Lect 23

Three term recurrence relation

$$P_{-1}(x) = 0$$

$$S_i = \langle P_i, P_i \rangle \neq 0 \qquad i = 0, \dots, k-1$$

$$P_{i+1}(x) = (x - B_i) P_i(x) - C_i P_{i-1}(x)$$

$$i = 0, 1, \dots, k-1$$

$$B_i = \frac{\langle x P_i(x), P_i(x) \rangle}{S_i} \qquad i = 1, \dots, k-1$$

$$C_i = \begin{cases} 0 & i = 0 \\ \frac{S_i}{S_{i-1}} & i > 0 \end{cases}$$

Example $C[-1,1]$

$$\langle g, h \rangle = \int_{-1}^{1} g h \, dx$$

$$P_0(x) = 1$$

$$S_0 = \int_{-1}^{1} 1 \, dx = 2$$

$$B_0 = \frac{\langle x, 1 \rangle}{S_0} = 0$$

$$C_0 = 0$$

$$P_1 = (x - 0) 1 - 0 = x$$

$$S_1 = \int_{-1}^{1} x^2 dx = \frac{x^3}{3} \Big|_{-1}^{1} = \frac{2}{3}$$

$$B_1 = \frac{\langle x P_1, P_1 \rangle}{S_1} = \frac{1}{S_1} \int_{-1}^{1} x^2 \cdot x = 0$$

$$c_1 = \frac{s_1}{s_0} = \frac{1}{3}$$

$$P_2(x) = x^2 - \frac{1}{3}$$

$$s_2 = \int_{-1}^{1} (x^2 - \tfrac{1}{3})^2 dx = \frac{8}{45}$$

$$B_2 = \frac{1}{s_2} \int_{-1}^{1} x (x^2 - \tfrac{1}{3})^2 = 0$$

$$c_2 = \frac{s_2}{s_1}$$

$$= \frac{8}{45} \cdot \frac{3}{2} = \frac{4}{15}$$

$$P_3(x) = x P_2(x) - \frac{4}{15} P_1(x)$$

$$= x^3 - \frac{3}{5} x$$

Normal $P_k(1) = 1$

$A_k = \frac{2k+1}{k+1}$ $\quad B_k = 0$

$c_k = \frac{k}{k+1}$ $\quad k = 0, 1, 2$

$$P_{k+1}(x) = \frac{(2k+1) x P_k(x) - k P_{k-1}(x)}{k+1}$$

$$s_i = \langle P_i, P_0 \rangle = \frac{2}{2i+1}$$

$f : [a,b] \longrightarrow \mathbb{R}$ $\quad \omega : (a,b) \to \mathbb{R}_{>0}$

want to approximate f on $(a,b]$ by a polynomial of degree $\leq k$

$$\langle f-P, f-P\rangle = \begin{cases} \int_a^b (f-P)^2 \omega(x)\,dx \\ \sum_{n=1}^{N} \left(f(x_n)-P(x_n)\right)^2 \omega(x) \end{cases}$$

P such that $\|f-P\| = \sqrt{\langle f-P, f-P\rangle}$ is as small as possible

$$P(x) = d_0 P_0 + \cdots + d_k P_k$$

$$E = \langle f-P, f-P\rangle = \langle f - \textstyle\sum d_i P_i \,,\, f - \sum d_i P_i\rangle$$

$$\frac{\partial E}{\partial d_j} = \langle -P_j, f - \textstyle\sum d_i P_i\rangle + \langle f - \sum d_i P_i, -P_j\rangle$$

$$= \langle -P_j, f\rangle + d_j \langle P_j, P_j\rangle - \cancel{\langle P_i, f\rangle} \langle f, -P_j\rangle + \langle P_i, P_i\rangle d_j$$

$$d_j \langle P_i, P_i\rangle = \langle f, P_i\rangle$$

$$d_j = \frac{\langle f, P_i\rangle}{\langle P_i, P_i\rangle} \quad \text{if } S_i = \langle P_i, P_i\rangle \neq 0$$

