Tutorial 11

1. Find General solutions to the following difference equations

a) $y_{n+2} + 2y_{n+1} + 2y_n = 0$

b) $y_{n+3} = y_{n+2} + 2y_{n+1} - 2y_n = 0$

c) $y_{n+2} - y_{n+1} - y_n = 0$

2. Find the general solution of

$y_{n+3} - 6y_{n+2} + 11y_{n+1} - 6y_n = 0$

3. Show that the Adam Bashforth method of order 2 is strongly stable (use test eq^n $y' = \lambda y$)

4) Let V be the vectorspace of all real polynomials in x of degree ≤ 10.

Let T = linear operator $\frac{d}{dx}$

Let A be matrix of T w.r.t basis $\{1, x, x^2, \ldots, x^{10}\}$

Show A does not have a complete set of eigenvectors.

(5) Consider the symmetric matrix

$$A = \begin{bmatrix} 10 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Find an approximation to the ~~dom~~ eigenvalue λ of A s.t $|\lambda| > |\mu|$ for all other eigenvalues μ of A.

Solution to Tutorial 11

1 (a) Char. polynomial

$$\beta^2 + 2\beta + 2 = 0$$

roots $\quad -2 \pm \sqrt{4-8}$

$$= -2 \pm \sqrt{-4}$$

$$= -2 \pm 2i$$

$$r = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$$

$$\theta = \tan^{-1}\left(\tfrac{y}{x}\right) = \tan^{-1}\left(\tfrac{2}{-2}\right) = \tan^{-1}(-1) = -\pi/4$$

Gen Sol^n $= (2\sqrt{2})^n \left(c_1 \cos\left(-\tfrac{\pi}{4} n\right) + c_2 \sin\left(-\tfrac{\pi}{4} n\right) \right)$

$$= (2\sqrt{2})^n \cdot \left(c_1 \cos\left(\tfrac{\pi}{4} n\right) + c_2' \sin \tfrac{\pi}{4} n \right)$$

(b) Char polynomial

$$\beta^3 + \beta^2 + 2\beta - 2 = 0$$

$\beta = 1$ is a root

$$\beta - 1 \;\big)\; \beta^3 - \beta^2 + 2\beta - 2 \;\big(\; \beta^2 + 2$$

$$\beta^3 - \beta^2$$

$$2\beta - 2$$

char polynomial

$= (\beta - 1)(\beta^2 + 2)$

$= (\beta - 1)(\beta - 2i)(\beta + 2i)$

Gen sol^n $= c_1(1) + 2^n\left(c_1 \cos\left(\frac{\pi}{2}n\right) + c_2 \sin\left(\frac{\pi}{2}n\right)\right)$

c) char polynomial is

$\beta^2 - \beta - 1 = 0$

roots $\beta = \dfrac{1 \pm \sqrt{1 - 4(-1)(1)}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

$\alpha = \dfrac{1+\sqrt{5}}{2} \qquad \beta = \dfrac{1-\sqrt{5}}{2}$

Gen sol^n $= c_1 \alpha^n + c_2 \beta^n$

(d) char polynomial

$\beta^3 - 6\beta^2 + 11\beta - 6 = 0$

roots $\beta = 1, 2, 3.$

~~$y_n = \alpha$~~

~~$\alpha = \dfrac{2^n}{1^n - 6 + 11 - 6}$~~

Gen sol^n $y_n = c_1 + c_2 2^n + c_3 3^n$

(3) AB method of order 2

$$y_{n+1} = y_n + \frac{h}{2}(3f_n - f_{n-1})$$

Test eqn $y' = f = \lambda y$

$$y_{n+1} - y_n - \frac{h\lambda}{2}(3y_n - y_{n-1}) = 0$$

Char polynomial

$$P(\beta) = \beta^2 - \beta - \frac{h\lambda}{2}(3\beta - 1) = 0$$

$$p(\beta) = \alpha(\beta) - \frac{h\lambda}{2}(\gamma(\beta))$$

$$\alpha(\beta) = \beta^2 - \beta$$
$$\gamma(\beta) = 3\beta - 1$$

$h \to 0$ $\quad p(\beta) = \beta^2 - \beta = \alpha(\beta)$

roots $= 1, 0 = \beta_2$
$\quad\quad\;\; \| \quad\;\; \|$
$\quad\quad\;\; \beta_1 \quad\;\; \beta_2$

