Tutorial 6

1) $g : I \to I$ be a twice continuously differentiable function and $|g'(x)| \le k < 1 \quad \forall x \in I$
$I = [a, b]$
Let ξ be a the unique fixed pt of g. If $g'(\xi) = 0$ and $g''(\xi) \neq 0$ then show that fixed point iteration has quadratic rate of convergence

2) If f has a double zero at ξ
Then show that the iteration

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)}$$

converges quadratically to ξ
(x_0 suff close to ξ).

3) Let A be a given positive constant.
Set $g(x) = 2x - Ax^2$.
(a) Show that if the fixed pt iteration converges to a non-zero limit then the limit is $p = 1/A$.
(b) Find an interval about $1/A$ for which fixed pt iteration converges

4) Use fixed pt iteration to find a root of $2\sin \pi x + x = 0$ on $[1, 2]$

(5) Show that if A is any +ve number then the sequence defined by

$$x_n = \frac{1}{2} x_{n-1} + \frac{A}{2x_{n-1}} \quad \text{for } n \geq 1$$

converges to $\sqrt{A}$ whenever $x_0 > 0$

Solutions to Tutorial 6

1) $g(x) = g(\xi) + (x-\xi)\, g'(\xi) + (x-\xi)^2 \, g''(\eta_\cdot)$

η between x and ξ

$x = x_n \qquad g(x_n) = x_{n+1}$

$x_n - \xi = e_n \qquad g(\xi) = \xi$

$$e_{n+1} = e_n\, g'(\xi) + e_n^2\, g''(\eta_n)$$

$$g'(\xi) = 0$$

$$e_{n+1} = e_n^2\, g''(\eta_n)$$

η_n between x_n and ξ

$$\left| \frac{e_{n+1}}{e_n^2} \right| = |g''(\eta_n)|$$

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^2} = |g''(\xi)| \neq 0$$

($\because x_n \to \xi$, $\eta_n \to \xi$ and g'' is ct

Thus the order of fixed pt convergence is quadratic

(2) $g(x) = x - 2\dfrac{f(x)}{f'(x)}$

$f(\xi) = 0$
$f'(\xi) = 0$
$f''(\xi) \neq 0$ (2)

$$\lim_{x\to\xi} g(x) = \xi - 2\lim_{x\to\xi} \frac{f(x)}{f'(x)}$$

$$= \xi - 2\lim_{x\to\xi} \frac{f'(x)}{f''(x)} \quad \text{"L'Hospital rule}$$

$$= \xi - 2\frac{f'(\xi)}{f''(\xi)}$$

$$= \xi$$

$g(\xi) = \xi$ Thus ξ is a fixed pt of g

$$g'(x) = -1 + 2\frac{f(x)}{f'(x)^2} f''(x)$$

$$\lim_{x\to\xi} g'(x) = -1 + 2\left(\lim_{x\to\xi} \frac{f(x)}{f'(x)^2}\right) f''(\xi)$$

$$= -1 + 2\left(\lim_{x\to\xi} \frac{f'(x)}{2f'(x)f''(x)}\right) \cdot f''(\xi)$$

by L'Hoptials rule

$$= -1 + 2\left(\lim_{x\to\xi} \frac{1}{2f''(x)}\right) f''(\xi)$$

$$= -1 + 1$$

$g'(\xi) = 0$

So by (1) convergence is Quadratic

~~3 (a) $g(x) = x$~~ ~~$x \neq 0$~~

~~$2x - Ax^2 = x$~~

3 (a) $g(\xi) = \xi$ $\quad \xi \neq 0$

$2\xi - A\xi^2 = \xi$

$\Rightarrow \quad 2 - A\xi = 1$

$A\xi = 1$

$\Rightarrow \quad \xi = \frac{1}{A}$.

