Tutorial 12

Q1 Let Q be a $n \times n$ orthogonal matrix

Show

(a) $\|Q\|_2 = 1$

(b) If λ is a real eigen value of Q then $\lambda = \pm 1$

Q2 Find the dominant eigen value of the following matrix by power method

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$

Note eigen vals are 4, 1

iteration = 5 take $x_0 = (1, 0)$

Q3 Find the dominant eigen value of the following symmetric matrix by the Symmetric power method

$$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$$

eigen vals are

3.414

5.858 E-1

iteration = 5

take $x_0 = (1, 0)$

Solution to Tutorial 12

(1) (a) Let $\bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ be s.t $\|\bar{x}\|_2 = 1$

$Q = [v_1, \ldots, v_n]$

Q orthogonal $\Rightarrow \{v_1, \ldots, v_n\}$ is orthonormal basis of $\mathbb{R}^n$

$Q\bar{x} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n$

$\|Q\bar{x}\|_2^2 = Q\bar{x}^t \cdot Q\bar{x}$ $\quad (\because v_i^t v_j = \delta_{ij})$

$= x_1^2 + x_2^2 + \cdots + x_n^2$

$= 1$

$\|Q\bar{x}\|_2 = 1$

$\|Q\|_2 = \max_{\|\bar{x}\|_2 = 1} \|Q\bar{x}\|_2 = 1.$

(b) λ real eigenvalue of Q

$|\lambda| \le \|Q_{\bar{x}}\|_2$

So $|\lambda| \le 1$

if $|\lambda| < 1 \Rightarrow \frac{1}{|\lambda|} > 1$

Note $\frac{1}{\lambda}$ is Eigenvalue of $Q^{-1} = Q^t$ which is also orthogonal

this gives $\|Q^t\|_2 \ge \frac{1}{|\lambda|} > 1$ A contradiction

So $|\lambda| = 1 \Rightarrow \lambda = \pm 1$

(2) $A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$.

it	x	y	p_m	ν
0	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 2 \end{pmatrix}$	1	
1	$\begin{pmatrix} 1 \\ 6.667 \cdot E-1 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 2 \end{pmatrix}$	1	3
2	$\begin{pmatrix} 1 \\ 9.089E-1 \end{pmatrix}$	$\begin{pmatrix} 3.667 \\ 3.333 \end{pmatrix}$	1	3.667
3	$\begin{pmatrix} 1 \\ 9.767 E-1 \end{pmatrix}$	$\begin{pmatrix} 3.909 \\ 3.818 \end{pmatrix}$	1	3.909
4	$\begin{pmatrix} 1 \\ 9.940 E-1 \end{pmatrix}$	$\begin{pmatrix} 3.977 \\ 3.953 \end{pmatrix}$	1	3.977
5	$\begin{pmatrix} 1 \\ 9.985 E-1 \end{pmatrix}$	$\begin{pmatrix} 3.994 \\ 3.988 \end{pmatrix}$	1	3.994

So 3.994 is approximate eigenvalue with approximate eigenvector $\begin{pmatrix} 1 \\ 9.985\,E-1 \end{pmatrix}$

③ $A = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$

it	x	y	ν
0	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$		
1	$\begin{pmatrix} 9.487\,E{-}1 \\ 3.162\,E{-}1 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$	3
2	$\begin{pmatrix} 9.285\,E{-}1 \\ 3.714\,E{-}1 \end{pmatrix}$	$\begin{pmatrix} 3.162 \\ 1.265 \end{pmatrix}$	3.4
3	$\begin{pmatrix} 9.247\,E{-}1 \\ 3.807\,E{-}1 \end{pmatrix}$	$\begin{pmatrix} 3.157 \\ 1.3 \end{pmatrix}$	3.414
4	$\begin{pmatrix} 9.240\,E{-}1 \\ 3.824\,E{-}1 \end{pmatrix}$	$\begin{pmatrix} 3.155 \\ 1.305 \end{pmatrix}$	3.414
5	$\begin{pmatrix} 9.239\,E{-}1 \\ 3.826\,E{-}1 \end{pmatrix}$	$\begin{pmatrix} 3.154 \\ 1.306 \end{pmatrix}$	3.414

3.414 is approximate dominant eigenvals

with approximate eigenvector

$$\begin{pmatrix} 9.239\,E{-}1 \\ 3.826\,E{-}1 \end{pmatrix}$$