Tutorial Sheet 1[1]

1. Let $y(x) = \left\{ \begin{array}{ll} e^x - 1, & x \geq 0 \\ 1 - e^{-x}, & x < 0 \end{array} \right\}$. Check that the derivative of y is continuous. Verify that $y(x)$ is a solution of $y' = |y| + 1$ on $(-\infty, \infty)$.

2. Find the general solution for the following equations.

   (a) $y' + 3\,y = \cos 10x$

   (b) $y' + 2\,y = x^2$

   (c) $y' + y - \sin^2 x = 0$

   (d) $y' + 2y - (1 + x^3) = 0$

3. Find the general solution for the following equations.

   (a) $y' - \dfrac{2x}{1+x^2} y = 0$

   (b) $e^{10x^2} y' - xy = 0$

   (c) $(1 + \cos^2 x)\, y' - \sin 2x\, y = 0$

   (d) $y' + e^{2x} \cos 3x\, y = 0$

4. Find the general solution for the following equations.

   (a) $xy' + 2y = 8x^2$

   (b) $(x-2)(x-1)y' - (4x-3)y = (x-2)^3$

   (c) $x^2 y' + 3xy = e^x$

5. Solve the following non-linear differential equations.

   (a) $y' = 2y - 10y^2$

   (b) $5x^2 y' - 3xy + e^x y^6 = 0$

   (c) $xy' + 4y = 16x^2 y^{1/2}$

6. Solve the following differential equations.

   (a) $\dfrac{dy}{dx} = \dfrac{x+3y}{x-y}$

   (b) $y' = \dfrac{x^3 + y^3}{xy^2}$

7. Following may not be separable but can be made separable by substitution.

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(a) $y' = \dfrac{-6x + y - 3}{2x - y - 1}$

(b) $y' = \dfrac{-x + 3y - 14}{x + y - 2}$.

(c) $(3x + 2y + 2)y' - (2x + 3y + 10) = 0$

(d) $(x + y - 2)y' - (2x - y - 3) = 0$

8. Show that the initial value problem $y' = \sqrt{y}$, $y(0) = 0$ has more than one solution by finding at least two solutions explicitly.

9. Find all initial conditions such that $(x^2 - x)y' = (2x - 1)y$ has no solution, precisely one solution, and more than one solution.

10. Let $xy' - 2y = -1$.

(a) Find a general solution to the above problem on $\mathbb{R} - \{0\}$.

(b) Show that y is a general solution for the above ODE if and only if

$$y = \begin{cases} \dfrac{1}{2} + c_1 x^2, & x \geq 0 \\ \\ \dfrac{1}{2} + c_2 x^2, & x < 0 \end{cases}$$

where c_1, c_2 are arbitrary constants.

(c) Conclude that all solutions of the ODE on $\mathbb{R}$ are solutions of the initial value problem $xy' - 2y = -1, \quad y(0) = \dfrac{1}{2}$

(d) Show that if $x_0 \neq 0$ and y_0 is arbitrary, then the initial value problem

$xy' - 2y = -1, \quad y(x_0) = y_0$ has infinitely many solutions on $\mathbb{R}$.

Why does this not contradict existence and uniqueness theorem for linear ODEs?

11. Solve the following IVP's

(a) $(1 + 2y)y' = 2x$, $y(0) = -2$.

(b) $y' = \dfrac{(1 + 3x^2)}{3y^2 - 6y}$, $y(0) = 1$.

(c) $y' = 2\cos 2x/(3 + 2y)$, $y(0) = -1$.

12. In each of following problems determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist.

(a) $y' + (\tan\ x)y = \sin\ x$, $y(\pi) = 0$.

(b) $(4 - x^2)y' + 2xy = 3x^2$, $y(1) = -3$.

13. In each of following problems solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value y_0.

(a) $y' + y^3 = 0, \quad y(0) = y_0$

(b) $y' = \dfrac{x^2}{y(1+x^3)}, y(0) = y_0$

14. (a) Verify that both $y_1(x) = 1 - x$ and $y_2(x) = -x^2/4$ are solutions of the initial value problem

$$y' = \frac{-x + (x^2 + 4y)^{1/2}}{2}, \quad y(2) = -1$$

Where are these solutions valid?