3(b) $g'(x) = 2 - 2Ax$

$g'(\frac{1}{A}) = 0$ so converges is quadratic

~~$g(x)$~~ $|2 - 2Ax| \leq 4/5$

$|1 - Ax| \leq 2/5$

$-2/5 \leq 1 - Ax \leq 2/5$

$-\frac{2}{5A} \leq \frac{1}{A} - x \leq 2/5A$

Interval $|x - 1/A| \leq 2/5A$

$|g'(x)| \leq 4/5$

$I = [1/A - 2/5A,\ 1/A + 2/5A]$ (4)

Note

$$|g(x) - \tfrac{1}{A}| = |g(x) - g(1/A)| \qquad x \in I$$

$$= |g'(\eta)|\,|x - 1/A| \qquad \eta \text{ between } 1/A \text{ and } x$$

$$\leq 4/5\,|x - 1/A|$$

$$\leq \frac{4}{5} \cdot \frac{2}{5A} = \frac{8}{10A}$$

Thus $g(x) \in I$

$g : I \longrightarrow I$.

fixed pt iteration will converge to $1/A$.

(4) $2\sin \pi x + x = 0$

$\sin \pi x = -\frac{x}{2}$

$\pi x = \sin^{-1}\left(-\frac{x}{2}\right)$

$x = \frac{1}{\pi} \sin^{-1}\left(-\frac{x}{2}\right)$

~~$\frac{1}{2} \leq x$~~ $\quad 1 \leq x \leq 2$

$-1 \geq -x \geq -2$

$-\frac{1}{2} \geq -\frac{x}{2} \geq -1$

So $\sin^{-1}\left(-\frac{x}{2}\right)$ is well defined

$x_0 = 1$

$x_1 = 1.7$ 2

4) $2 \sin \pi x + x = 0 \qquad x \in [1, 2]$

$$\sin \pi x = -\frac{x}{2}$$

$$\sin [\pi x - 2\pi] = -\frac{x}{2}$$

$$\pi x - 2\pi = \sin^{-1}\left(-\frac{x}{2}\right)$$

$$x = \frac{1}{\pi} \sin^{-1}\left(-\frac{x}{2}\right) + 2$$

$x_0 = 1$

$x_1 = 1.833$

$x_2 = 1.631$

$x_3 = 1.696$

$x_4 = 1.678$

$x_5 = 1.683$

$x_6 = 1.682$

$x_7 = 1.682$

$x_n = 1.682$ for $n \geq 7$

$f(x) = x + 2 \sin \pi x$

$f(1.682) = 4.174 \; E{-5}$

#5 if ~~$0 < x_0 < \sqrt{A}$~~ $x_0 \neq \sqrt{A}$

$$(x_0 - \sqrt{A})^2 > 0$$

$$x_0^2 - 2x_0\sqrt{A} + A > 0$$

divide by $2x_0$

$$\frac{x_0}{2} - \sqrt{A} + \frac{A}{2x_0} > 0$$

$$x_1 = \frac{x_0}{2} + \frac{A}{2x_0} > \sqrt{A}$$

$$g(x) = \frac{1}{2}x + \frac{A}{2x}$$

$g(\xi) = \xi \Rightarrow \xi = \frac{1}{2}\xi + \frac{A}{2\xi}$

$\xi > 0$

$\frac{1}{2}\cdot\xi = \frac{A}{2\xi}$

$\xi^2 = A$

$\xi = \sqrt{A}$

$$g'(x) = \frac{1}{2} - \frac{A}{2x^2}$$

$$= \frac{x^2 - A}{2x^2}$$

If $x > \sqrt{A}$

$$x^2 > A$$

$$\frac{1}{2} > \frac{A}{2x^2}$$

$$0 < g'(x) < 1/2.$$

$g: [\sqrt{A}, \infty) \longrightarrow [\sqrt{A}, \infty)$

Let $x_0 > \sqrt{A}$

$I = [\sqrt{A}, x_0]$

$x \in I$

$|g(x) - \sqrt{A}| \leq 1/2 |x - \sqrt{A}|$

as $g(x) > \sqrt{A}$

we have $g(x) \in I$

Also $|g'(x)| \leq 1/2 \quad \forall x \in I$

so fixed pt iteration converges to $\sqrt{A}$.

If $0 < x_0 < \sqrt{A}$ then $x_1 > \sqrt{A}$

So again fixed pt iteration converges

$x_0 = \sqrt{A} \Rightarrow x_n = \sqrt{A} \ \forall n$. So fixed pt iteration converges.