Lect 24

Fourier series

function 2π periodic

Interval $[-\pi, \pi]$

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f g \, dx$$

$1, \cos mx, \sin nx \qquad m, n \geq 1$

$$\int_{-\pi}^{\pi} \cos mx = + \frac{\sin mx}{m} \Big|_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi} \sin mx = \frac{1}{m} \cos mx \Big|_{-\pi}^{\pi}$$

$$= \frac{1}{m} \left[(-1)^m - (-1)^m \right] = 0$$

Ex show $\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0$

Show $\int_{-\pi}^{\pi} \cos mx \cos lx \, dx = 0 \quad$ if $m \neq l$

Show $\int_{-\pi}^{\pi} \sin mx \sin lx \, dx = 0 \quad$ if $n \neq l$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2 mx \, dx = 1 = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 mx \, dx$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} \, dx = \frac{2}{2}$$

$\left\{ \frac{1}{\sqrt{2}}, \cos mx, \sin nx \right\}$ are othonormal in $[-\pi, \pi]$

Trig poly of deg r

$$g_r = \frac{a_0}{2} + \sum_{n=1}^{r} a_n \cos nx + \sum_{m=1}^{r} \sin mx$$

Want a, b

s.t. $\|f - g_r\|_2$ as small as possible

$$E = \|f - g_r\|_2^2 = \langle f - g_r, f - g_r \rangle$$

$$0 = \frac{\partial E}{\partial a_0} = \langle \tfrac{f}{\pi} - \tfrac{1}{2}, f - g_r \rangle + \langle f - g_r, \tfrac{f}{\pi} - \tfrac{1}{2} \rangle$$

$$= -\frac{1}{2}\langle 1, f \rangle + \langle \tfrac{1}{2}, \tfrac{a_0}{2} \rangle - \frac{1}{2}\langle f, 1 \rangle + \langle \tfrac{1}{2}, \tfrac{a_0}{2} \rangle$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f = \langle 1, f \rangle = 2\,\frac{1}{4}\langle 1, a_0 \rangle = \frac{1}{2}\int_{-\pi}^{\pi} a_0\, dx = a_0$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$n \geq 1$

$$0 = \frac{\partial E}{\partial a_n} = \frac{\partial}{\partial a_n}\langle f - g_r, f - g_r \rangle$$

$$= \langle -\cos nx, f - g_r \rangle + \langle f - g_r, -\cos nx \rangle$$

$$= -2\langle \cos nx, f \rangle + 2\langle \cos nx, \cos nx \rangle a_n$$

$$a_n = \langle \cos nx, f \rangle$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(x)\, f(x)\, dx$$

$m \geq 1$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(mx)\, f(x)\, dx$$

Convergence

$f_n : [a,b] \to \mathbb{R}$ $\quad f : [a,b] \to \mathbb{R}$

1) $f_n \to f$ pointwise in $[a,b]$
if $\lim_{n\to\infty} f_n(x) = f(x) \quad \forall\, x \in [a,b]$

2) $f_n \to f$ uniformly (or via ∞-norm)
if given $\varepsilon > 0$, $\exists\, n_0$ s.t.
$n \geq n_0 \quad \|f_n - f\|_\infty < \varepsilon$
$\sup_{x\in[a,b]} |f_n(x) - f(x)| < \varepsilon$

3) If $\int_a^b f_n^2\, dx < \infty \qquad \int_a^b f_n^2\, dx < \infty$

$f_n \to f$ in 2-norm
if $\|f_n - f\|_2 \to 0$

$\int_a^b (f_n - f)^2\, dx \to 0$

Example

(1) $f_n(x) = x^n$ $\quad x \in [0,1]$
$0 \leq x < 1$

$\lim_{n\to\infty} x^n = 0$

$f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$

$x = 1 \qquad \lim_{n\to\infty} 1^n = 1$

$f_n \to f$

- $f_n \to f$ in L^2 norm

$$\int_0^1 |f_n(x) - f(x)|^2 \, dx = \int_0^1 x^{n2} \, dx$$

$$= \frac{x^{2n+1}}{2n+1} \Big|_0^1$$

$$= \frac{1}{2n+1} \to 0 \quad \text{as } n \to \infty$$

- $f_n \not\to f$ uniformly

$\varepsilon = \frac{1}{2}$

$\|f_n - f\|_{[0,1)}$ $\quad \therefore \|x^n - \| = |x^n| > \frac{1}{2}$

$x > \left(\frac{1}{2}\right)^{1/n}$

Ex $f_n \to f$ uniformly f_n, f int $\Rightarrow f_n \to f$ in L^2 norm

$\int_a^b |f_n - f|^2 \le (b-a) \|f_n - f\| \to 0$

Thm $f_n \to f$ uniformly $x \in [a,b]$

f_n continuous $\forall n \Rightarrow f$ is also continuous

Pf Let $x_0 \in [a,b]$ $\quad$ let $\varepsilon > 0$

$\|f_{n_0} - f\| < \varepsilon/3$

$\exists \delta > 0$ $\quad \|f_{n_0}(x) - f_{n_0}(x_0)\| < \varepsilon/3$

$x \in (x_0 - \delta, x_0 + \delta) \cap [a,b]$

$x \in (x_0 - \delta, x_0 + \delta) \cap [a,b]$

$|f(x) - f(x_0)|$

$\le |f(x) - f_{n_0}(x) + f_{n_0}(x) - f_{n_0}(x_0) + f_{n_0}(x_0) - f(x_0)|$

$\le |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x_0)|$

$\le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$

Thm

$f_n \to f$ unif f_n cts

$\Rightarrow \int_a^b f_n(x)\,dx \longrightarrow \int_a^b f(x)\,dx$

Pf $\left| \int_a^b f_n(x) - \int_a^b f(x)\,dx \right| = \left| \int_a^b (f_n(x) - f(x))\,dx \right|$

$\leq \int_a^b |f_n(x) - f(x)|\,dx$

$\leq (b-a)\,\|f_n - f\| \to 0$

f cts so $\int_a^b f(x)\,dx$ exist

So $\int_a^b f_n(x)\,dx \longrightarrow \int_a^b f(x)\,dx$

Thm f cts f' cts, f'' cts on $[-\pi, \pi]$ (*)

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{m=1}^{\infty} b_m \sin mx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$n \geq 1$ $\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$

$m \geq 1$ $\quad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx$

Then (*) is uniformly convergent.

