Lecture 3

Recall

1) Let $f : [a,b] \to \mathbb{R}$ be a function and let $x_0, x_1, \cdots, x_n$ be distinct points in $[a,b]$. Then there exists a unique polynomial $P_n(x)$ such that

$$P_n(x_i) = f(x_i) \quad \text{for } i = 0, 1, \cdots, n$$

$P_n(x) \to$ Interpolating polynomial of f (w.r.t $x_0, x_1, \cdots, x_n$)

We had two forms of $P_n(x)$

Lagrange form

$$l_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{x_k - x_i} \qquad k = 0, 1, \cdots, n$$

$$P_n(x) = \sum_{i=0}^{n} l_i(x) f(x_i)$$

Advantage of Lagrange form

It is simple

Disadvantage of Lagranges' form

There is no way of using Lagrange form of $P_n(x)$ to determine $P_{n+1}(x)$.

Newtons form of interpolating polynomial

$$P_n(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots + f[x_0, x_1, x_2, \ldots, x_n] \prod_{i=0}^{n-1} (x - x_i)$$

$$P_n(x) = P_{n-1}(x) + f[x_0, \ldots x_n] \prod_{i=0}^{n-1} (x - x_i)$$

Here $$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f[x_0, \ldots, x_k] = \frac{f[x_1, \ldots x_k] - f[x_0, \ldots x_{k-1}]}{x_k - x_0}$$

Recall

Error of interpolating polynomial

$$e_n(x) = f(x) - P_n(x)$$

$$e_n(x) = f[x_0, x_1, \ldots, x_n, x] \prod_{i=0}^{n} (x - x_i)$$

$$f[x_0, x_1, \ldots, x_n, x] = \frac{f^{n+1}(\xi)}{(n+1)!}$$

$$\left\{ \psi_{n+1}(x) = \prod_{i=0}^{n} (x - x_i) \right.$$

↑ Not in syllabus on how to minimize $\psi_{n+1}(x)$.

Osculatory Interpolation

Sometimes we have

$$x_0, x_1, \cdots, x_k$$

$$f(x_0), f(x_1), \cdots, f(x_k)$$

$$f'(x_0), f'(x_1), \cdots, f'(x_k)$$

We need a polynomial $p(x)$ such that

$$\left.\begin{aligned} p(x_i) &= f(x_i) \\ p'(x_i) &= f'(x_i) \end{aligned}\right\} \; i = 0, 1, \cdots, n$$

Note $\deg p(x) \leq 2n+1$

Example

$$\frac{dy}{dx} = g(x,y)$$

$$y(x_0) = y_0$$

Note $y'(x_i) = g(x_i, y_i)$ can be readily computed.

Last time I had given an example to calculate $P(x)$

Algorithim for computing $P(x)$

Set $z_0 = x_0$

$z_1 = x_0$

$z_2 = x_1$

$z_3 = x_1$

$\vdots$

$z_{2i} = x_i$

$z_{2i+1} = x_i$

$\vdots$

$z_{2n} = x_n$

$z_{2n+1} = x_n$

$$\text{Set } f[z_i, z_{i+1}] = \begin{cases} f'(z_i) & \text{if } z_i = z_{i+1} \\ \dfrac{f(z_{i+1}) - f(z_i)}{z_{i+1} - z_i} & \text{if } z_i \neq z_{i+1} \end{cases}$$

for higher order compute

$$f[z_i, z_{i+1}, \ldots, z_{i+k}] = \frac{f[z_{i+1}, \ldots, z_{i+k}] - f[z_i, \ldots, z_{i+k-1}]}{z_{i+k} - z_i}$$

$$P_{2n+1} = f[z_0] + f[z_0, z_1](x - z_0) + f[z_0, z_1, z_2](x - z_0)(x - z_1) + \cdots + f[z_0, \ldots z_{2n+1}] \prod_{j=0}^{2n} (x - z_j)$$