(b) Explain why the existence of two solutions of the given problem does not contradict the uniqueness part of the existence uniqueness theorem for ODE.

Tutorial Sheet 2

1. Determine if the following equations are exact and solve them.

   (a) $(3y\cos x + 4xe^x + 2x^2e^x)\ dx + (3\sin x + 3)\ dy = 0.$

   (b) $(\frac{1}{x} + 2x)\ dx + (\frac{1}{y} + 2y)\ dy = 0.$

   (c) $(y\sin(xy) + xy^2\cos(xy))\ dx + (x\sin(xy) + xy^2\cos(xy))\ dy = 0.$

   (d) $(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x)\ dx + (xe^{xy}\cos 2x - 3)\ dy = 0.$

   (e) $\dfrac{x}{(x^2+y^2)^{3/2}}\ dx + \dfrac{y}{(x^2+y^2)^{3/2}} = 0.$

2. Solve the following IVP.

   (a) $(4x^3y^2 - 6x^2y - 2x - 3)dx + (2x^4y - 2x^3)\ dy = 0 \quad y(1) = 3.$

   (b) $(y^3 - 1)e^x\ dx + 3y^2(e^x + 1)\ dy = 0, \quad y(0) = 0.$

   (c) $(9x^2 + y - 1)\ dx - (4y - x)\ dy = 0, \quad y(1) = 0.$

3. Find all the functions M such that the following equation is exact.

$$M(x, y)\ dx + 2xy\sin x\cos y\ dy = 0$$

4. Find all the functions N such that the equation is exact.

$$(\ln(xy) + 2y\sin x\ dx + N(x, y)\ dy = 0.$$

5. Suppose M and N are continuous and have continuous partial derivatives M_y and N_x that satisfy the exactness condition $M_y = N_x$ on an open rectangle R around (x_0, y_0). Show that if (x, y) is in R and

$$F(x, y) = \int_{x_0}^{x} M(s, y_0)\ ds + \int_{y_0}^{y} N(x, t)\ dt.$$

   then $F_x = M$ and $F_y = N$. (HINT: Use Leibniz Rule for differentiation under the integral sign)

6. Solve using the previous exercise. $(x^2 + y^2)\ dx + 2xy\ dy = 0.$

7. Solve the initial value problem $y' + \dfrac{2}{x}y = -\dfrac{2xy}{x^2 + 2x^2 + 1}, \quad y(1) = -2.$

8. Solve the following after finding an integrating factor.

   (a) $(27xy^2 + 8y^3)\ dx + (18x^2y + 12xy^2)\ dy = 0.$

   (b) $-y\ dx + (x^4 - x)\ dy = 0.$

(c) $y \sin y \; dx + x(\sin y - y \cos y) \; dy = 0$.

(d) $y(1 + 5 \ln |x|) \; dx + 4x \ln |x| \; dy = 0$.

(e) $(3x^2y^3 - y^2 + y) \; dx + (-xy + 2x) \; dy = 0$.

(f) $y \; dx + (2x - ye^y) \; dy = 0$.

(g) $(a \cos(xy) - y \sin(xy)) \; dx + (b \cos(xy) - x \sin(xy)) \; dy = 0$.

9. Let $y' + p(x)y = f(x)$. Show that $\mu = \pm e^{\int p(x) \; dx}$ is an integrating factor. Find the explicit solution using this integrating factor.

10. Show that if $(N_x - M_y)/(xM - yN) = R$, where R depends on the quantity xy only, then the differential equation $M + Ny' = 0$ has an integrating factor of the form $\mu(xy)$. Find a general formula for this integrating factor.

11. Use the previous problem to solve $(3x + \frac{6}{y}) + (\frac{x^2}{y} + 3\frac{y}{x})\frac{dy}{dx} = 0$.

12. Consider the initial value problem $y' = y^{1/3}$, $y(0) = 0$.

(a) Is there a solution that passes through the point $(1, 1)$? If so, find it.

