Lecture 4

Recall

Last time we did osculatary interpolation

Simplest and often occuring case

$$x_0, x_1, \ldots, x_k$$
$$f(x_0), f(x_1), \ldots, f(x_k)$$
$$f'(x_0), f'(x_1), \ldots, f'(x_k)$$

Algorithm

Set $z_{2i} = x_i \qquad i = 0, \ldots, k$

$z_{2i+1} = x_i$

$$f[z_j, z_{j+1}] = \begin{cases} f'(z_j) & \text{if } z_{j+1} = z_j \\ \dfrac{f(z_{j+1}) - f(z_j)}{z_{j+1} - z_j} & \text{if } z_{j+1} \neq z_j \end{cases}$$

$$f[z_i, z_{i+1}, \ldots, z_{i+s}] = \frac{f[z_{i+1}, \ldots, z_s] - f[z_i, \ldots z_{i+s-1}]}{z_{i+s} - z_i}$$

$$P_{2k+1}(x) = f[z_0) + f[z_0, z_1](x - z_0) + f[z_0, z_1, z_2](x - z_0)(x - z_1)$$
$$+ \cdots$$
$$+ f[z_0, \ldots, z_{2k+1}] \prod_{j=0}^{2k} (x - z_j)$$

For interpolation problems it is a good idea to create the Newton divided-difference ~~table~~ table.

We then did the general osculatory interpolating formula

Let $x_0, x_1, \ldots, x_m$ be not necessarily distinct pts

we say two functions $f(x)$ and $g(x)$ agree at the points $x_0, \ldots, x_m$ if

$$f^{(j)}(z) = g^{(j)}(z) \quad \text{for } j = 0, 1, \ldots, k-1$$

for every point z which occurs k times in the sequence $x_0, x_1, \ldots, x_m$

Last time we proved

that if $f(x)$ has n continuous derivatives

and no point in the sequence $x_0, x_1, \dots, x_m$

appears more than n times then $\exists$

exactly one polynomial $P_m(x)$ of degree $\leq m$

which agrees with $f(x)$ at $x_0, \dots, x_m$.

Uniqueness was an exercise.

I hope you have done it.

Disadvantages of interpolation

Usually the number of data points $x_0, x_1, \ldots, x_m$ is very large.

Large degree polynomials oscillate a lot. It also creates lot of round off errors

To avoid this problem we study piecewise-polynomial interpolation

We did

1. piecewise linear interpolation
2. piecewise cubic interpolation
   - a) piecewise-cubic Hermite ✓
   - b) piecewise-cubic spline (we will do this today)

Piecewise cubic Hermite

$$a = x_1 < x_2 < \cdots < x_{N+1} = b$$

in $[x_i, x_{i+1}]$ $g_i(x)$ is given by polynomial $P_i(x)$ which interpolates $f(x)$ at

$x_i, x_i, x_{i+1}, x_{i+1}$

i.e $P_i(x_i) = f(x_i)$ $\quad P_i'(x_i) = f'(x_i)$

$P_i(x_{i+1}) = f(x_{i+1})$ $\quad P_i'(x_{i+1}) = f'(x_{i+1})$

$$P_i(x) = f[x_i] + f[x_i, x_i](x - x_i) +$$
$$+ f[x_i, x_i, x_{i+1}](x - x_i)^2 +$$
$$+ f[x_i, x_i, x_{i+1}, x_{i+1}](x - x_i)^2(x - x_{i+1})$$

$x \in [x_i, x_{i+1}]$

The piecewise Hermite polynomial $g(x)$ is continuously differentiable in $[a, b]$

Disadvantage of piecewise Hermite polynomial

$f'(x)$ might not be available

Cubic Spline Interpolation

Def^n Given a function f defined on $[a,b]$ and a set of nodes

$$a = x_0 < x_1 < \cdots < x_n = b$$

A cubic spline interpolant S for f is a function that satisfies the following conditions.