Problem

$f(0) = 1$ $\quad f(1) = 1.9$

$f'(0) = 2$ $\quad f'(1) = 2.5$

Approximate $f(0.4)$

$z_0 = 0, \quad z_1 = 0, \quad z_2 = 1, \quad z_3 = 1$

z_i	$f(z)$	$f[,]$	$f[,,]$	$f[,,,]$
0	1	2	-1.1	2.7
0	1	0.9	1.6	
1	1.9	2.5		
1	1.9			

Here $f[0,0] = f'(0) = 2$

$f[1,1] = f'(1) = 2.5$

$f[0,1] = \dfrac{f(1) - f(0)}{1-0} = \dfrac{1.9 - 1}{1} = 0.9$

$f[0,0,1] = \dfrac{f[0,1] - f[0,0]}{1-0}$

$= 0.9 - 2 = -1.1$

$f[0,1,1] = \dfrac{f[1,1] - f[0,1]}{1-0}$

$= 2.5 - 0.9 = 1.6$

finally

$$f[0,0,1,1] = \frac{f[0,1,1] - f[0,0,1]}{1-0}$$

$$= 1.6 - (-1.1)$$

$$= 2.7$$

$$P_3(x) = 1 + 2(x-0) + (-1.1)(x-0)^2 + 2.7(x-0)^2(x-1)$$

$$= 1 + 2x - 1.1x^2 + 2.7x^2(x-1)$$

$$f(0.4) \simeq P_3(0.4)$$

$$= 1.365$$

Theory of Osculatory Interpolation

Convention

Let $x_0, x_1, \ldots, x_m$ be not necessarily distinct points.

We say two functions $f(x)$ and $g(x)$ agree at the points $x_0, \ldots, x_m$ if

$$f^{(j)}(z) = g^{(j)}(z) \quad \text{for } j = 0, 1, \ldots, k-1$$

for every point z which occurs k times in the sequence $x_0, x_1, \ldots, x_m$

Example

$\underline{x} = 1, 2, 2, 1, 3, 2$

$f(x)$ and $g(x)$ agree at $1, 2, 2, 1, 3, 2$

if

$$f(1) = g(1)$$
$$f'(1) = g'(1)$$

$$f(2) = g(2)$$
$$f'(2) = g'(2)$$
$$f''(2) = g''(2)$$

$$f(3) = g(3)$$

Problem Given $x_0, x_1, \ldots, x_m$ not necessarily distinct points and $f : [a,b] \to \mathbb{R}$

We need a polynomial $p(x)$ of degree $\leq m$ such that

$p(x)$ and $f(x)$ agree at $x_0, x_1, \ldots x_m$.

Exercise: Show that if two polynomials of degree $\leq m$ agree at $x_0, x_1, \ldots, x_m$, then they are equal.

So it makes sense to talk about the polynomial of degree $\leq m$ which agrees with $f(x)$ at the $m+1$ pts $x_0, \ldots x_m$.

Theorem:- If $f(x)$ has r continuous derivatives and no point in the sequence $x_0, \ldots, x_m$ occurs more than r times then there exists exactly one polynomial $P_m(x)$ of degree $\leq m$ which agrees with $f(x)$ at $x_0, x_1, \ldots, x_m$.

Proof

Uniqueness — Exercise

Existence

Assume ~~$x_0 \leq \dots$~~

$$x_0 \leq x_1 \leq x_2 \leq \dots \leq x_m$$

for $m = 0$. there is nothing to show.

Assume the statement for $m = k-1$

and consider it for $m = k$

There are two cases

Case 1 $x_0 = x_k$.

Then $x_0 = x_1 = \dots = x_k$.

So $r \geq k$. By assumption f has at-least k continuous derivatives

Then the Taylor polynomial $P_k(x)$ for $f(x)$ around the center $c = x_0$ does the job.

Case 2 Note that its leading coefficient is

$$\frac{f^{(k)}(x_0)}{k!}$$

Case 2 $x_0 < x_k$

Then by induction hypothesis we can find polynomial $P_{k-1}(x)$ of degree $\leq k-1$ which agrees with $f(x)$ at $x_0, x_1, \ldots, x_k$ and a polynomial $q_{k-1}(x)$ of degree $\leq k-1$ which agree with $f(x)$ at $x_1, x_2, \ldots, x_k$.

Verify

$$P_k(x) = \frac{x - x_0}{x_k - x_0} q_{k-1}(x) + \frac{x_k - x}{x_k - x_0} P_{k-1}(x)$$

does the job.