(b) Is there a solution that passes through the point $(2, 1)$? If so, find it.

13. Apply the Picard's iteration method to the following initial value problems and get four iterations:

(a) $y' = x + y$, $y(0) = 0$

(b) $y' = 2y^2$, $y(0) = 1$

(c) $y' = 2\sqrt{y}$, $y(1) = 0$

Tutorial Sheet 3

1. Find the general solution of $y'' - 2y' + 2y = 0$. Solve it with initial conditions
   (a) $y(0) = 3, y'(0) = -2$
   (b) $y(0) = k_0, y'(0) = k_1$.
2. Compute the Wronskians of the given set of functions.
   (a) $\{e^x, e^x \sin x\}$
   (b) $\{x^{1/2}, x^{-1/3}\}$
   (c) $\{x \ln|x|, x^2 \ln|x|\}$.
3. Find the Wronskian of a given set of solutions of $y'' + 3(x^2 + 1)y' - 2y = 0$, given that $W(\pi) = 0$.
4. Find the Wronskian of a given set of solutions of $(1 - x^2)y'' - 2xy' + a(a + 1)y = 0$, given that $W(0) = 1$.
5. Find the Wronskian of a given set of solutions of $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, given that $W(1) = 1$.
6. Given one solution y_1, find other solution y_2 s.t. $\{y_1, y_2\}$ is linearly independent set.
   (a) $y'' - 6y' + 9y = 0$; $y_1 = e^{3x}$,
   (b) $x^2y'' - xy' + y = 0$; $y_1 = x$.
   (c) $(x^2 - 4)y'' + 4xy' + 2y = 0$; $y_1 = 1/(x - 2)$.
7. Suppose p_1, p_2, q_1, q_2 are continuous on (a, b) and the equations $y'' + p_1(x)y' + q_1(x)y = 0$ and $y'' + p_2(x)y' + q_2(x)y = 0$ have the same solutions on (a, b). Show that $p_1 = p_2$ and $q_1 = q_2$ on (a, b). [Hint. Use Abel's formula.]
8. Find a linear homogeneous ODE for which the given functions form a fundamental set of solutions on some interval.
   (a) $e^x \cos 2x,\ e^x \sin 2x$
   (b) $x,\ e^{2x}$
   (c) $\cos(\ln x),\ \sin(\ln x)$.
9. Solve the following IVPs.
   (a) $y'' + 14y' + 50y = 0,\ y(0) = 2, y'(0) = -17$.
   (b) $6y'' - y' - y = 0,\ y(0) = 10, y'(0) = 0$.
   (c) $4y'' - 4y' - 3y = 0,\ y(0) = \frac{13}{12},\ y'(0) = \frac{23}{24}$

(d) $4y'' - 12y' + 9y = 0,\ y(0) = 3, y'(0) = \dfrac{5}{2}$

10. Find a particular solution of $x^2y'' + xy' - 4y = 2x^4$.

11. (Principle of Superposition) Assume y_1 is a solution of $a(x)y'' + b(x)y' + c(x)y = f_1(x)$ and y_2 is a solution of $a(x)y'' + b(x)y' + c(x)y = f_2(x)$. Show that $y_1 + y_2$ is a solution of $a(x)y'' + b(x)y' + c(x)y = f_1(x) + f_2(x)$.

12. Find the general solution of

(a) $x^2y'' - 3xy' + 3y = x$

(b) $y'' - 3y' + 2y = 1/(1 + e^{-x})$

(c) $x^2y'' + xy' - 4y = -6x - 4$

(d) $x^2y'' - 2xy' + 2y = x^{9/2}$

(e) $y'' - 2y' + y = 14x^{3/2}e^x$

(f) $y'' + 4xy' + (4x^2 + 2)y = 4e^{-x(x+2)}$, given that $y_1 = e^{-x^2}$, $y_2 = xe^{-x^2}$ are solutions of homogeneous part.