1. $S(x)$ is a cubic polynomial denoted by $S_j(x)$ on the sub-interval $[x_j, x_{j+1}]$ for each $j = 0, 1, \ldots, n-1$

2. $S_j(x_j) = f(x_j)$, $S_j(x_{j+1}) = f(x_{j+1})$ for each $j = 0, 1, \ldots, n-1$

3 a) $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$

b) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$

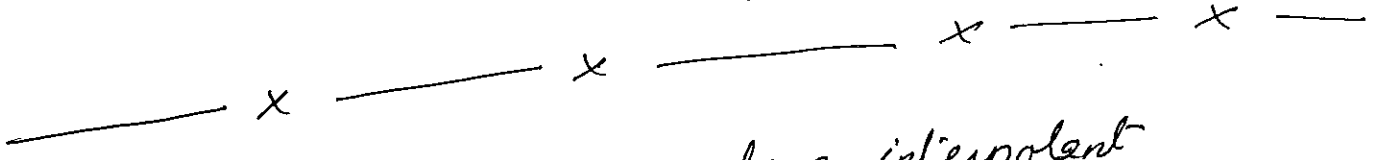
c) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$

} for $j = 0, 1, \ldots, n-2$

b) One of the following sets of boundary conditions is satisfied

(i) $S''(x_1) = S''(x_n) = 0$ (free boundary)

(ii) $S'(x_0) = f'(x_0)$, $S'(x_n) = f'(x_n)$ (clamped boundary)



Construction of cubic spline interpolant on $[x_j, x_{j+1}]$

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$$

$S_j(x_j) = f(x_j)$. So $a_j = f(x_j)$

Set $h_j = x_{j+1} - x_j$

$$\boxed{a_{j+1} = S_j(x_{j+1}) = a_j + b_j h_j + c_j h_j^2 + d_j h_j^3} \rightarrow (1)$$

$$S_j'(x) = b_j + 2c_j(x - x_j) + 3d_j(x - x_j)^2$$

$S_j'(x_j) = b_j$

$b_{j+1} = S_{j+1}'(x_{j+1}) = S_j'(x_{j+1})$

$$\boxed{b_{j+1} = b_j + 2c_j h_j + 3d_j h_j^2} \rightarrow (2)$$

Define $b_n = S'(x_n)$

$$S_j''(x) = 2c_j + 6d_j(x - x_j)$$

$$c_j = \frac{S_j''(x_j)}{2}$$

$$c_{j+1} = \frac{S_{j+1}''(x_{j+1})}{2} = \frac{S_j''(x_{j+1})}{2}$$

$$\boxed{c_{j+1} = c_j + 3d_j h_j} \longrightarrow (3)$$

Define $c_n = \frac{S''(x_n)}{2}$

By 3 $$d_j = \frac{1}{3h_j}(c_{j+1} - c_j)$$

by 1 $$a_{j+1} = a_j + b_j h_j + c_j h_j^2 + d_j h_j^3$$

$$= a_j + b_j h_j + c_j h_j^2 + \frac{1}{3}(c_{j+1} - c_j) h_j^2$$

$$\boxed{a_{j+1} = a_j + b_j h_j + \frac{1}{3}(2c_j + c_{j+1}) h_j^2} \longrightarrow (4)$$

Plugging value of d_j in (2) we get

$$b_{j+1} = b_j + 2c_j h_j + h_j (c_{j+1} - c_j)$$

$$\boxed{b_{j+1} = b_j + h_j (c_j + c_{j+1})} \rightarrow (5)$$

By (4) we get

$$b_j = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1}) \rightarrow (6)$$

$$b_{j-1} = \frac{1}{h_{j-1}}(a_j - a_{j-1}) - \frac{h_{j-1}}{3}(2c_{j-1} + c_j)$$

Plugging these values of b_j and b_{j-1} in (5)

$$\frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1})$$
$$= \frac{1}{h_{j-1}}(a_j - a_{j-1}) - \frac{h_{j-1}}{3}(2c_{j-1} + c_j)$$
$$+ h_{j-1}(c_{j-1} - c_j$$

Collecting terms of c_j, c_{j+1} and c_{j-1} we get

$$h_{j-1} c_{j-1} + 2(h_{j-1} + h_j) c_j + h_j c_{j+1}$$
$$= \frac{3}{h_j}(a_{j+1} - a_j) - \frac{3}{h_{j-1}}(a_{j-1} - a_j)$$
$$j = 1, 2, \ldots, n-1$$