Example poly p of deg ≤ 3

which min $\int_{-1}^{1} [e^x - p(x)]^2 dx$

$f(x) = e^x$

$\langle f, P_0 \rangle = \int_{-1}^{1} e^x dx = e - \frac{1}{e}$

$\langle f, P_1 \rangle = \int_{-1}^{1} e^x x \, dx = \frac{2}{e}$

$\langle f, P_2 \rangle = \frac{3}{2} \int_{-1}^{1} e^x \left(x^2 - \frac{1}{3}\right) dx = e - \frac{7}{e}$

$\langle f, P_3 \rangle = \frac{5}{2} \int_{-1}^{1} e^x \left(x^3 - \frac{3}{5}x\right) dx = -5e + \frac{37}{e}$

$e = 2.71828183$

$S_0 = 2$

$S_1 = \frac{2}{3}$

$S_2 = \frac{2}{5}$

$S_3 = \frac{3}{7}$

$p^*(x) = 1.1752 P_0 + 1.1036 P_1 + 0.3578 P_2 + 0.0704 P_3$

$|p(x) - e^x| \leq 0.011$

$p(x) \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{3}$

$p(x) = 0.996 + 0.9979 x + 0.5367 x^2 + 0.17613 x^3$

Algorithm

$x_1, \dots, x_N$ N distinct pts

orth. poly.

$$\langle h, g \rangle = \sum_{n=1}^{N} g(x_n)\, h(x_n)\, w(x_n)$$

$w : (a,b) \to \mathbb{R} > 0$

set $P_0 = 1$

$$\delta_0 = \langle P_0, P_0 \rangle = \sum_{n=1}^{N} w(x_n) > 0$$

$$\beta_0 = \frac{\langle x P_0, P_0 \rangle}{\delta_0} = \sum_{n=1}^{N} \frac{x_n\, w(x_n)}{\delta_0}$$

$$P_1(x) = (x - \beta_0) P_0 = x - \beta_0$$

$|P_j| = j \qquad j \le n \ll N$

$$\gamma_j = \langle P_j, P_j \rangle = \sum_{n=1}^{N} \left[P_j(x_n)\right]^2 w(x_n)$$

N dash. pts

$\gamma_j = 0 \Rightarrow P_j(x_n) = 0$

$\Rightarrow P_j = 0$ ✗

$$\beta_j = \frac{\langle x P_j, P_j \rangle}{\gamma_j}$$

$$\alpha_j = \frac{\gamma_j}{\gamma_{j-1}}$$

$$P_{j+1} = (x - \beta_j) P_j - \alpha_j P_{j-1}$$

Ex

$$f(x) = 10 - 2x + \frac{x^2}{10}$$

$$x_N = 10 + \frac{n-1}{5} \qquad f_n = f(x_n) \qquad n = 1, \ldots 6$$

$$\sum_{n=1}^{6} \left[f_n - P(x_n) \right]^2$$

$$P_0 = 1 \qquad S_0 = 6$$

$$B_0 = \frac{1}{6} \left[\sum 10 + \frac{n-1}{5} \right] / S_0 = 10.5$$

$$P_1(x) = x - 10.5$$

$$S_1 = 0.7$$

$$B_1 = 10.5$$

$$C_1 = \frac{0.7}{6} = 0.116667$$

$$P_2(x) = (x - 10.5)^2 - 0.1166667$$

$$S_2 = 0.059733332$$

$$d_0^* = \frac{\sum_{n=1}^{6} f_n}{6} = 0.03666667$$

$$d_1^* = 0.1$$

$$d_2^* = 0.099999$$

$$P_9^*(x) = d_0^* P_0 + d_1^* P_1 + d_2 P_2$$

$$= 9.99998 - 1.999998\, x + 0.099999\, x^2$$

Previous

$C_1^* = 8492$

$C_2^* = -1.712$

$C_3^* = 0.0863$

Approximation by trig poly.