Tutorial Sheet 4

1. Solve the following differential equations

   (a) $y''' - y = 0$.

   (b) $y^{(4)} + 64y = 0$.

   (c) $y^{(5)} + y^{(4)} + y''' + y'' + y' + y = 0$.

   (d) $y''' - 2y'' + 4y' - 8y = 0, \quad y(0) = 0,\ y'(0) = -2,\ y''(0) = 0$

   (e) $y''' - 6y'' + 12y' - 8y = 0, \quad y(0) = 1,\ y'(0) = -1,\ y''(0) = -4$

   (f) $y^{(4)} + 2y''' - 2y'' - 8y' - 8y = 0, \quad y(0) = 5,\ y'(0) = -2,\ y''(0) = 6,\ y'''(0) = 8$.

   (g) $y^{(4)} + 2y'' + y = 0$.

2. Find the fundamental set of solutions for the following equations.

   (a) $(D^2 + 9)^3 D^2 y = 0$.

   (b) $D^3(D - 2)^2(D^2 + 4)^2 y = 0$.

   (c) $[(D - 1)^4 - 16]y = 0$

3. Find a particular solution using Anhilator method. Write down the Anhilator explicitly. Do not evaluate the coefficients.

   (a) $y''' - 2y'' + y' = t^3 + 2e^t$

   (b) $y^{(4)} - y''' - y'' + y' = t^2 + 4 + t\sin t$.

   (c) $y^{(4)} + 4y'' = \sin 2t + te^t + 4$.

   (d) $y''' - 2y'' + y' - 2y = -e^x[(9 - 5x + 4x^2)\cos 2x - (6 - 5x - 3x^2)\sin 2x]$

   (e) $y^{(4)} - 7y''' + 18y'' - 20y' + 8y = e^{2x}(3 - 8x - 5x^2)$.

   (f) $y^{(4)} + 5y''' + 9y'' + 7y' + 2y = e^{-x}(30 + 24x) - e^{-2x}$.

4. Find the general solution using the annihilator method (method of undetermined coefficients).

   (a) $y'' - 2y' - 3y = e^x(-8 + 3x)$.

   (b) $y'' + y = e^{-x}(2 - 4x + 2x^2) + e^{3x}(8 - 12x - 10x^2)$.

   (c) $y'' + 3y' - 2y = e^{-2x}[(4 + 20x)\cos 3x + (26 - 32x)\sin 3x]$.

   (d) $y'' + 2y' + y = 8x^2\cos x - 4x\sin x$.

   (e) $y''' - y'' - y' + y = 2e^{-t} + 3$

   (f) $y^{(4)} - 4y'' = 3t + \cos t$.

   (g) $y''' - y'' - y' + y = e^x(7 + 6x)$.

   (h) $4y^{(4)} - 11y'' - 9y' - 2y = -e^x(1 - 6x)$.

   (i) $y''' + 3y'' + 4y' + 12y = 8\cos 2x - 16\sin 2x$.

(j) $y^{(4)} + 3y''' + 2y'' - 2y' - 4y = -e^{-x}(\cos x - \sin x)$

5. Let $P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x)$. Let y_1 be a solution to the corresponding homogeneous equation. Then making the substitution uy_1 in the differential gives a second order equation of the form $Q_0(x)u'' + Q_1(x)u' = F$. This is really a first order equation in variable $z = u'$ and can be solved using the variation of parameters method. This is called the method of **reduction of order**. Use the method of reduction of order to solve $(2-x)y''' + (2x-3)y'' - xy' + y = 0$ given that $y_1(x) = e^x$ is a solution.

Tutorial Sheet 5

1. Determine if the following improper integrals exist.

   (a) $\int_0^\infty (t^2+1)^{-1}\ dt$, (b) $\int_1^\infty t^{-2}e^t\ dt$

2. Find the Laplace transform of following functions.

   (a) $\cosh t \sin t$

   (b) $\cosh^2 t$

   (c) $t \sinh 2t$

   (d) $\sin(t + \dfrac{\pi}{4})$

   (e) $f(t) = \begin{cases} e^{-t}, & 0 \le t < 1 \\ e^{-2t}, & t \ge 1 \end{cases}$

   (f) $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$

3. (a) Prove that if $L(f(t)) = F(s)$, then $L(t^k f(t)) = (-1)^k F^{(k)}(s)$.