$n-1$ eqns and $n+1$ unknowns

$c_0, c_1, \ldots, c_n$

Case 1 free boundary

$$S''(x_0) = S''(x_n) = 0$$

$$S_j''(x) = 2c_j + 6d_j(x - x_j)$$

$$0 = S_j''(x_0) = 2c_0 + 6d_j(0)$$

$$\text{so } c_0 = 0$$

$$c_n = \frac{S''(x_n)}{2} = 0$$

So we have a system $Ax = b$

where A is $(n+1) \times (n+1)$ matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ h_0 & 2(h_0+h_1) & h_1 & & & \\ 0 & h_1 & 2(h_1+h_2) & h_2 & & \\ & & & & & \\ & & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 & 0 & \cdots & & 0 & 0 & 1 \end{bmatrix}$$

$$\underline{X} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 0 \end{bmatrix}$$

Question 1. Does $Ax = b$ have a solution?

~~Defn A mat~~

Question 2. Any comments on the structure of A?

Def^n A matrix $T = (t_{ij})_{n\times n}$ is called strictly diagonally dominant if

$$|t_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |t_{ij}|$$

We will see later that a strictly diagonally dominant matrix T is invertible

Our matrix A is strictly diagonally dominant

So we can solve $Ax = b$ to get

$c_0, c_1, \to c_n$.

Then we obtain $d_0, d_1, \to d_{n-1}$

and $b_0, b_1, \to b_n$

Example Approximate $f(x) = e^x$ in the interval $[0,3]$

$x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$

$f(x_0) = 1$, $f(x_1) = e$, $f(x_2) = e^2$, $f(x_3) = e^3$

Find cubic spline with free boundary

Ans $n = 3$

$h_0 = h_1 = h_2 = 1$

$a_0 = 1 \quad a_1 = e \quad a_2 = e^2 \quad a_3 = e^3$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} 0 \\ 3(e^2-2e+1) \\ 3(e^3-2e^2+e) \\ 0 \end{bmatrix}$$

$$\underline{X} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

free boundary. So $c_0 = c_3 = 0$

$$4c_1 + c_2 = 3(e^2-2e+1)$$
$$c_1 + 4c_2 = 3(e^3-2e^2+e)$$

$c_1 = 0.7569 \quad c_2 = 5.83$

$$d_0 = \frac{1}{3h_0}(c_1 - c_0) = \frac{1}{3}c_1 = 0.2523$$

$$d_1 = \frac{1}{3h_1}(c_2 - c_1) = 1.691$$

$$d_2 = \frac{1}{3h_2}(c_3 - c_2) = -1.943$$

$$b_0 = \frac{1}{h_0}(a_1 - a_0) - \frac{h_0}{3}(2c_0 + c_1)$$

$$= a_1 - a_0 - \frac{1}{3}c_1$$

$$= 1.466$$

$$b_1 = \frac{1}{h_1}(a_2 - a_1) - \frac{h_1}{3}(c_2 + 2c_1)$$

$$= 2.223$$

$$b_2 = \frac{1}{h_2}(a_3 - a_2) - \frac{h_2}{3}(c_3 + 2c_2)$$

$$= 8.81$$

$$S(x) = \begin{cases} 1 + 4.466x + 0x^2 + 0.2523x^3 & 0 \le x \le 1 \\ 2.718 + 2.223(x-1) + 0.7569(x-1)^2 + 1.691(x-1)^3 & 1 \le x \le 2 \\ 7.389 + 8.81(x-2) + 5.83(x-2)^2 - 1.943(x-2)^3 & 2 \le x \le 3 \end{cases}$$

Case 2 **Clamped boundary**

$S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$

$S'(x_0) = f'(a) = b_0$

$$f'(a) = \frac{1}{h_0}(a_1 - a_0) - \frac{h_0}{3}(2c_0 + c_1)$$

$$\boxed{2h_0c_0 + h_0c_1 = \frac{3}{h_0}(a_1 - a_0) - 3f'(a)}$$

$$f'(b) = b_n = b_{n-1} + h_{n-1}(c_{n-1} + c_n)$$

So equation (6) with $j = n-1$ gives

$$f'(b) = \frac{1}{h_{n-1}}(a_n - a_{n-1}) - \frac{h_{n-1}}{3}(2c_{n-1} + c_n) + h_{n-1}(c_{n-1} + c_n)$$

