Tutorial 8

(1) Let $A = (a_{ij})$ be a $n \times n$ - matrix

Show $\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$

(2) Let $\bar{x} \in \mathbb{R}^n$.

Show $\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\, \|x\|_\infty$

(3) Let S be a positive definite matrix

$\bar{x} \in \mathbb{R}^n$. Define $\|\bar{x}\| = \sqrt{x^t S x}$

Show this defines a norm on $\mathbb{R}^n$

(4) Let S be a non-singular matrix and $\|\ \|$ any norm on $\mathbb{R}^n$.

Set $\|x\|' = \|Sx\|$.

Then show $\|x\|'$ defines a norm on $\mathbb{R}^n$.

Solutions to Tutorial 8

1) Say $\max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}| = \sum_{i=1}^{n} |a_{ip}|$

e_p = p^{th} unit vector.

note $\|e_p\|_1 = 1$

$$\|Ae_p\|_1 = \|(a_{1p}, a_{2p}, \ldots, a_{np})\|_1$$

$$= \sum_{i=1}^{n} |a_{ip}|$$

So $\|A\|_1 \ge \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$

Now let $\bar{x} \in \mathbb{R}^n$ $\|\bar{x}\|_1 = 1$

$\bar{x} = (x_1, \ldots x_n)$ so $|x_1| + \ldots + |x_n| = 1$

$$\|Ax\|_1 = \left\| \sum_{j=1}^{n} a_{1j} x_j, \sum_{j=1}^{n} a_{2j} x_j, \ldots, \sum_{j=1}^{n} a_{nj} x_j \right\|_1$$

$$= \left| \sum_{j=1}^{n} a_{1j} x_j \right| + \left| \sum_{j=1}^{n} a_{2j} x_j \right| + \ldots + \left| \sum_{j=1}^{n} a_{nj} x_j \right|$$

$$\le \sum_{j=1}^{n} |a_{1j}||x_j| + \sum_{j=1}^{n} |a_{2j}||x_j| + \ldots + \sum_{j=1}^{n} |a_{nj}||x_j|$$

$$= \left(\sum_{i=1}^{n} |a_{i1}| \right) |x_1| + \left(\sum_{i=1}^{n} |a_{i2}| \right) |x_2| + \ldots$$

$$+ \left(\sum_{i=1}^{n} |a_{in}| \right) |x_n|$$

Note $\sum_{i=1}^{n} |a_{ij}| \leq \sum_{i=1}^{n} |a_{ip}|$ by our assumption for all j

So $\|Ax\|_1 \leq \left(\sum_{i=1}^{n} |a_{ip}|\right)(|x_1| + \cdots + |x_n|)$

$$\leq \sum_{i=1}^{n} |a_{ip}|$$

Thus $\|A\|_1 = \max_{\|x\|_1 = 1} \|Ax\|_1$

$$\leq \sum_{i=1}^{n} |a_{ip}|$$

Thus $\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|$

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(2) $\bar{x} = (x_1, \ldots, x_n)$

$\|\bar{x}\|_\infty = \max_{1 \leq i \leq n} |x_i| = |x_p|$ (say)

Clearly $|x_p| \leq \sqrt{x_1^2 + \cdots + x_n^2} = \|x\|_2$

Thus $\|x\|_\infty \leq \|x\|_2$.

Also $x_i^2 \leq x_p^2$

So $x_1^2 + \cdots + x_n^2 \leq n\, x_p^2$

$\Rightarrow \|x\|_2 \leq \sqrt{n}\, |x_p| = \sqrt{n}\, \|x\|_\infty$

(3) (i) As S is positive definite

$\forall \bar{x} \neq 0 \quad x^t S x > 0$

Thus $\|x\| > 0$ and $= 0$ iff $\bar{x} = 0$.

(ii) Also $\|\alpha \bar{x}\| = \sqrt{\alpha x^t S \alpha x}$

$= \sqrt{\alpha^2 x^t S x}$

$= |\alpha| \sqrt{x^t S x} = |\alpha| \|x\|$.

(iii) Triangle inequality.

Note S has cholesky decomposition $S = LL^t$

$y = L^t x$

then $x^t S x = y^t y = \|y\|_2^2$

Thus $\|x\| = \|y\|_2$.

Let $x_1, x_2 \in \mathbb{R}^n$

$y_1 = L^t x_1 \qquad y_2 = L^t x_2$

Note $y_1 + y_2 = L^t (x_1 + x_2)$

$\|x_1 + x_2\| = \|y_1 + y_2\|_2$

$\leq \|y_1\|_2 + \|y_2\|_2$

$= \|x_1\| + \|x_2\|$

(4) (i) $\|x\|' = \|Sx\| \geq 0$

$\|x\|' = 0 \iff \|Sx\| = 0$

$\iff Sx = 0$ $\because \|\ \|$ is a norm

$\iff x = 0$ $\because S$ is invertible

(ii) $\|\alpha x\|' = \|S\alpha x\|$

$= \|\alpha Sx\|$

$= |\alpha|\,\|Sx\|$

$= |\alpha|\,\|x\|'$

(iii) $\|x_1 + x_2\|' = \|S(x_1 + x_2)\|$

$= \|Sx_1 + Sx_2\|$

$\leq \|Sx_1\| + \|Sx_2\|$

$= \|x_1\|' + \|x_2\|'$

Thus $\|x_1 + x_2\|' \leq \|x_1\|' + \|x_2\|'$.