   [Hint. Assume that we can differentiate the integral $\int_0^\infty e^{-st} f(t)dt$ with respect to s under the integral sign.]

   (b) Using $L(1) = 1/s$, show that $L(t^n) = \dfrac{n!}{s^{n+1}}$, n an integer.

4. Show that if f is piecewise continuous and of exponential order, then $\lim_{s\to\infty} F(s) = 0$.

5. Show that if f is continuous on $[0,\infty)$ and of exponential order $s_0 > 0$, then

$$L\left(\int_0^t f(\tau)d\tau\right) = \frac{1}{s}L(f), \quad s > s_0.$$

6. Suppose f is piecewise continuous and of exponential order, and $\lim_{t\to 0+} f(t)$ exists. Show that

$$L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(r)dr.$$

7. Suppose f is piecewise continuous on $[0,\infty)$.

   (a) If the integral $g(t) = \int_0^t e^{-s_0\tau} f(\tau)d\tau$ satisfies the inequality $|g(t)| \le M,\ t \ge 0$, then f has a Laplace transform $F(s)$ defined for $s > s_0$.

   [Hint. Use integration by parts to show that

$$\int_0^T e^{-st} f(t)dt = e^{-(s-s_0)T} g(T) + (s - s_0)\int_0^T e^{-(s-s_0)t} g(t)dt$$

   (b) Show that if $L(f)$ exists for $s = s_0$, then it exists for $s > s_0$.

8. Find the Laplace transform of the following functions.

   (a) $\dfrac{\sin \omega t}{t}, \ \ \omega > 0,$

   (b) $\dfrac{e^{at} - e^{bt}}{t}$

   (c) $\dfrac{\cosh t - 1}{t},$

   (d) $\dfrac{\sinh^2 t}{t}.$

9. Suppose f is continuous on $[0, T]$ and $f(t + T) = f(t)$ for all $t \geq 0$. We say f is periodic with period T.

   (a) Show that the Laplace transform $L(f)$ is defined for $s > 0$.

   (b) Show that

$$F(s) = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt, \quad s > 0$$

10. Find the Laplace transform of the following periodic functions.

    (a) $f(t) = \begin{cases} t, & 0 \leq t < 1 \\ 2 - t, & 1 \leq t < 2 \end{cases}, \quad f(t+2) = f(t), \ t \geq 0.$

    (b) $f(t) = \begin{cases} 1, & 0 \leq t < 1/2 \\ -1, & 1/2 \leq t < 1 \end{cases}, \quad f(t+1) = f(t), \ t \geq 0.$

    (c) $f(t) = \begin{cases} \sin t, & 0 \leq t < \pi \\ 0, & \pi \leq t < 2\pi \end{cases}, \quad f(t+2\pi) = f(t), \ t \geq 0.$

    (d) $f(t) = |\sin t|.$

11. Find the inverse Laplace transform of the following functions.

    (a) $\dfrac{3}{(s-7)^4},$

    (b) $\dfrac{2s - 4}{s^2 - 4s + 13},$

    (c) $\dfrac{s^2 - 1}{(s^2 + 1)^2},$

    (d) $\dfrac{s^2 - 4s + 3}{(s^2 - 4s + 5)^2},$

    (e) $\dfrac{s^3 + 2s^2 - s - 3}{(s+1)^4},$

    (f) $\dfrac{3 - (s+1)(s-2)}{(s+1)(s+2)(s-2)},$

    (g) $\dfrac{3 + (s-2)(10 - 2s - s^2)}{(s-2)(s+2)(s-1)(s+3)},$

(h) $\dfrac{2+3s}{(s^2+1)(s+2)(s+1)}$,

(i) $\dfrac{3s+2}{(s^2+4)(s^2+9)}$,

(j) $\dfrac{17s-15}{(s^2-2s+5)(s^2+2s+10)}$,

(k) $\dfrac{2s+1}{(s^2+1)(s-1)(s-3)}$.

