Lect 22

QR factorization

$A \in M_n(\mathbb{R})$ invertible

QR-factorization $A = QR$ Q orthogonal R upper triang.

Two methods

Gram-Schmidt orthog. process

$A = [a_1, \dots, a_n]$ inv

$u_1 = a_1$ $\quad e_1 = \frac{u_1}{\|u_1\|}$

$u_2 = a_2 - \langle e_1, a_2 \rangle e_1$ $\quad e_2 = \frac{u_2}{\|u_2\|}$

$u_k = a_k - \sum_{j=1}^{k-1} \langle e_j, a_k \rangle e_j$ $\quad e_k = \frac{u_k}{\|u_k\|}$

$a_1 = \langle e_1, a_1 \rangle e_1$

$a_2 = \langle e_1, a_2 \rangle e_1 + \langle e_2, a_2 \rangle e_2$

$a_k = \sum_{j=1}^{k} \langle e_j, a_k \rangle e_j$

$$A = QR = [e_1 \dots e_n] \begin{bmatrix} \langle e_1, a_1 \rangle & \langle e_1, a_2 \rangle & \\ & \langle e_2, a_2 \rangle & \\ & & \end{bmatrix}$$

$$R = \begin{bmatrix} \langle e_1, a_1 \rangle & \langle e_1, a_2 \rangle & \langle e_1, a_3 \rangle & \cdots & \langle e_1, a_n \rangle \\ 0 & \langle e_2, a_2 \rangle & \langle e_2, a_3 \rangle & \cdots & \langle e_2, a_n \rangle \\ 0 & 0 & \langle e_3, a_3 \rangle & & \\ 0 & & 0 & & \\ \vdots & & & & \\ 0 & 0 & 0 & & \langle e_n, a_n \rangle \end{bmatrix}$$

Ex

$$A = \begin{bmatrix} 12 & -51 & 4 \\ 6 & 167 & -68 \\ -4 & 24 & -41 \end{bmatrix}$$

$$Q = \begin{bmatrix} 6/7 & -69/175 & -58/175 \\ 3/7 & 158/175 & 6/175 \\ -2/7 & 6/35 & -33/35 \end{bmatrix}$$

Adv Easy to compu

Dis Not numeri stable

$$A = QR$$

$$Q^t A = Q^t Q R = R = \begin{pmatrix} 14 & 21 & -14 \\ 0 & 175 & -70 \\ 0 & 0 & 35 \end{pmatrix}$$

~~Householder technique~~

QR factorizations using Householder reflection

A
x first column

$\|x\| = |\alpha|$

$\operatorname{sgn}\alpha = -\operatorname{sgn} x_1$

$u = x - \alpha e_1$

$v = \frac{u}{\|u\|}$ if $u \neq 0$

$Q = I - 2vv^t$

$u^t x$ scalar

$Qx = x - \frac{2uu^t x}{\|u\|^2}$

$= x - \frac{2(x-\alpha e_1)u^t x}{\|u\|^2}$

$= \left(1 - \frac{2u^t x}{\|u\|^2}\right)x + \frac{2\alpha e_1 u^t x}{\|u\|^2}$

$\|u\|^2 = x^t x - 2\alpha x_1 + \alpha^2$

$= 2\alpha^2 - 2\alpha x_1$

$u^t x = x^t x - \alpha x_1 = \alpha^2 - \alpha x_1$

$Qx =$

first ter $= \left(\frac{2\alpha^2 - 2\alpha x_1 - 2\alpha^2 + 2\alpha x_1}{\|u\|^2}\right) x_1$

$= 0$

first term = 0

second term of $Q_1 x$

$$= \begin{bmatrix} 2\alpha e_1 (\alpha^2 - \alpha x_1) \\ \hline 2\alpha^2 - 2\alpha x_1 \end{bmatrix} e_1$$

$$= \frac{\alpha(\alpha - x_1)}{(\alpha - x_1)} e_1$$

$$= \alpha e_1$$

α same sign as x_1 ← (or of round of err)

if ~~$\|x_1\| >$~~ $\|x\| > |x_1|$ then α opp sign as x_1

A column does not

$$A = \begin{pmatrix} 12 & -51 & 4 \\ 6 & 167 & -68 \\ -4 & 24 & -41 \end{pmatrix}$$