Pointwise

Uniform conv of series

We say $\sum_{n=1}^{\infty} f_n(x) \xrightarrow{\text{pointwise}} f(x)$ $\quad x \in [a,b]$

if $\sum_{n=1}^{\infty} f_n(x) = f(x)$

$\sum_{n=1}^{\infty} f_n \xrightarrow{\text{unif}} f$

if $S_N = \sum_{n=1}^{N} f_n \xrightarrow{\text{unif}} f$

Weierstrass M-test

$|f_n(x)| \le M_n$

$M_n \ge 0 \quad \forall n \ge 1$

$\sum_{n=1}^{\infty} M_n < \infty$

$\Rightarrow \sum_{n=1}^{\infty} f_n(x)$ converges absolutely and uniformly on A

Proof

$\sum_{n=n+1}^{m} M_n < \varepsilon$

$n, m \ge n_0$

$|S_n(x) - S_m(x)| = \left| \sum_{n+1}^{m} f_n(x) \right| \le \sum_{k=n+1}^{m} M_n < \varepsilon$

~~$S_n(x)$~~

$\{S_n(x)\}$ is Cauchy

$\lim_{n\to\infty} S_n(x) = S(x)$

$\varepsilon > 0$ $\quad |S_n(x) - S_m(x)| \le \varepsilon/2 \quad m > n$

Take lim as $m \to \infty$

$|S_n(x) - S(x)| \le \varepsilon/2 < \varepsilon$

$\uparrow$ ind of x

so $S_n \to S(x)$ uniformly

Thm $\left(\hat{f} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right)$ is cgt series unif

Proof

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$ integrating by parts

$$= \frac{1}{\pi} f(x) \frac{\sin nx}{n\pi} \Big|_{-\pi}^{\pi} - \frac{1}{n\pi} \int_{-\pi}^{\pi} f'(x) \sin nx \, dx$$

$$a_n = +\frac{1}{n^2\pi} f'(x) \cos nx \Big|_{-\pi}^{\pi} - \frac{1}{n^2\pi} \int_{-\pi}^{\pi} f''(x) \cos nx \, dx$$

$$a_n = \frac{-1}{n^2\pi} \int_{-\pi}^{\pi} f''(x) \cos(nx) \, dx$$

f'' cts on $[a,b]$ $\quad |f''(x)| \le C$ $\forall x \in [a,b]$

$|\cos nx| \le 1$

$$|a_n| \le \frac{C}{n^2\pi} \cdot 2\pi$$

$$|a_n| \le \frac{1}{n^2\pi} \int_{-\pi}^{\pi} C \, dx = \frac{2C}{n^2}$$

Ily $|b_n| \le \frac{2C}{n^2}$

So by $\sum_{n=1}^{\infty} \frac{2C}{n^2}$ converges

(*) conv absolut and uniform

By M-test

Pointwise Convergence of Fourier series

$f: [a,b] \to \mathbb{R}$

$P = \{ x_0 \le x_1 \le \dots \le x_{n_p} \}$ partition

$$V(f,P) = \sup \sum_{i=0}^{n_p - 1} |f(x_{i+1}) - f(x_i)|$$

$$V(f) = \sup_{P \text{ partition of } [a,b]} V(f,P)$$

$f: [a,b] \to \mathbb{R}$ diff

$$|f(x_{i+1}) - f(x_i)| = |f'(x_i)|\,|x_{i+1} - x_i|$$

$$V(f) = \int_a^b |f'(x)|\,dx$$

We say f is bounded variat if $V(f) < \infty$

It can be shown f is of bounded variation on $[a,b]$
$\iff f = f_1 - f_2$ f_1, f_2 montone increasing fcts
(Dirichlet)

Thm $f: [a,b] \to \mathbb{R}$
f bounded variation
$\forall x$ s.t. f is cont at x
$\lim_{n \to \infty} S_n(f)(x) \longrightarrow f(x)$

Thm f BV, f ct $\forall x \in [a,b]$ $S_n(f) \longrightarrow f$ uniformly

Riesz-theorem $f \in L^2$ $\int_a^b f^2 dx < \infty$

$$\lim_{n\to\infty} \| S_n f - f \|_2 = 0$$

$$\int_{-\pi}^{\pi} (S_n(f) - f)^2 dx \longrightarrow 0 \text{ as } n \to \infty$$

Remark If $f : [a,b] \to \mathbb{R}$ cts then ~~it~~ Fourier-series need not converge

Carleson theorem (Very difficult)

A subset $U \subseteq \mathbb{R}$ is said to be of measure zero if given $\varepsilon > 0$ $\exists$ a $V_n = (a_n, b_n)$

$$U \subseteq \bigcup V_n$$

$$\sum_{n=1}^{\infty} (b_n - a_n) < \varepsilon$$

A property P of real is said to hold ~~if~~ almost every $x \in [a,b]$ if $f(x)$ satisfies P for all but a subset of measure zero

Thm f Square integrable

$$S_n(f)(x) \longrightarrow f(x) \quad \text{a.e in } [-\pi, \pi]$$