Simplify to get

$$h_{n-1}c_{n-1} + 2h_{n-1}c_n = 3f'(b) - \frac{3}{h_{n-1}}(a_n - a_{n-1})$$

Thus we obtain

$$Ax = b \quad \text{where}$$

$$A = \begin{bmatrix} 2h_0 & h_0 & 0 & \cdots & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 \cdots & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 \cdots & 0 \\ & & & & \\ & & & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 \cdots & & h_{n-2} & h_{n-1} & 2h_{n-1} \end{bmatrix}$$

$$\underline{X} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$$\underline{b} = \begin{bmatrix} \frac{3}{h_0}(a_1 - a_0) - 3f'(a) \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 3f'(b) - \frac{3}{h_{n-1}}(a_n - a_{n-1}) \end{bmatrix}$$

The matrix A is again strictly diagonally dominant. So A is invertible. Thus we can solve $Ax = b$ to get $c_0, \ldots c_n$. Then we find $\underline{d}$ and $\underline{b}$ to get the final answer.

Numerical Integration

Reasons for Numerical Integration

(1) We cannot solve $\int_a^b f(x)\,dx$ exactly.

examples 1. $\int_0^1 \sin(x^2)\,dx$

2. $\int_0^1 e^{-x^2}\,dx$

3. $\int_0^1 \sqrt{1+\cos^4 x}\,dx$

(2) f is not known explicitly

Only values of f at some points are known

~~Eg~~ Example

$$\frac{dy}{dx} = \sin(x^2+y^2)$$

$$y(0) = 1$$

find $\int_0^1 y(t)\,dt$

Note we do not know y exactly.

By Numerical Sol^n to ODE's we can approximate $y(0.1), y(0.2), \ldots, y(0.9), y(1)$

Derivation of Numerical Integration formulae

Let $P_k(x)$ be the polynomial which interpolates $f(x)$ at points $x_0, x_1, \ldots, x_k$

We approximate $I(f) = \int_a^b f(x)\,dx$

by $I(P_k) = \int_a^b P_k(x)\,dx$

$$f(x) = P_k(x) + f[x_0, x_1, \ldots, x_k, x]\,\psi_k(x)$$

$$\psi_k(x) = \prod_{j=0}^{k} (x - x_j)$$

Error in computing integral $= E(f)$

$$= I(f) - I(P_k)$$

$$= \int_a^b f[x_0, x_1, \ldots, x_k, x]\,\psi_k(x)\,dx$$

Simplification of error term

Case 1 $\psi_k(x)$ is of one sign on (a,b)

Then by MVT for integrals

$$E(f) = \int_a^b f[x_0,\ldots,x_k,x]\,\psi_k(x)\,dx$$

$$= f[x_0,\ldots,x_k,\xi]\int_a^b \psi_k(x)\,dx$$

If in addition $f(x)$ is $k+1$ times differentiable

$$E(f) = \frac{f^{(k+1)}(\eta)}{(k+1)!}\int_a^b \psi_k(x)\,dx$$

for some $\eta \in (a,b)$

Case 2 $\int_a^b \psi_k(x)\,dx = 0$

We use the identity

$$f[x_0, x_1, \ldots, x_k, x] = f[x_0, \ldots x_k, x_{k+1}] + f[x_0, x_1, \ldots, x_{k+1}, x](x - x_{k+1})$$

$$E(f) = \int_a^b f[x_0, \ldots, x_k, x]\,\psi_k(x)\,dx$$

$$= \int_a^b f[x_0, \ldots x_k, x_{k+1}]\,\psi_k(x)\,dx$$

$$+ \int_a^b f[x_0, \ldots x_{k+1}, x](x - x_k)\,\psi_k(x)\,dx$$

$$= \int_a^b f[x_0, \ldots x_{k+1}, x]\,\psi_{k+1}(x)\,dx$$

$$\psi_{k+1}(x) = (x - x_{k+1})\,\psi_k(x)$$

$$= \prod_{j=0}^{k+1} (x - x_j)$$

$$E(f) = \int_a^b f[x_0, x_1, \ldots, x_{k+1}, x]\,\psi_{k+1}(x)\,dx$$

If we can choose x_{k+1} such that $\psi_{k+1}(x)$ is of one sign on (a,b)