[slightly tricky to show
See textbook Conte & de Boor Pg 64]

Convention

$x_0, x_1, \cdots, x_m$ not necessarily distinct points

$P_m(x)$ = unique polynomial which agrees with $f(x)$ at $x_0, x_1, \cdots, x_m$

$f[x_0, x_1, \cdots, x_m]$ = leading coefficient of $P_m(x)$
= coefficient of x^m in $P_m(x)$

We have

$$P_m(x) = P_{m-1}(x) + f[x_0, \cdots, x_m] \prod_{i=0}^{m-1} (x - x_i)$$

Proof $P_m(x) - f[x_0, \cdots, x_m] \prod_{i=0}^{m-1} (x - x_i)$ has degree $\leq m-1$ and agrees with $f(x)$ at $x_0, x_1, \cdots, x_{m-1}$. So by uniqueness of interpolating polynomial the result follows.

Thus we can write $P_m(x)$ as

$$P_m(x) = f(x_0) + f[x_0, x_1](x - x_0) +$$
$$+ f[x_0, x_1, x_2](x - x_0)(x - x_1)$$
$$\vdots$$
$$+ f[x_0, \ldots x_m] \prod_{i=0}^{m-1} (x - x_i)$$

Note we are assuming $x_0 \le x_1 \le \ldots \le x_m$.

Case 1 $\quad x_0 = x_1 = \ldots = x_m$

$P_m(x)$ = Taylor polynomial with center x_0

$$= f(x_0) + \frac{f'(x_0)}{1}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2$$
$$+ \ldots + \frac{f^{(m)}(x_0)}{m!}(x - x_0)^m$$

So

$$f[\underbrace{x_0, x_0, \ldots, x_0}_{m+1 \text{ times}}] = \frac{f^{(m)}(x_0)}{m!}$$

Case 2 $x_m \neq x_0$

Then $f[x_0, x_1, \ldots, x_m] = \dfrac{f[x_1, \ldots x_m] - f[x_0, \ldots x_{m-1}]}{x_m - x_0}$.

Example

$f(0) = 1 \qquad f'(0) = 0 \qquad f''(0) = 1$

$f(0.1) = 9.95\ E{-}1$

Approximate $f(0.05)$

z	$f(z)$	$f[,]$	$f[,,]$	$f[,,,]$
0	1	0	0.5	−10
0	1	0	−5 E-1	
0	1	−5 E-2		
0.1	9.95 E-1			

$p(x) = 1 + 0(x-0) + 0.5(x-0)^2 - 10x^3$

$p(0.05) = 1 + 0.5x^2 - 10x^3$

$= 1.$

Calculations

$z_0 = 0 \qquad z_1 = 0 \qquad z_2 = 0 \qquad z_3 = 0.1$

$f[z_0, z_1] = f'(0) = 0$

$f[z_1, z_2] = f'(0) = 0$

$$f[z_2, z_3] = \frac{f(z_3) - f(z_2)}{z_3 - z_2} = -5\ E-2$$

$$f[z_0, z_1, z_2] = \frac{f''(z_0)}{2} = 0.5$$

$$f[z_1, z_2, z_3] = \frac{f[z_2, z_3] - f[z_1, z_2]}{z_3 - z_1}$$

$$= \frac{-5\ E-2 - 0}{.1}$$

$$= -5\ E-1$$

$$f[z_0, z_1, z_2, z_3] = \frac{f[z_1, z_2, z_3] - f[z_0, z_1, z_2]}{z_3 - z_0}$$

$$= -10$$

Examples where osculatory interpolation is useful

1. $\frac{dy}{dx} = f(x,y)$

$y(x_0) = y_0$

Note Num methods for solving O.D.Es yield

x	y
x_1	$y(x_1)$
x_2	$y(x_2)$
⋮	
x_n	$y(x_n)$

Note $y'(x_i) = f(x_i, y_i)$ can be calculated easily

2. $f(x) = \int_0^x g(t)\,dt$

Numerical integration techniques yield $f(x_0), f(x_1) \cdots, f(x_n)$

Note $f'(x) = g(x)$

$f'(x_i) = g(x_i)$ can be calculated easily

As the following example will be used often I give a direct proof.

Example a, b distinct points

We know $f(a), f(b), f'(a), f'(b)$

$$P_3(x) = f(a) + f[a,a](x-a) + f[a,a,b](x-a)^2 + f[a,a,b,b](x-a)^2(x-b)$$

We prove by direct computation that $P_3(x)$ agree with $f(x)$ at a, a, b, b.

$P_3(a) = f(a)$ clear.

