6 Tutorial 13

1. Let $P_0, P_1, \ldots, P_k, \ldots$ be Legendre polynomials

Let $x_0, x_1, \ldots, x_k$ be roots of $P_{k+1}(x)$

$$l_i(x) = \prod_{j \neq i} \frac{(x - x_j)}{(x_i - x_j)} \qquad i = 0, 1, \ldots, k.$$

Then $\int_{-1}^{1} l_i(x) l_j(x)\, dx = 0 \qquad i \neq j$

Pf Note if $i \neq j$ $x_0, x_1, \ldots x_k$ are not of $l_i(x) l_j(x)$

$\deg P_{k+1}(x) = k+1$ and $x_0, x_1, \ldots x_k$ are roots of $P_{k+1}(x)$

So $l_i(x) l_j(x) = P_{k+1}(x)\, g(x)$

$\deg l_i(x) l_j(x) = 2k$

So $\deg g(x) = 2k - (k+1)$

$= k - 1$

$$\int_{-1}^{1} l_i(x) l_j(x) = \int_{-1}^{1} P_{k+1}(x) g(x)\, dx$$

$= 0$ by property of Legendre polynomials

(2) Calculate the poly of degree ≤ 2

which minimizes $\int_{-1}^{1} (\sin \pi x - p(x))^2 \, dx$

$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = x^2 - 1/3$

$p(x) = d_0 + d_1 x + d_2 P_2$

$E = \langle \sin \pi x - p(x), \sin(\pi x) - p(x) \rangle$ min

$0 = \frac{\partial E}{\partial d_0} = \langle 1, \sin(\pi x) - p(x) \rangle + \langle \sin \pi x - p(x), 1 \rangle$

$= 2\langle 1, \sin \pi x \rangle - 2\langle 1, p \rangle$

$2\langle 1, p \rangle = \langle 1, d_0 \rangle = d_0 \int_{-1}^{1} 1^2 dx = 2 d_0$

$2 d_0 = \langle 1, \sin \pi x \rangle = \int_{-1}^{1} \sin \pi x \, dx = 0$ $\sin \pi x$ odd

$d_0 = 0$

$0 = \frac{\partial E}{\partial d_1} = \langle x, \sin(\pi x) - p(x) \rangle + \langle \sin \pi x - p(x), x \rangle$

$= 2\langle x, \sin \pi x \rangle - 2 d_1 \langle x, x \rangle$

$\langle x, x \rangle = \int_{-1}^{1} x^2 dx = \frac{x^3}{3} \Big|_{-1}^{1} = 2/3$

$d_1 \cdot \frac{2}{3} = \langle x, \sin \pi x \rangle = \int_{-1}^{1} (x \sin \pi x) \, dx$

$$\frac{2}{3} d_1 = 2\int_0^1 x \sin \pi x \, dx$$

$u = x \qquad du = 1$

$dv = \sin \pi x \qquad v = -\frac{\cos \pi x}{\pi}$

$$\frac{1}{3} d_1 = -x\frac{\cos \pi x}{\pi}\Big|_0^1 + \int_0^1 \frac{\cos \pi x}{\pi} dx$$

$$= 1 + \frac{\sin \pi x}{\pi^2}\Big|_0^1$$

$$= 1$$

$$d_1 = 3$$

$$0 = \frac{\partial E}{\partial d_2} = \langle P_2, \sin(\pi x) - p(x) \rangle + \langle \sin \pi x - p(x), P_2 \rangle$$

$$= 2\langle P_2, \sin(\pi x) \rangle - 2 d_2 \langle P_2, P_2 \rangle$$

$$\langle P_2, \sin(\pi x) \rangle = \int_{-1}^{1} \underbrace{P_2 \sin \pi x}_{\text{odd}} \, dx$$

$$= 0$$

so $d_2 = 0$

$$p(x) = 3 d_1 = \underline{3x}$$

(3) Let $f(x) = \begin{cases} 1 & 0 \le x \le \pi \\ -1 & -\pi \le x < 0 \end{cases}$

Calculate its Fourier series

Solution

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{f \cos(nx)}_{\text{odd}} \, dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f \sin(nx) \, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} -\sin(nx) \, dx + \int_{0}^{\pi} 1 \cdot \sin(nx) \, dx \right]$$

$$= \frac{1}{\pi} \left\{ \left[\frac{\cos nx}{n} \right]_{-\pi}^{0} + \left[\frac{\cos nx}{n} \right]_{0}^{\pi} \right\}$$

$$= \frac{1}{\pi} \left\{ \left[\frac{1}{n} - \frac{(-1)^n}{n} \right] - \left[\frac{(-1)^n}{n} - \frac{1}{n} \right] \right\}$$

$$= \frac{1}{\pi} \left\{ \frac{2}{n} - \frac{2(-1)^n}{n} \right\}$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd} \end{cases}$$