Tutorial 9

1) For the following system

$$\begin{aligned} 5x_1 - x_2 &= 8 \\ x_1 + 5x_2 - 2x_3 &= 4 \\ 3x_2 + 5x_3 &= 6 \end{aligned}$$

find first four iterations of Jacobi method with initial approximation $(0,0,0)$

2) For the previous system find the first four iterations of Gauss-Seidel method with initial approximation $(0,0,0)$

3) Show that if A is an invertible upper triangular matrix then Jacobi-method converges in finitely many steps.

4) Suppose $A = (a_{ij})_{n \times n}$ $a_{ii} = 1$

and $1 > \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}|$ $\forall j = 1, 2, \cdots, n$

then show that the Jacobi-iteration converges

5) Solve $y' = y - x^3 + 1$ $0 \leq x \leq 1$

$y(0) = 1$

$h = 0.2$

by Euler-Method.

Solutions to Tutorial 9

1) $$x_1^{(m)} = 0.2 x_2^{(m-1)} + 1.6$$
$$x_2^{(m)} = -0.2 x_1^{(m-1)} + 0.4 x_3^{(m-1)} + 0.8$$
$$x_3 = -0.6 x_2^{(m-1)} + 1.2$$

m	0	1	2	3	4
$x_1^{(m)}$	0	1.6	1.760	1.792	1.746
$x_2^{(m)}$	0	0.8	9.6 E-1	7.296 E-1	6.912 E-1
$x_3^{(m)}$	0	1.2	7.2 E-1	6.24 E-1	7.622 E-1

Exact ans $x = 1.750$
$y = 7.5\,E{-}1$
$z = 7.5\,E{-}1$

2) Gauss Siedel

$$x_1^{(m)} = 0.2 x_2^{(m-1)} + 1.6$$
$$x_2^{(m)} = -0.2 x_1^{(m)} + 0.4 x_3^{(m-1)} + 0.8$$
$$x_3^{(m)} = -0.6 x_2^{(m)} + 1.2$$

m	0	1	2	3	4
$x_1^{(m)}$	0	1.6	1.696	1.765	1.746
$x_2^{(m)}$	0	4.8 E-1	8.256 E-1	7.288 E-1	7.559 E-1
$x_3^{(m)}$	0	9.12 E-1	7.046 E-1	7.627 E-1	7.464 E-1

(3) take initial approximation $(0, 0, 0, \ldots 0)$

Let exact soln of $Ux = b$

$= u = (u_1, u_2, \ldots, u_n)$

Note $x_n^{(1)} = \dfrac{b_n}{a_{nn}} = u_n$

also $x_n^{(m)} = x_n^{(1)}$ for $m \geq 2$.

Notice $x_{n-1}^{(2)} = \dfrac{b_{n-1} - a_{n-1,n}\, x_n^{(1)}}{a_{n-1,n-1}} = u_{n-2}$

also note $x_{n-1}^{(m)} = x_{n-1}^{(2)}$ for $m \geq 3$.

in general

$$x_{n-r}^{(r+1)} = \frac{b_{n-r} - \sum_{j > n-r} a_{n-r,j}\, x_j^{(r)}}{a_{n-r,n-r}} = u_{n-r}$$

and $x_{n-r}^{(m)} = x_{n-r}^{(r+1)}$ for $m \geq r+2$.

Thus after n iteration we get the exact answer and the iteration terminates

(4) $\hat{D} = I$

$I - \hat{D}^{-1}A = I - A$

note $\|I - A\|_1 = \max_{\substack{i=1 \\ i \neq j}} \sum^{n} |a_{ij}| < 1$

So the Jacobi-iteration converges.

(5) $y_{n+1} = y_n + 0.2\,(y_n - 0.008\,n^3 + 1)$

$x_n = 0.2\,n \qquad y_0 = 1$

x_n	y_n
0	1
0.2	1.4
0.4	1.878
0.6	2.441
0.8	~~2.441~~ 3.086
1	~~2.991~~ 3.801