$\|x_1\| = 14$

$\alpha = \pm 14$

$u = x - 14 e_1$

$= 2[-1, 3, -2]$

$v = \frac{1}{\sqrt{14}}[-1, 3, -2]$

$Q_1 = I_1 - 2vv^t$

$$Q_1 A = \begin{pmatrix} 14 & 21 & -14 \\ 0 & -49 & -14 \\ 0 & 168 & -77 \end{pmatrix}$$ ← iterate by process

$$A = QR$$

$$A_1 = RQ$$

$$\Rightarrow Q A_1 Q^{-1} = QRQQ^{-1}$$
$$= A$$

$$A_1 = Q^{-1} A Q$$

$$Q =$$

Data fitting

$y = f(x)$

$$f(x_n) = f_n + \varepsilon_n \qquad n = 1, \dots, N$$

(f_n: approx value; ε_n: errors)

N is large.

Task: to approximate f by F s.t.

F contain most of inf of f and little if any of the error.

$$F(x) = F(x, c_1, \dots, c_n)$$

$$F(x) = c_1 \phi_1(x) + c_2 \phi_2(x) + \dots + c_k \phi_k(x)$$

normally k is small compared to N

ϕ_i known apriori

$$d_n = f_n - F(x_n, c_1, \dots, c_k)$$

want to make d_n simult. as small as possible

Minim $\|d\|$

(a) the 1-norm
$$\|d\|_1 = \sum_{i=1}^{N} |d_i|$$

the ∞-norm
$$\|d\|_\infty = \max_{1 \le n \le N} |d_n|$$

These two lead into sys of non-linear system of eq-

Use $\|d\|_2 = \left(\sum |d_n|^2 \right)^{1/2}$

Least squares approximation

min $\|d\|_2$ $\longrightarrow$ same as $\min \|d\|_2^2$

$$E(c) = \sum_{n=1}^{N} \left[f_n - F(x_n, c) \right]^2$$

$$\nabla E(c^*) = 0$$

$$\frac{\partial}{\partial c_i} \left[f_n - F(x_n, c) \right] = -\phi_i(x_n)$$

$$-2 \sum_{n=1}^{N} \left(f_n - F(x_n, c^*) \right] \phi_i(x_n) = 0 \qquad i = 1, \dots k$$

$$\underline{\phi}_i = \left[\phi_i(x_1), \quad , \phi_i(x_N) \right]^T$$

$$e = [e_1, \dots, e_N]^T \qquad e_n = f_n - F(x_n, c^*)$$

~~$-2 e^T \phi_i$~~

~~$F(x) = \sum c_i \phi_i$~~

~~$e^T (c_1 \phi_1 \dots c_k \phi_k) = 0$~~

$$\sum_{n=1}^{N} f_n \phi_i(x_n) = \sum_{n=1}^{N} F(x_n, c^*) \phi_i(x_n)$$

$$f^T \phi_i = \sum_{j=1}^{k} c_j^* \phi_j^T \phi_i \qquad i = 1, \dots k$$

x_n	f_n
1	0
2	0.6
3	1.77
4	1.82
5	3.31
6	3.52
7	4.59
8	5.31
9	5.79
10	7.06
11	7.17

$\phi_1(x) = 1$

$\phi_2(x) = x$

$F = c_1 + c_2 x \qquad k = 2$

$11 c_1^* + 66 c_2^* = 41.04$

$66 c_1^* + 506 c_2^* = 328.05$

$c_1^* = -0.7314$

$c_2^* = 0.747$

Example $x_n = 10 + \frac{n-1}{5} \quad n = 1, \dots, 6$

$\phi_1(x) = 1 \qquad \phi_2(x) = x \qquad \phi_3(x) = x^2$

$$A = \begin{bmatrix} 6 & 63 & 662.2 \\ 63 & 662.2 & 6967.8 \\ 662.2 & 6967.8 & 73393.5664 \end{bmatrix}$$