then by MVT of integrals

$$E(f) = f[x_0, \ldots, x_{k+1}, \xi] \int_a^b \psi_{k+1}(x)\,dx$$

If $f(x)$ is $k+2$ times continuously differentiable then

$$E(f) = \frac{f^{(k+2)}(\eta)}{(k+2)!} \int_a^b \psi_{k+1}(x)\,dx$$

for some $\eta \in (a,b)$

Examples

Let $k = 0$

$$P_0(x_0) = f(x_0)$$

$$f(x) = f(x_0) + f[x_0, x](x - x_0)$$

$$I = \int_a^b f(x)\,dx = \int_a^b f(x_0)\,dx + \int_a^b f[x_0, x](x - x_0)\,dx$$

$$I(P_0(x)) = (b-a) f(x_0)$$

Case (1)

$x_0 = a$

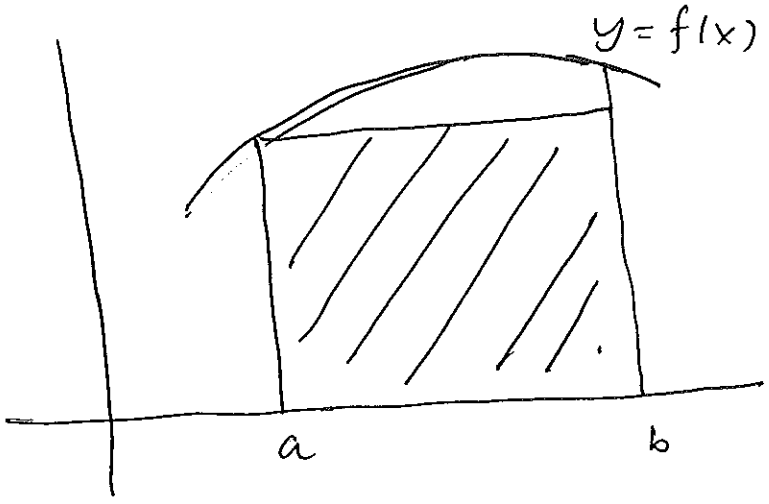
$I(f) \approx R = (b-a) f(a)$

$\psi_0(x) = x - a$ has one sign on $[a, b]$

$$E^R = f'(\eta) \int_a^b (x-a)\,dx$$

$$= f'(\eta) \frac{(b-a)^2}{2}$$

$R \leftarrow$ Rectangle rule



Case 2 $\quad x_0 = \frac{a+b}{2}$.

$\psi_0(x) = x - x_0$ not of one sign

However $\int_a^b (x - \frac{a+b}{2})\,dx = 0$

Choose $x_1 = x_0$

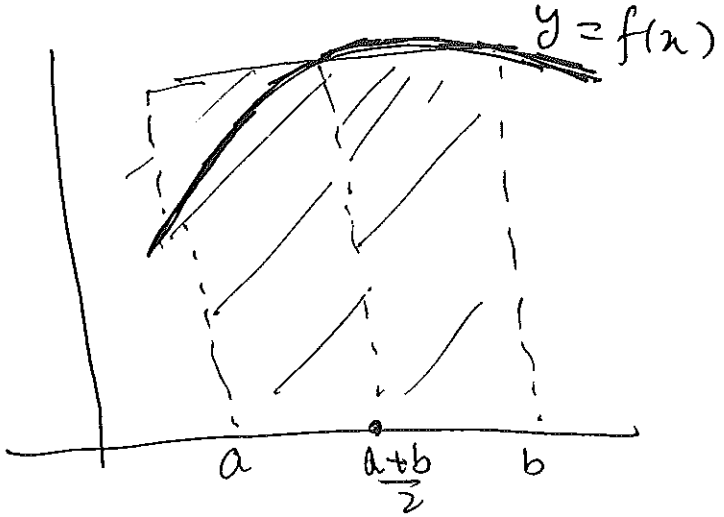
$\psi_1(x) = (x-x_0)^2$ is of one sign

$$I(f) \simeq M = (b-a) f\left(\frac{a+b}{2}\right)$$

midpoint rule

$$E^M = \frac{f''(\eta)}{2} \int_a^b \left(x - \frac{a+b}{2}\right)^2 dx$$

$$= \frac{f''(\eta)}{2} \frac{(b-a)^3}{24} \quad \text{for some } \eta \in [a,b]$$