$P_3'(a) = f[a,a] = f'(a)$

$$f[a,b] = \frac{f(b)-f(a)}{b-a}$$

$$f[a,a,b] = \frac{f[a,b]-f[a,a]}{b-a}$$

$$= \frac{\frac{f(b)-f(a)}{b-a} - f'(a)}{b-a}$$

$$= \frac{f(b)-f(a)-(b-a)f'(a)}{(b-a)^2}$$

$$P_3(b) = f(a) + f'(a)(b-a) + \frac{f(b) - f(a) - (b-a)f'(a)}{(b-a)^2}(b-a)^2$$

$$= f(b)$$

Verify

$$f[a,b,b] = \frac{f'(b)(b-a) - (f(b)-f(a))}{(b-a)^2}$$

$$f[a,a,b,b] = \frac{(f'(b)+f'(a))(b-a) - 2(f(b)-f(a))}{(b-a)^3}$$

$$P_3'(b) = f'(a) + 2f[a,a,b](x-a) + f[a,a,b,b](b-a)^2$$

$$\overset{\text{check}}{=} f'(b)$$

Continuity of divided differences

Theorem $f[x_0, x_1, \ldots, x_n]$ is a continuous function of $x_0, x_1, \ldots, x_n$.

(Assuming f has n continuous derivatives)

i.e for each r $x_0^{(r)}, \ldots, x_n^{(r)}$ are $n+1$ pts in $[a,b]$ and

$$\lim_{r \to \infty} x_i^{(r)} = y_i \quad \text{for } i = 0, 1, \ldots, n$$

Then

$$\lim_{r \to \infty} f[x_0^{(r)}, x_1^{(r)}, \ldots, x_n^{(r)}] = f[y_0, y_1, \ldots, y_n]$$

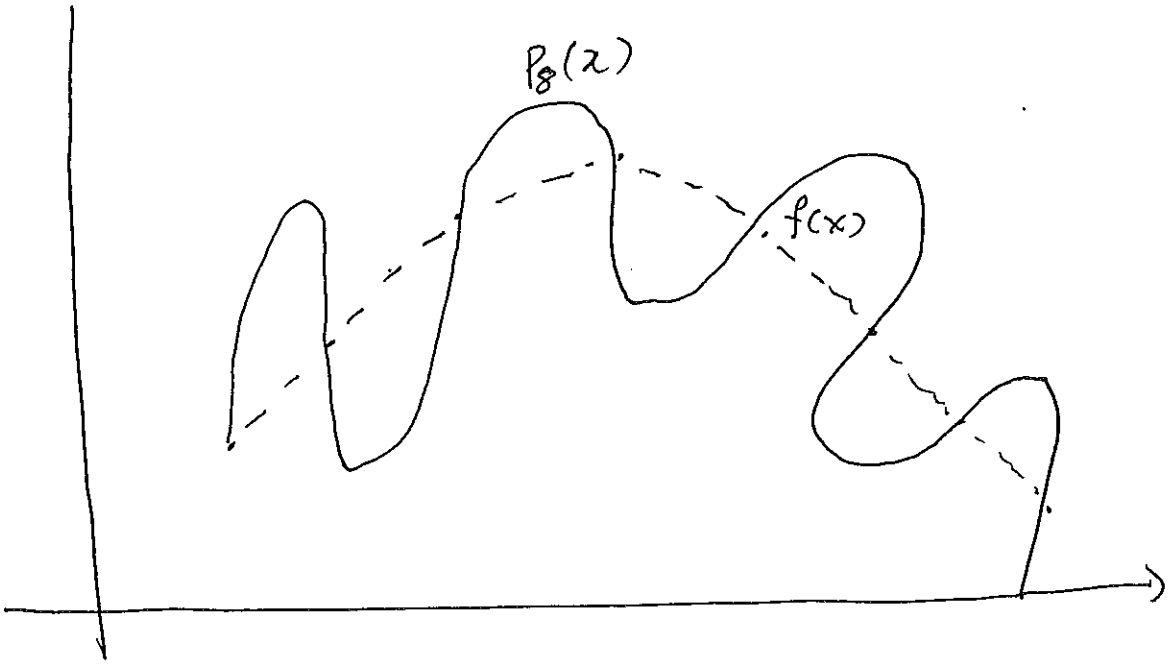
Proof See textbook Conte & de Boor page 65.