many phenoms such as light & sound have periodic character

$$f(x+\tau) = f(x) \quad \forall x$$

τ of $x+\tau$ ↑ period of

$$g(x) = f\left(\frac{\tau x}{2\pi}\right)$$

$$g(x+2\pi) = f\left(\frac{\tau(x+2\pi)}{2\pi}\right)$$

$$= f\left(\frac{\tau x}{2\pi} + \tau\right)$$

$$= f\left(\frac{\tau x}{2\pi}\right)$$

$$= g(x)$$

g periodic with period 2π

Trig poly of order n

$$p(x) = \frac{a_0}{2} + \sum_{j=1}^{n} (a_j \cos jx + b_j \sin jx)$$

more conve

$$p(x) = \sum_{j=-n}^{n} c_j e^{ijx} \qquad i = \sqrt{-1}$$

$$e^{ix} = \cos x + i \sin x$$

$$\sum_{j=-n}^{n} c_j e^{ijx} = \sum_{j=-n}^{n} c_n (\cos jx + i \sin jx)$$

$$= c_0 + \sum_{0=1}^{n} (c_j + c_j) \cos jx + i(c_j - c_j) \sin jx$$

$$a_j = c_j + c_{-j}$$
$$b_j = i(c_j - c_{-j})$$

If f is real

$$\sum_{j=-n}^{n} c_j e^{ijx} = \sum_{j=-n}^{n} \bar{c}_j e^{-ijx} = \sum_{j=-n}^{n} \bar{c}_{-j} e^{ijx}$$
$$= p(x)$$

so $c_j = \bar{c}_{-j} \quad \forall j$

$$a_j = 2 \operatorname{Re} c_j$$
$$b_j = -2 \operatorname{Im} c_j$$

Fourier series $\sum_{j=-\infty}^{\infty} \hat{f}(j) e^{ijx} =$

$$f(x) \sim \sum_{j=-\infty}^{\infty} \hat{f}(j) e^{ijx}$$

$$\hat{f}(j) = \frac{1}{2\pi} \int_0^{2\pi} f e^{-ijx} \, dx$$

$$\sum_{-\infty}^{\infty} \hat{f}(j) e^{ijx} \longrightarrow f(x) \quad \text{unif if}$$

f is cts

and f' is piecewise continuous

$$\int_0^{2\pi} e^{ijx}\overline{e^{ikx}}\,dx = \begin{cases} 2\pi & k=j \\ 0 & k\neq j \end{cases}$$

$1, e^{\pi i x}, e^{\pi i 2x}, \ldots$

$$\langle g,h\rangle = \frac{1}{2\pi}\int_0^{2\pi} h(x)\,g(x)\,dx$$

$$\langle e^{ijx}, e^{ikx}\rangle = \begin{cases} 1 & k=j \\ 0 & k\neq j \end{cases}$$

Thm $\sum_{j=-n}^{n} \hat{f}(j)\,e^{ijx}$ is best approx of $f(x)$ by trig poly of order n w.r.t $\|g\| = \|g\|_2 = \sqrt{\frac{1}{2\pi}\int_0^{2\pi}|g|^2\,dx}$

Pf $\left\langle f - \sum_{j=-n}^{n} d_j e^{ijx},\ f - \sum_{j=-n}^{n} d_j e^{ijx}\right\rangle = E$

$$\frac{\partial E}{\partial d_j} = \left\langle 0 - d_j e^{ijx},\ f - \sum_{j=-n}^{n} d_j e^{ijx}\right\rangle + \left\langle f - \sum_{j=-n}^{n} d_j e^{ijx},\ -d_j e^{ijx}\right\rangle$$

$$= -\overline{d_j}\langle f, e^{ijx}\rangle + |d_j|^2 - d_j\langle f, e^{ijx}\rangle$$

$$= -\overline{d_j}\langle e^{ijx}, f\rangle + |d_j|^2 + |d_j|^2$$

d_j real

$$d_j = 2\,\mathrm{Re}\left\langle \frac{1}{2\pi}\int e^{ijx} f\,dx\right\rangle$$