Midpoint rule is usually better than Rectangle rule

Now let $k=1$

$$f(x) = f(x_0) + f[x_0, x_1](x-x_0) + f[x_0, x_1, x]\,\psi_1(x)$$

$$\psi_1(x) = (x-x_0)(x-x_1)$$

$x_0 = a, \quad x_1 = b$

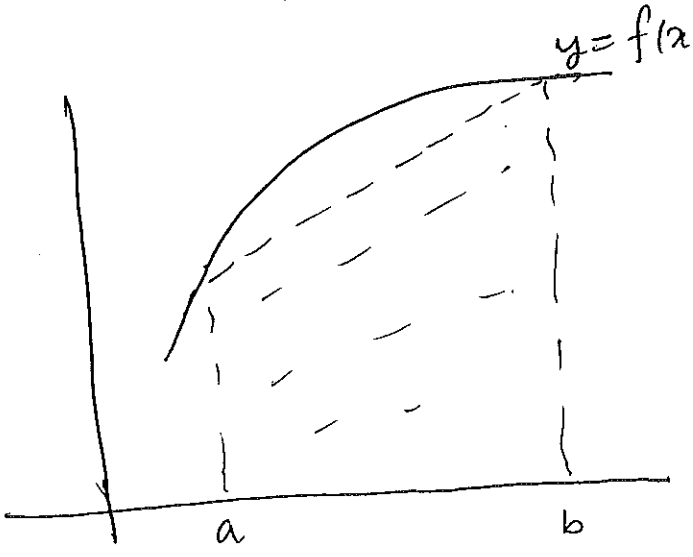
$\psi_1(x) = (x-a)(x-b) \le 0 \quad$ on $[a,b]$

$$I(f) = \int_a^b \{ f(x_0) + f[a,b](x-a) \}\,dx + \frac{1}{2} f''(\eta) \int_a^b (x-a)(x-b)\,dx$$

$$I(f) \simeq T = \frac{1}{2}(b-a)[f(a)+f(b)]$$

Trapezoidal rule

$$E^T = -\frac{f''(\eta)(b-a)^3}{12} \qquad \eta \in (a,b)$$



Now let $k=2$

$$f(x) = P_2(x) + f[x_0, x_1, x_2, x]\, \psi_k(x)$$

For distinct pts x_0, x_1, x_2

$\psi_2(x) = (x-x_0)(x-x_1)(x-x_2)$ is NOT of one sign on $[a,b]$

However for

$x_0 = a, \quad x_1 = \frac{a+b}{2}, \quad x_2 = b$

$$\int_a^b \psi_2(x)\,dx = \int_a^b (x-a)\left(x - \frac{a+b}{2}\right)(x-b)\,dx$$

$$u = x - \frac{a+b}{2}$$

$$= \int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \left(u + \frac{b-a}{2}\right) u \left(u - \frac{b-a}{2}\right) du$$

$$= 0$$

put $x_3 = x_1 = \frac{a+b}{2}$

$$\psi_3(x) = (x-a)\left(x - \frac{a+b}{2}\right)^2 (x-b)$$

$$\leq 0 \quad \text{on } [a,b]$$

$$I(f) = I(P_2) + \frac{1}{4!} f^{(4)}(\eta) \int_a^b \psi_3(x)\,dx$$

One calculate

$$\int_a^b \psi_3(x)\,dx = \int_a^b (x-a)\left(x - \frac{a+b}{2}\right)^2 (x-b)\,dx$$

$$= -\frac{4}{15}\left(\frac{b-a}{2}\right)^5$$

$$\boxed{E^S(f) = -\frac{1}{90} f^{(4)}(\eta)\left(\frac{b-a}{2}\right)^5 \qquad \eta \in (a,b)}$$

$$I(f) \approx I(P_2)$$

$$P_2(x) = f(a) + f[a,b](x-a) + f\left[a,b,\frac{a+b}{2}\right](x-a)(x-b)$$

$$\int_a^b P_2(x)\,dx = f(a)(b-a) + f[a,b]\left(\frac{b-a}{2}\right)^2 + f\left[a,b,\frac{a+b}{2}\right]\int_a^b (x-a)(x-b)\,dx$$

$$\int_a^b (x-a)(x-b)\,dx = -\frac{(b-a)^3}{6}$$

Note $\quad f\left[a,b,\frac{a+b}{2}\right] = f\left[a,\frac{a+b}{2},b\right]$

(why?)

