Tutorial 7

1. Show that an upper triangular matrix $U = (u_{ij})$ is invertible iff all $u_{ii} \neq 0$.

2. Suppose U is an invertible upper-triangular matrix. Prove U^{-1} is also upper-triangular.

3. Find LU factorization of the following matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 3 & 5 & 0 \\ 3 & 9 & 15 \end{bmatrix}$$

4. Show that LU-factorization of a matrix is unique.

5. Find cholesky factorization of

$$\begin{bmatrix} 4 & 0 & 8 \\ 0 & 9 & 0 \\ 8 & 0 & 41 \end{bmatrix}$$

Solution to Tutorial 7

1. If $u_{ii} = 0$ then $\det U = u_{11} u_{22} \cdots u_{nn} = 0$

$\Rightarrow U$ is singular.

Conversely the result holds as $\det U \neq 0$

2. Let $U^{-1} = [c_1, c_2, \ldots, c_n]$.

$$U U^{-1} = I$$

$$e_i = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow i^{th} \text{ place}$$

So $U c_i = e_i$

we have $u_{nn} c_n = 0$ ①

$u_{n-1,n-1} c_{n-1} + u_{n-1,n} c_n = 0$ ②

$\vdots$

$u_{i,i} c_i + u_{i,i+1} c_{i+1} + \cdots + u_{i,n} c_n = 0$ (n-i)

By eqn 1 we have $c_n = 0$

by eqn 2 $c_{n-1} = 0$

$\vdots$

by eqn n-i-1 $c_{i+1} = 0$

by eqn n-i $c_i = \frac{1}{u_{ii}}$

Thus $c_i = \begin{pmatrix} * \\ * \\ \vdots \\ * \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th}$ row

It follows that U^{-1} is upper triangular

3) $$A = \begin{bmatrix} 3 & 1 & 1 \\ 3 & 5 & 0 \\ 3 & 9 & 15 \end{bmatrix}$$

$R_2 - R_1$ $R_3 - R_1$ yields

$$\begin{bmatrix} 3 & 1 & 1 \\ 0 & 4 & -1 \\ 0 & 8 & 14 \end{bmatrix}$$

$R_3 - 2R_2$ yields

$$\begin{bmatrix} 3 & 1 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 16 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

4) Suppose $A = L_1 U_1 = L_2 U_2$

Note L_1 has 1 in diagonal. Similarly L_2 has 1 in diagonal.

$$L_2^{-1} L_1 = U_2 U_1^{-1} = D$$

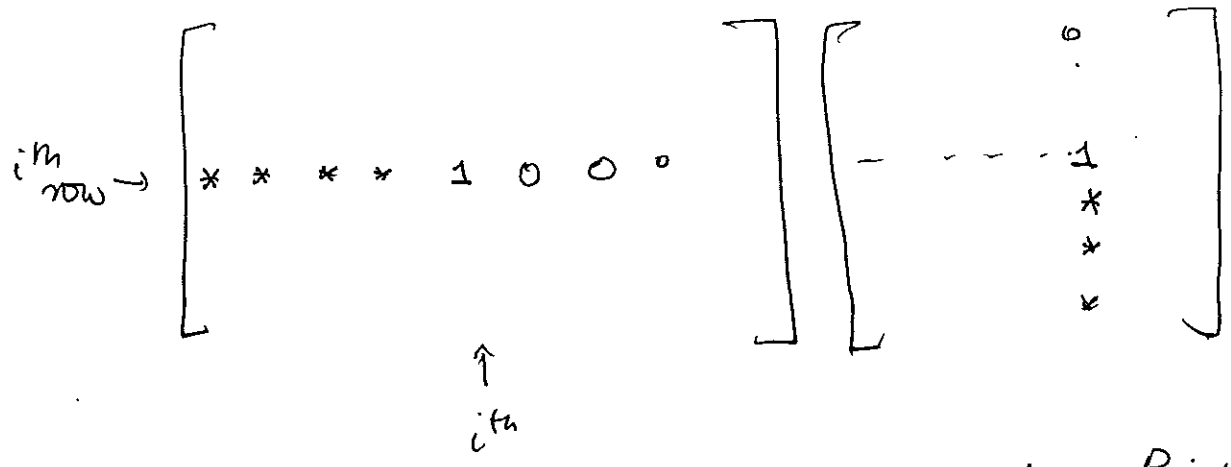
Left hand side is lower ~~dea~~ triangular matrix

~~Upper~~ Right hand side is upper triangular matrix

So D is a diagonal matrix

Note L_2^{-1} has 1 in diagonal.

It follows that $(L_2^{-1} L_1)_{ii} = R_i \cdot C_i$



(i,i) entry of $L_2^{-1} L_1$ is $R_i \cdot C_i$ where R_i is i^{th} row of L_2^{-1} and C_i is i^{th} colum of L_1

Note $R_i \cdot C_i = 1$.

Thus $D = I$

It follows $L_2 = L_1$ and hence $U_2 = U_1$.

(5) $\begin{bmatrix} 4 & 0 & 8 \\ 0 & 9 & 0 \\ 8 & 0 & 41 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$

$4 = l_{11}^2 \Rightarrow l_{11} = 2$

$l_{21} l_{11} + = 0 \Rightarrow l_{21} = 0$

$l_{31} l_{11} = 8 \Rightarrow l_{31} = 4$

$l_{21}^2 + l_{22}^2 = 9 \Rightarrow l_{22} = 3$

$l_{31} l_{21} + l_{32} l_{22} = 0 \Rightarrow l_{32} = 0$

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 41$

$16 + 0 + l_{33}^2 = 41$

$l_{33}^2 = 25$

$l_{33} = 5$

$L = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 4 & 0 & 5 \end{bmatrix}$