1.163$$

$$P_3(x) = P_2(x) + 9.15\,E{-}13\,(x-2)(x-3)(x-4)$$

$$P_3(3.2) = 1.166 + 9.15\,E{-}3\,(1.2)(0.2)(-0.8)$$

$$= 1.164$$

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Example 2

$$f(x) = \int_0^x \sin(t^2)\,dt$$

| x | f(x) | f[,] | f[,,] | f[,,,] |
|---|---|---|---|---|
| 0.8 | 1.657 E-1 | 6.62 E-1 | 6.1 E-1 | -2.5 E-1 |
| 0.9 | 2.319 E-1 | 7.84 E-1 | 5.35 E-1 | |
| 1.0 | 3.103 E-1 | 8.91 E-1 | | |
| 1.1 | 3.994 E-1 | | | |

$$P_2(x) = f[0.8) + f[0.8,0.9](x-0.8) + f[0.8,0.9,1](x-0.8)(x-0.9)$$

$$= 1.657\,E{-}1 + 6.62\,E{-}1\,(x-0.8) + 6.1\,E{-}1\,(x-0.8)(x-0.9)$$

$$P_2(0.85) = 1.973\,E{-}1$$

$$f(0.85) = 1.974\,E{-}1$$

$$P_3(x) = P_2(x) + f[0.8, 0.9, 1.0, 1.1](x-0.8)(x-0.9)(x-1)$$

$$P_3(0.85) = P_2(0.85) - 2.5\,E{-}1\,(0.05)(-0.05)(-0.15)$$

$$= 1.973\,E{-}1$$

Remark 1) In the above example error of $P_3(x)$ was same as that of $P_2(x)$.

2) It is possible that interpolating error to increase if we increase number of points

error $e_n(x) = f(x) - P_n(x)$

i.e it is possible that

$$\max_{x\in[a,b]} |e_{n+1}(x)| > \max_{x\in[a,b]} |e_n(x)|$$

See example 2.4 pg 44 of your textbook Conte de Boor

# The error of the interpolating polynomial

$x_0, x_1, \ldots, x_n$ — $n+1$ distinct points

$f(x_0), f(x_1), \ldots, f(x_n)$ — function value

$f : [a,b] \longrightarrow \mathbb{R}$.

$P_n(x)$ = polynomial which interpolates $f(x)$ at $x_0, x_1, \ldots, x_n$

error $e_n(x) = f(x) - P_n(x)$

Let $\bar{x} \in [a,b]$ be distinct from $x_0, x_1, \ldots, x_n$

We need to estimate

$e_n(\bar{x})$.

Let $P_{n+1}(x)$ = polynomial which interpolates $f(x)$ at the points

$x_0, x_1, \ldots, x_n, \bar{x}$

($n+2$ points)

$$P_{n+1}(x) = P_n(x) + f[x_0, x_1, \ldots, x_n, \bar{x}] \prod_{j=0}^{n} (x - x_j)$$

$f(\bar{x}) = P_{n+1}(\bar{x})$ by definition

$$e_n(\bar{x}) = f(\bar{x}) - P_n(\bar{x})$$
$$= P_{n+1}(\bar{x}) - P_n(\bar{x})$$

$$\boxed{e_n(x) = f[x_0, x_1, \ldots, x_n, \bar{x}] \prod_{j=0}^{n} (\bar{x} - x_j)}$$

Thus to estimate error we need to approximate

a) $f[x_0, x_1, \ldots, x_n]$

b) $\prod_{j=0}^{n} (\bar{x} - x_j)$

Theorem: Let $f(x)$ be a continuous function in $[a, b]$ and $k$ times differentiable in $(a, b)$.

**Theorem** Let $f:[a,b] \to \mathbb{R}$ be continuous and $k$ times differentiable in $(a,b)$. If $x_0, \ldots, x_k$ are $k+1$ distinct pts in $[a,b]$ then there exists $\xi \in (a,b)$ such that

$$f[x_0, x_1, \ldots, x_k] = \frac{f^{(k)}(\xi)}{k!}$$

**Proof** By $k=1$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(\xi)$$ by M.V.T.

$$e_k(x) = f(x) - P_k(x)$$

has $(k+1)$ zeros $x_0, x_1, \ldots, x_k$

$e_k^{(k)}(x)$ will have a zero say $\xi$

$$0 = e_k^{(k)}(\xi) = f^{(k)}(\xi) - P_k^{(k)}(\xi)$$

$$0 = f^{(k)}(\xi) - k!\, f[x_0, \ldots x_k]$$

So $$f[x_0, x_1, \ldots x_k] = \frac{f^{(k)}(\xi)}{k!}$$

**Corollary**

$$e_n(\bar{x}) = f(\bar{x}) - P_n(\bar{x})$$

$$= \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (\bar{x} - x_j)$$

**Estimating**

$$\psi_{n+1}(x) = \prod_{j=0}^{n} (x - x_j)$$

It is possible to choose $x_0, x_1, \ldots, x_n$ in $[a,b]$ such that $|\psi_{n+1}(x)|$ is as small as possible.

This choice of pts are called chebyshev points of $[a,b]$.

(Unfortunately it is not in syllabus)

## Osculatory interpolation

Sometimes we have the following situation

We have

$x_0, x_1, \ldots, x_k$

$f(x_0), f(x_1), \ldots, f(x_k)$

$f'(x_0), f'(x_1), \ldots, f'(x_k)$

We need a polynomial $p(x)$ such that

$$\left.\begin{aligned} p(x_i) &= f(x_i) \\ p'(x_i) &= f'(x_i) \end{aligned}\right\} \; i = 0, 1, \ldots, n$$

Note $\deg p(x) \leq 2n+1$

Example where this situation arises

$$\frac{dy}{dx} = g(x, y)$$

$$y(x_0) = y_0$$

| | | |
|---|---|---|
| $x_0$ | $y(x_0)$ | $y'(x_0) = g(x_0, y_0)$ |
| $x_1$ | $y(x_1)$ | $y'(x_1) = g(x_1, y_1)$ |
| ⋮ | | |
| $x_n$ | $y(x_n)$ | $y'(x_n)$ |

Remark
$y(x_i)$ is calculated by some Numerical method

## Remark

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$\lim_{x_1 \to x_0} f[x_0, x_1] = f'(x_0)$$

definition $\quad f[x_0, x_0] = f'(x_0)$

## Example

$f(1) = 0 \qquad f'(1) = 1$

$f(2) = 6.931\ E{-}1 \qquad f'(2) = 0.5$

We need cubic polynomial $P_3(x)$ such that

$P_3(1) = f(1) \qquad P_3'(1) = f'(1)$

$P_3(2) = f(2) \qquad P_3'(2) = f'(2)$

| $y$ | $f(\ )$ | $f[\ ,\ ]$ | $f[\ ,\ ,\ ]$ | $f[\ ,\ ,\ ,\ ]$ |
|---|---|---|---|---|
| $y_0 = 1$ | 0 | 1 | $-0.3069$ | 0.1137 |
| $y_1 = 1$ | 0 | $6.931\ E{-}1$ | $-0.1931$ | |
| $y_2 = 2$ | $6.931\ E{-}1$ | 0.5 | | |
| $y_3 = 2$ | $6.931\ E{-}1$ | | | |

$$P_3(x) = 0 + 1(x-1) - 0.3069(x-1)^2 + 0.1137(x-1)^2(x-2)$$