Tutorial 3

1) Give an example of a polynomial $P_{2n+2}(x)$ of degree $2n+2$ such that Gaussian Quadrature (with $n+1$ nodes) is not exact for $P_{2n+2}(x)$

2) Approximate $\int_0^1 e^{-x^2}\,dx$

by
1) Trapezoidal rule $h = 0.25$
2) Simpsons rule $h = 0.5$
3) Gaussian Quadrature with 3 nodes.

3) Suppose $f''(x) \geq 0 \quad \forall x \in [a,b]$.
Let α = approximation of $\int_a^b f(x)\,dx$ by composite Mid point rule
β = approximation of $\int_a^b f(x)\,dx$ by composite trapezoidal rule.
Let $I = \int_a^b f(x)\,dx$ (exact value)
Show I is between α and β

4) Consider $I = \int_0^1 \sin(x^3)\,dx$

a) How many subdivisen of the interval ~~I~~ $= [0,1]$ is needed so that Trapezoid rule gives an error of 10^{-4} (or less)

b) (Same Question as b) for Simpson's rule

5) ~~Find~~ Approximate $\int_0^1 \frac{\cos x}{\sqrt{x}}\,dx$ by Taylor series method

Solutions to Tutorial 3

1. Let $x_0, x_1, \ldots, x_n$ be roots of $Q_{n+1}(x)$ the $(n+1)^{st}$ Legendre polynomial.

Set $P_{2n+2}(x) = \prod_{i=0}^{n} (x - x_i)^2$

$\deg P_{2n+2}(x) = 2n+2$

approximation of $\int_{-1}^{1} P_{2n+2}(x)\,dx$ by Gaussian Quadrature

$$= \sum_{i=0}^{n} P_{2n+2}(x_i)\, C_i \qquad C_i = \int_{-1}^{1} l_i(x)\,dx$$

$$= 0$$

But $\int_{-1}^{1} P_{2n+2}(x)\,dx > 0$ (why?)

2)

x	$f(x)$
0	1
0.25	~~.0647~~ 9.394 E-1
0.5	7.788 E-1
0.75	5.698 E-1
0.1	3.679 E-1

2 1) Approximation by Trapezoidal rule

$$T = \frac{0.25}{2}\left[f(0) + 2f(0.25) + 2f(0.5) + 2f(0.75) + f(1)\right]$$

$$= 7.430 \; E-1$$

2) Approximation by Simpson's Rule

$$S = \frac{0.5}{6}\left[f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + f(1)\right]$$

$$= 7.469 \; E-1$$

3) $$I = \int_0^1 e^{-x^2} dx$$

$$= \int_{-1}^{1} \frac{1}{2} e^{-\left(\frac{t+1}{2}\right)^2} dt$$

$t = 1 + 2(x-1)$

$dt = 2\,dx$

$\frac{t-1}{2} = x-1$

$x = \frac{t-1}{2} + 1$

$= \frac{t+1}{2}$

$x_0 = -0.7746$

$x_1 = 0$

$x_2 = 0.7746$

$b_0 = 0.5556$

$c_1 = 0.8889$

$c_2 = 0.5556$

$$I \approx \sum_{i=0}^{2} f(x_i)\, c_i$$

x_0	$4.937\ E-1$
x_1	$3.894\ E-1$
x_2	$2.275\ E-1$

$$I \approx 7.468\ E-1$$

3) $\int_{x_i}^{x_{i+1}} f(x)\,dx \cong (x_{i+1}-x_i)\, f\left(\frac{x_i+x_{i+1}}{2}\right)$ · midpt rule

$$\text{Error}^{M} = \frac{f''(\eta_i)\,(x_{i+1}-x_i)^3}{24} \geq 0$$

Then $\int_{x_i}^{x_{i+1}} f(x)\,dx \geq (x_{i+1}-x_i)\, f\left(\frac{x_i+x_{i+1}}{2}\right)$ $(\because f'' \geq 0)$