12. Solve the following IVP's using Laplace transforms.

(a) $y'' + 3y' + 2y = e^t,\ y(0) = 1,\ y'(0) = -6,$

(b) $y'' - 3y' + 2y = 2e^{3t},\ y(0) = 1,\ y'(0) = -1$

(c) $y'' + y = \sin 2t,\ y(0) = 0,\ y'(0) = 1,$

(d) $y'' + 4y = 3\sin t,\ y(0) = 1,\ y'(0) = -1.$

(e) $y'' + y = t,\ y(0) = 0,\ y'(0) = 2,$

(f) $y'' + 2y' + y = 6\sin t - 4\cos t,\ y(0) = -1,\ y'(0) = 1.$

(g) $y'' - 5y' + 6y = 10e^t \cos t,\ y(0) = 2,\ y'(0) = 1,$

(h) $y'' + 4y' + 5y = e^{-t}(\cos t + 3\sin t),\ y(0) = 0,\ y'(0) = 4.$

Tutorial Sheet 6

1. Find the Laplace transform of the following functions using the Laplace transform of step functions.

   (a) $f(t) = \begin{cases} te^t, & 0 \le t < 1 \\ e^t, & t \ge 1 \end{cases}$

   (b) $f(t) = \begin{cases} t, & 0 \le t < 1 \\ t^2, & 1 \le t < 2 \\ 0, & t \ge 2 \end{cases}$

2. Find the inverse Laplace transform of the following functions.

   (a) $H(s) = \dfrac{e^{-\pi s}(1-2s)}{s^2+4s+5}$.

   (b) $H(s) = \dfrac{1}{s} - \dfrac{2}{s^3} + e^{-2s}\left(\dfrac{3}{s} - \dfrac{1}{s^2}\right) + e^{-3s}\left(\dfrac{4}{s} + \dfrac{3}{s^2}\right)$.

3. Solve the following IVPs using Laplace transform.

   (a) $y'' - y = \begin{cases} e^{2t}, & 0 \le t < 2 \\ 1, & t \ge 2 \end{cases}$. $\quad y(0) = 3, \;\; y'(0) = -1.$

   (b) $y'' - 5y' + 4y = \begin{cases} 1, & 0 \le t < 1 \\ -1, & 1 \le t < 2 \\ 0, & t \ge 2 \end{cases}$. $\quad y(0) = 3, \;\; y'(0) = -5.$

   (c) $y'' + 9y = \begin{cases} \cos t, & 0 \le t < \dfrac{3\pi}{2} \\ \sin t, & t \ge \dfrac{3\pi}{2} \end{cases}$ $\quad y(0) = 0, \;\; y'(0) = 0.$

   (d) $y'' + y = \begin{cases} t, & 0 \le t < \pi \\ -t, & t \ge \pi \end{cases}$. $\quad y(0) = 0, \;\; y'(0) = 0.$

   (e) $y'' - 3y' + 2y = \begin{cases} 0, & 0 \le t < 2 \\ 2t-4, & t \ge 2 \end{cases}$. $\quad y(0) = 0, \;\; y'(0) = 0.$

   (f) $y'' + 2y' + y = \begin{cases} e^t, & 0 \le t < 1 \\ e^t - 1, & t \ge 1 \end{cases}$. $\quad y(0) = 3, \;\; y'(0) = -1.$

   (g) $y'' + 2y' + 2y = \begin{cases} t^2, & 0 \le t < 1 \\ -t, & 1 \le t < 2 \\ -1, & t \ge 3\pi \end{cases}$. $\quad y(0) = 2, \;\; y'(0) = -1.$

4. Solve the IVP and find a formula in terms of f for the solution that does not involve any step functions and represents y on each interval of continuity of f

   (a) $y'' + y = f(t) \;\; y(0) = 0, \;\; y'(0) = 0;$
   $f(t) = m+1, \;\; m\pi \le t < (m+1)\pi, \;\; m = 0, 1, \ldots.$