$\|A\|_\infty \cong 8 \cdot 10^4$

$$x = \begin{bmatrix} 10.07 \\ -2 \\ 0.099 \end{bmatrix} \qquad Ax = \begin{bmatrix} -0.02 \\ -0.18 \\ -1.28 \end{bmatrix}$$

$\|A^{-1}\|_\infty \geq 78$

$\|Ax\| \geq \frac{\|x\|}{\|A^{-1}\|_\infty}$

$$cond(A) = \|A\|_\infty \|A^{-1}\|_\infty \geq 10^5$$

$$f(x) = 10 - 2x + \frac{x^2}{10}$$

$f_n = f(x_n)$

we should go for seven digit floating arithmetic

$c_1^* = 8.492$

$c_2^* = -1.712$

$c_3^* = 0.0863$

A possible sol^n

Choose $x_1 \ldots x_N$ orthog. w.r.t $\phi_1 \ldots \phi_K$

$$\phi_j^T \phi_i = \sum_{n=1}^{N} \phi_i(x_n)\phi_j(x_n) = 0 \qquad i \neq j$$

$$c_i^* \phi_i^T \phi_i = f^T \phi_i$$

How to comb orthogonal functions?

Orthogonal polynomials

$[a,b]$

$w : (a,b) \longrightarrow \mathbb{R} > 0$

$$\langle g, h \rangle = \int_a^b g(x) h(x) w(x)\, dx$$

$$\langle g, h \rangle = \sum_{n=1}^{N} g(x_n) h(x_n) w(x_n)$$

$g = 1, \quad h = x$

$$\langle g, h \rangle = \int_{-1}^{1} g h\, dx = 0$$

$$\langle g, h \rangle = \sum_{n=-10}^{10} g(n) h(n) = 0$$

$$\langle g, h \rangle = \int_{-1}^{1} \frac{gh}{\sqrt{1-x^2}}\, dx$$

$[a,b] = [0, 2\pi]$

$g(x) = \sin mx \qquad m \neq n$

$h(x) = \sin nx$

$$\int_0^{2\pi} gh = 0$$

$P_0(x), P_1(x), \ldots, P_k(x), \ldots$

$\deg P_i = i$

(1) $P_i(x_i) = a_i x^i +$ poly of deg $< i$

$a_i \neq 0$

(2) $i \neq j \quad \langle P_i, P_j \rangle = 0$

$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = 3x^2 - x$

$$\langle u, v \rangle = \int_{-1}^{1} uv \, dx$$

Prop

(1) $p(x)$ poly deg $p \leq k$

$p(x) = d_0 P_0(x) + \cdots + d_k P_k(x)$

d_i uniq det by p

(2) $\deg p < k$

$\Rightarrow \langle p, P_k \rangle = 0$

$$\langle g, h \rangle = \int_a^b g h \, w \, dx$$

(3) P_k has k simple zeros in (a, b) $\quad k > 0$

$P_k(x) = a_k (x - \xi_{1,k})(x - \xi_{2,k}) \cdots (x - \xi_{k,k})$

$\xi_{1,k}, \ldots, \xi_{k,k}$ distinct in (a, b)

$k > 0$

$\xi_{1,k}, \ldots, \xi_{n,k}$ be pts in (a, b) where $P_k(x)$ change sign

$n \geq k$

$\bar{x} \in (\max \xi_{1,k}, b)$

$P(x) = P_k(\bar{x})\,(x-\xi_{1,k}) \cdots (x-\xi_{n,k})$

$\int_a^b p(x)\,P_k(x)\,\omega(x) = \langle P, P_k \rangle \Rightarrow 0$

$p(x)\,P_k(x)\,\omega(x) > 0$

$\forall\, x \in (a,b)$ except at $\xi_1 \cdots \xi_n$

$r \geq k$

P_k poly of deg k

so $r = k$ $\quad \xi_1 \cdots \xi_k$ are zeros of P_k

Note if P_k has no zero in (a,b)

~~$u = f(a$~~

$u = f\left(\frac{a+b}{2}\right)$

$\langle u, P_k \rangle = \int u P_k \omega\, dx > 0$ ✳

so P_k has a zero in (a,b)

$\xi_1 \cdots \xi_r$ ~~$r < k$~~ zeros of P_k in (a,b)

$r < b$