$$f\left[a, \frac{a+b}{2}, b\right](b-a)^2$$

$$= \left(f\left[\frac{a+b}{2}, b\right] - f\left[a, \frac{a+b}{2}\right] \right)(b-a)$$

$$= \left\{ \frac{f(b) - f\left(\frac{a+b}{2}\right)}{\frac{b-a}{2}} - \frac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{b-a}{2}} \right\}(b-a)$$

$$= 2\left(f(b) - 2f\left(\frac{a+b}{2}\right) + f(a) \right)$$

$$I(P_2) = \int_a^b P_2(x)\,dx$$

$$= f(a)(b-a) + \left(f(b) - f(a)\right)\left(\frac{b-a}{2}\right)$$

$$= 2\left(f(b) - 2f\left(\frac{a+b}{2}\right) + f(a)\right)\frac{b-a}{6}$$

$$= \frac{b-a}{6}\left\{ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right\}$$

Simpson's Rule

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left\{ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right\}$$

$$E^s = -\frac{f^{(4)}(\eta)}{90}\left(\frac{b-a}{2}\right)^5$$

$k=3$ "Corrected Trapezoidal rule"

$$f(x) = P_3(x) + f[x_0, x_1, x_2, x_3, x]\,\psi_3(x)$$

$x_0 = x_1 = a \qquad x_2 = x_3 = b$

$\psi_3(x) = (x-a)^2(x-b)^2$ is of one sign in $[a,b]$

$$E(f) = \frac{1}{4!} f^{(4)}(\eta) \int_a^b (x-a)^2(x-b)^2\,dx$$

$$= \frac{f^{(4)}(\eta)\,(b-a)^5}{720}$$

$$P_3(x) = f(a) + f[a,a](x-a) + f[a,a,b](x-a)^2 + f[a,a,b,b](x-a)^2(x-b)$$

$$\int_a^b P_3(x)\,dx = f(a) + f'(a)\frac{(b-a)^2}{2} + f[a,a,b]\frac{(b-a)^3}{3} + f[a,a,b,b]\left\{\frac{(b-a)^4}{4} - \frac{(b-a)^4}{3}\right\}$$

$$f[a,a,b] = \frac{f[a,b] - f'(a)}{b-a}$$

$$f[a,a,b,b] = \frac{f'(b) - 2f[a,b] + f'(a)}{(b-a)^2}$$

$$\int_a^b \beta_3(x)\,dx = f(a)(b-a) + f'(a)\frac{(b-a)^2}{2} + \left\{f[a,b] - f'(a)\right\}\frac{(b-a)^2}{3} - \left\{f'(b) - 2f[a,b] + f'(a)\right\}\frac{(b-a)^3}{12}$$

replace $f[a,b]$ by $\frac{f(b)-f(a)}{b-a}$

So we get

$$I(f) \simeq CT = \frac{b-a}{2}\left(f(a)+f(b)\right) + \frac{(b-a)^2}{12}\left(f'(a) - f'(b)\right)$$

Corrected Trapezoidal rule

$$E^{CT} = f^{(iv)}(\eta)\frac{(b-a)^5}{720}$$

Rules for Numerical Integration

Rectangle rule $I \simeq (b-a) f(a)$

Midpoint rule $I \simeq (b-a) f\left(\frac{a+b}{2}\right)$

Trapezoidal rule $I \simeq \frac{b-a}{2} \{ f(a) + f(b) \}$

Simpsons rule $I \simeq \frac{b-a}{6} \{ f(a) + 4 f\left(\frac{a+b}{2}\right) + f(b) \}$

Corrected Trapezoidal rule

$$I \simeq \frac{b-a}{2} \{ f(a) + f(b) \} + \frac{(b-a)^2}{12} (f'(a) - f'(b))$$