Disadvantages of Interpolation

Note that if $x_0, x_1, \ldots, x_k$ are points in $[a,b]$ then the interpolating polynomial has degree k.

In practise k is large

Furthermore a polynomial of degree k with k large oscillates a lot



For example if there are 101 points then it is not advisable to work with a degree 100 interpolating polynomial as this also creates lot of round-off error

Strategy

Use piecewise-polynomial approximation

Simplest case

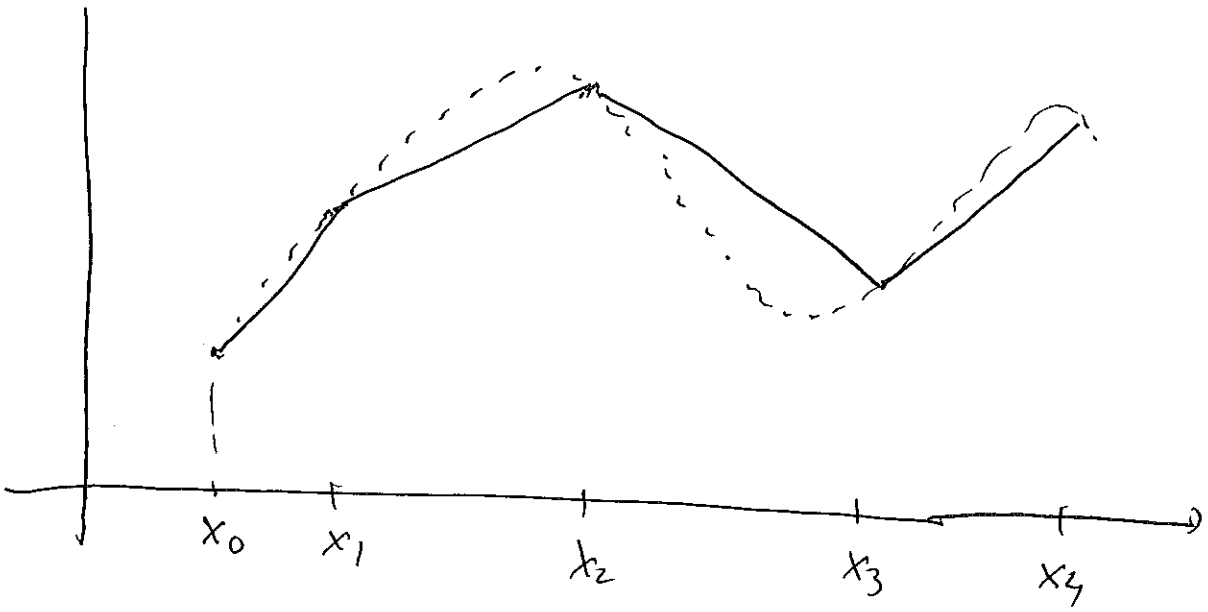
Piecewise-linear interpolation

$a = x_1 < x_2 < x_3 < \cdots < x_N = b$

$f(\bar{x})$ is approximated at a pt $\bar{x}$ by first locating the interval $[x_k, x_{k+1}]$ containing $\bar{x}$ and then taking

$$p(\bar{x}) = f(x_k) + f[x_k, x_{k+1}](\bar{x} - x_k)$$

Graphical representation



Note that if N is large and so $|x_i - x_{i+1}|$ is small then this is a good approximation of $f(x)$.

If we use higher degree (say cubic) piecewise-polynomial approximation then we get a better approximation

Construction of piecewise-cubic function $g_3(x)$ at the pts $x_1, \ldots x_{N+1}$ where

$$a = x_1 < x_2 < \ldots < x_{n+1} = b$$

On each $[x_i, x_{i+1}]$ we construct $g_3(x)$ as a cubic polynomial $P_i(x)$ $i = 1, \ldots, N$

$$P_i(x) = c_{1,i} + c_{2,i}(x - x_i) + c_{3i}(x - x_i)^2 + c_{4i}(x - x_i)^3 \qquad i = 1, \ldots, N$$

Since $g_3(x_i) = f(x_i)$ for $i = 1, \ldots N+1$

We have

$$\left.\begin{aligned} P_i(x_i) &= f(x_i) \\ \& \quad P_i(x_{i+1}) &= f(x_{i+1}) \end{aligned}\right\} \quad i = 1, 2, \ldots, N$$