roots of $p(\beta)$ are continuous functions of h

& so for h small enough

root of $p(\beta) = \beta_1(h), \beta_2(h)$

$|\beta_2(h)| < 1$. So AB (order 2) is strongly stable

(4) Note $T^{11} = 0$

So $A^{11} = 0$

if λ is an eigenvalue of A

$x \neq 0$

$$Ax = \lambda x$$

$$\Rightarrow A^2 x = \lambda^2 x$$

$$\therefore A^{11} x = \lambda^{11} x$$

$$0 = \lambda^{11} x \Rightarrow \lambda = 0$$

Thus any eigenvalue of A is zero

if A has a complete set of eigenvectors then $\exists$ invertible matrix C such that

$C^{-1} A C = D$ ← diagonal matrix

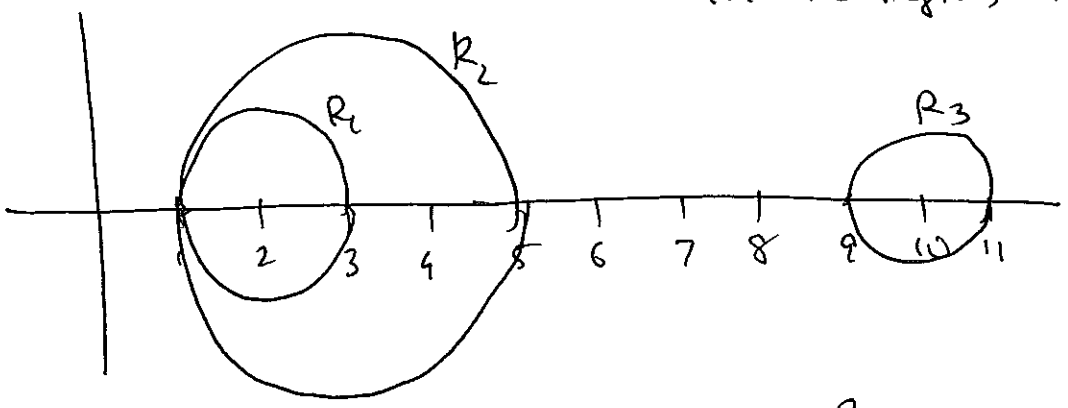
(also diagonal entries of D are eigenvalues of A.

So $D = 0$

$\Rightarrow A = 0$ ✗ (so $A \neq 0$)

($T \neq 0$ operator

(5) By Gresch gorins theorem eigen values lies in the regions R_1, R_2, R_3



R_3 is disjoint from $R_2 \cup R_1$

So R_3 contains one eigen value.

Char polynomial of A

$$|tI - A| = \begin{vmatrix} t-10 & -1 & 0 \\ -1 & t-3 & -1 \\ 0 & -1 & t-2 \end{vmatrix}$$

$$= (t-10)\begin{vmatrix} t-3 & -1 \\ -1 & t-2 \end{vmatrix} + 1\begin{vmatrix} -1 & -1 \\ 0 & t-2 \end{vmatrix}$$

$$= (t-10)((t-3)(t-2)-1) - (t-2)$$

$$= (t-10)(t^2-5t+5) - (t-2)$$

$$= t^3 - 5t^2 + 5t - 10t^2 + 50t - 50 - t + 2$$

$$= t^3 - 15t^2 + 54t - 48$$

$$p(t) = t^3 - 15t^2 + 54t - 48$$

$p(9) = -48$ $\quad p(11) = 62$

So a root between 9 and 11

bisection

a_n	b_n	$f(a_n)$	$f(b_n)$
9	11	$-4.8 E 1$	$6.2 E 1$
10	11	-8	$6.2 E 1$
10	10.5	-8	$2.288 E 1$
10	10.25	-8	6.453

now do Newton

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$x_0 = 10$

$x_1 = 1.015 \; E1$

$x_2 = 1.014 \; E1$

$x_3 = 1.014 \; E1$

10.14 is the approximation to maximal eigenvalue