$$\int_a^b f(x)\,dx \geq \alpha$$

$$\int_{x_i}^{x_{i+1}} f(x)\,dx \approx \frac{1}{2}(x_{i+1}-x_i)\left[f(x_i)+f(x_{i+1})\right]$$

$$\text{Error}^{T} = -\frac{f''(\eta_i)\,(x_{i+1}-x_i)^3}{12}$$

$$\leq 0$$

So $\int_a^b f(x)\,dx \leq \beta$

(4) (a) $f(x) = \sin(x^3)$

$f'(x) = \cos(x^3) \cdot 3x^2$

$f''(x) = 6x\cos(x^3) - 9x^4 \sin(x^3)$

$|f''(x)| \le 6 + 9 = 15$

$$E_N^T = -\frac{f''(\eta)\, h^2 (1-0)}{12}$$

$$|E_N^T| \le 15\frac{h^2}{12} \le 10^{-4}$$

$h = \frac{1}{N}$

N = # subdivisions

$$N^2 \ge \frac{15}{12} 10^4$$

$$N \ge 10^2 \sqrt{\frac{15}{12}} \approx 111.8$$

$N = 112$ will do the job

4 (b) $f'''(x) = 6\cos x^3 - 18x^3 \sin(x^3) - 36x^3 \sin(x^3) - 27x^6 \cos(x^3)$

$f^{(3)}(x) = 6\cos(x^3) - 54x^3 \sin(x^3) - 27x^6 \cos(x^3)$

$f^{(4)}(x) = -168x^2 \sin(x^3) - 162x^5 \cos x^3 - 162x^5 \cos x^3 + 81x^8 \sin(x^3)$

$$|f^{(4)}(x)| \leq 168 + 162 + 162 + 81 = 573$$

$$E_N^S = -\frac{f^{(iv)}(\xi)\,(h/2)^4\,(b-a)}{180}$$

$$|E_N^S| \leq \frac{573 \cdot h^4}{16 \times 180} \leq 10^{-4}$$

$$h = 1/N$$

$$N^4 \geq 10^4\,\frac{573}{16 \times 180}$$

$$N \geq 10\sqrt{\frac{573}{16 \times 180}}$$

$$= 4.46$$

so $N = 5$ will give the required accuracy.

5) Approximate $\int_0^1 \frac{\cos x}{\sqrt{x}}\,dx$

$P_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$ 4th Taylor polynomial of $\cos x$

$$I = \int_0^1 \frac{\cos x - P_4(x) + P_4(x)}{\sqrt{x}}\,dx$$

$$I_1 = \int_0^1 \frac{\cos x - P_4(x)}{\sqrt{x}} \qquad I_2 = \int_0^1 \frac{P_4(x)}{\sqrt{x}}\,dx$$

$= I_1 + I_2$

$$I_2 = \int_0^1 \left(x^{-1/2} - \frac{x^{3/2}}{2} + \frac{x^{7/2}}{24}\right) dx$$

$$= \frac{x^{1/2}}{1/2} - \frac{1}{2}\frac{x^{5/2}}{5/2} + \frac{1}{24}\cdot\frac{x^{9/2}}{9/2}\Big|_0^1$$

$$= 2 - \frac{1}{10} + \frac{1}{108}$$

$= 1.909$

$$I_2 = \int_0^1 \frac{\cos x - P_4(x)}{\sqrt{x}}$$

$h = 0.5$ simpson rule

X	G
0	0
0.25	-6.774 E-7
0.5	-3.055 E-5
0.75	-2.826 E-4
1	-1.364 E-3

$$G(x) = \begin{cases} 0 & x = 0 \\ \frac{\cos x - P_4(x)}{\sqrt{x}} & x \neq 0 \end{cases}$$

$h = 0.5$

$$\frac{I}{2} \approx \frac{h}{6}\left[G(0) + 4G(0.25) + 2G(0.5) + 4G(0.75) + G(1)\right]$$

≈ -2.132 E-4

$I = I_1 + I_2$

$= 1.909$