(b) $y'' + y = f(t)\ \ y(0) = 0,\ \ y'(0) = 0;$
$f(t) = (-1)^m,\ \ m\pi \le t < (m+1)\pi,\ \ m = 0, 1, \ldots.$

(c) $y'' - y = f(t)\ \ y(0) = 0,\ \ y'(0) = 0;$
$f(t) = m + 1,\ \ m\pi \le t < (m+1)\pi,\ \ m = 0, 1, \ldots.$

Hint: You will need the formula for $1 + r + \ldots + r^m = \dfrac{1 - r^{m+1}}{1 - r}\ (r \ne 1).$

(d) $y'' + 2y' + 2y = f(t)\ \ y(0) = 0,\ \ y'(0) = 0;$
$f(t) = (m+1)(\sin t + 2\cos t),\ \ 2m\pi \le t < 2(m+1)\pi,\ \ m = 0, 1, \ldots.$

5. Express the following inverse transform as an integral.

(a) $\dfrac{1}{s^2(s^2+4)}$

(b) $\dfrac{s}{s^2(s^2+4)}$

(c) $\dfrac{s}{(s+2)(s^2+9)}$

(d) $\dfrac{1}{(s+1)^2(s^2+4s+5)}$

(e) $\dfrac{1}{s^2(s-2)^3}$

6. Find the Laplace transform

(a) $\int_0^t \sin a\tau \cos b(t-\tau)\ d\tau.$

(b) $\int_0^t \sinh a\tau \cosh b(t-\tau)\ d\tau.$

(c) $e^t \int_0^t \sin \omega\tau \cos \omega(t-\tau)\ d\tau.$

(d) $e^t \int_0^t e^{2\tau} \sinh(t-\tau)\ d\tau.$

(e) $\int_0^t (t-\tau)^4 \sin 2\tau\ d\tau.$

(f) $\int_0^t (t-\tau)^7 e^{-\tau} \sin 2\tau\ d\tau.$

(g) $\int_0^t (t-\tau)^7 \tau^8\ d\tau$

(h) $\int_0^t (t-\tau)^6 \tau^7\ d\tau$

(i) $\int_0^t e^{-\tau} \sin(t-\tau)\ d\tau$

7. Find a formula for the solutions of the IVP.

(a) $y'' + 3y' + y = f(t),\ \ y(0) = 0,\ y'(0) = 0.$

(b) $y'' + 4y = f(t),\ \ y(0) = 0,\ y'(0) = 0.$

(c) $y'' + 6y' + 9y = f(t),\ \ y(0) = 0,\ y'(0) = -2.$

(d) $y'' + \omega^2 y = f(t),\ \ y(0) = a,\ y'(0) = b.$

(e) $y'' - 5y' + 6y = f(t),\ \ y(0) = 1,\ y'(0) = 3.$

8. Solve the integral equation

   (a) $y(t) = t - \int_0^t (t-\tau)y(\tau)\, d\tau$.

   (b) $y(t) = 1 + 2\int_0^t \cos(t-\tau)y(\tau)\, d\tau$.

   (c) $y(t) = t + \int_0^t y(\tau)e^{-(t-\tau)}\, d\tau$.

9. Show that $f * g = g * f$.

10. Show that if $p(s) = as^2 + bs + c$ has distinct real zeros r_1 and r_2 then the solution of

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \ y'(0) = k_1$$

is

$$y(t) = k_0 \frac{r_2 e^{r_1 t} - r_2 e^{r_2 t}}{r_2 - r_1} + k_1 \frac{e^{r_2 t} - e^{r_1 t}}{r_2 - r_1} + \frac{1}{a(r_2 - r_1)} \int_0^t (e^{r_2 \tau} - e^{r_1 \tau}) f(t-\tau)\, d\tau$$

11. For the above problem find a formula for the solution if the roots of $p(s)$ are repeated and is given by r, and when the roots are complex $\lambda \pm i\omega$.