In particular

$$P_{i-1}(x_i) = P_i(x_i) = f(x_i)$$

$i = 2, \ldots, N$

So $g_3(x)$ is continuous on $[a, b]$

Only constraint for $P_i(x)$ is

$P_i(x_i) = f(x_i)$ & $P_i(x_{i+1}) = f(x_{i+1})$

So we have some freedom in choosing $P_i(x)$

We study 2 cases

1. Piecewise - cubic Hermite interpolation
2. Cubic-Spline interpolation

1. Piecewise-cubic Hermite interpolation

One determines $P_i(x)$ so as to interpolate $f(x)$ at $x_i, x_i, x_{i+1}, x_{i+1}$

i.e we also have

$$P_i'(x_i) = f'(x_i) \quad \text{and} \quad P_i'(x_{i+1}) = f'(x_{i+1})$$

By Newtons formula

$$P_i(x) = f(x_i) + f[x_i, x_i](x - x_i) + f[x_i, x_i, x_{i+1}](x - x_0)^2 + f[x_i, x_i, x_{i+1}, x_{i+1})(x - x_i)^2(x - x_{i+1})$$

Write ~~$x = x_0$~~ $x - x_{i+1} = (x - x_i) + (x_i - x_{i+1})$

$$P_i(x) = f(x_i) + f'(x_i)(x - x_i) + \left(f[x_i, x_i, x_{i+1}] - f[x_i, x_i, x_{i+1}, x_{i+1}]\Delta x_i\right)(x - x_0)^2 + f[x_i, x_i, x_{i+1}, x_{i+1}](x - x_i)^3$$

Algorithm

$$f_i = f(x_i)$$

$$\Delta x_i = x_{i+1} - x_i$$

$$s_i = f'(x_i)$$

$$\Big| \quad i = 1, 2, \ldots, N+1$$

$$c_{1i} = f_i \qquad c_{2i} = s_i$$

$$c_{3i} = f[x_i, x_i, x_{i+1}] - f[x_i, x_i, x_{i+1}, x_{i+1}]\,\Delta x_i$$

$$= \frac{f[x_0, x_{i+1}] - s_i}{\Delta x_i} - c_{4i}\,\Delta x_i$$

$$c_{4i} = f[x_i, x_i, x_{i+1}, x_{i+1}]$$

$$= \frac{f[x_i, x_{i+1}, x_{i+1}] - f[x_i, x_i, x_{i+1}]}{\Delta x_i}$$

$$= \frac{s_{i+1} + s_i - 2f[x_i, x_{i+1}]}{\Delta x_i}$$

Example

$$\frac{dy}{dx} = y - x^2 + 1 \qquad 0 \le x \le 1$$

$$y(0) = 0.5$$

x_i	$y(x_i)$	$y'(x_i)$
0	0.5	1.5
0.2	0.826	1.786
0.4	1.207	2.047
0.6	1.637	2.277
0.8	2.110	2.470
1.0	2.618	2.618

Find $y(0.7)$, $y(0.9)$

Note $y(x_i)$ is found using a Numerical method

Remarks :- Usual oscillatory polynomial has degree 11

So we use piecewise Hermite interpolation

x	$y(x)$	$f[,]$	$f[,,]$	$f[,,,]$
0.6	1.637	2.277	4.4 E-1	4.25 E-1
0.6	1.637	2.365	5.25 E-1	
0.8	2.110	2.470		
0.8	2.110			

$$P_4(x) = 1.637 + 2.277\,(x-0.6) + 4.4\,E\text{-}1\,(x-0.6)^2 + 4.25\,E\text{-}1\,(x-0.6)^2(x-0.8)$$

$$P_4(0.7) = 1.869$$

x	$y(x)$	$f[,]$	$f[,,]$	$f[,,,]$
0.8	2.11	2.47	0.35	0.2
0.8	2.11	2.54	0.39	
1.0	2.618	2.618		
1.0	2.618			

$$P_5(x) = f[0.8] + f[0.8,0.8](x-0.8) + f[0.8,0.8,1](x-0.8)^2 + f[0.8,0.8,0.9,0.9](x-0.8)^2(x-1)$$

$$= 2.11 + 2.47(x-0.8) + 0.35(x-0.8)^2 + 0.2(x-0.8)^2(x-1)$$

$$P_5(0.9) = 2.360 \quad \text{in 4 sig digits}$$