Tutorial - 4

3. (c) $$y^{(4)} + 4y'' = \sin 2t + te^t + 4$$

$$L = D^2(D^2+4)$$

$$A = D(D^2+4)(D-1)^2$$

$$ALy = 0 \Rightarrow D^3(D^2+4)^2(D-1)^2$$

$$\cancel{1}, \cancel{x}, x^2, \quad \underbrace{\cancel{\cos 2x}, \cancel{\sin 2x}}_{e^x,\ xe^x}, \quad \underbrace{x\cos 2x, x\sin 2x}, $$

$$y(x) = c_1 x^2 + c_2 x\cos 2x + c_3 x\sin 2x + c_4 e^x + c_5 xe^x.$$

(a) $$L = D(D-1)^2$$

$$Ly = t^3 + 2e^t$$

$$A = D^4(D-1)$$

$$AL = D^5(D-1)^3$$

$$y(x) = c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^2 e^x.$$

④ (f) $y^{(4)} - 4y^{(2)} = 3t + \cos t.$

$Ly = 3t + \cos t$

where $L = D^2(D^2-4)$

$A = D^2(D^2+1)$

$AL = D^4(D^2+1)(D^2-4)$

let $y(t) = c_1 t^2 + c_2 t^3 + c_3 \sin t + c_4 \cos t.$

$\Rightarrow y^{(2)} = 2c_1 + 6c_2 t - c_3 \sin t - c_4 \cos t.$

& $y^{(4)} = c_3 \sin t + c_4 \cos t.$

$(c_3 \sin t + c_4 \cos t) - 4(2c_1 + 6c_2 t - c_3 \sin t - c_4 \cos t)$

$= 3t + \cos t.$

$c_3 = 0, \quad c_4 = \frac{1}{5}, \quad c_2 = -\frac{1}{8}, \quad c_1 = 0$

$\therefore y(t) = -\frac{t^3}{8} + \frac{1}{5}\cos t.$

Tutorial - 5

1. (a) $\displaystyle\int_0^\infty \frac{1}{t^2+1}\,dt$

$$= \int_0^1 \frac{dt}{t^2+1} + \int_1^\infty \frac{dt}{t^2+1}$$

$$\le \int_0^1 \frac{dt}{t^2+1} + \int_1^\infty \frac{dt}{t^2}$$

$$= \int_0^1 \frac{dt}{t^2+1} + -\frac{1}{t}\Big|_{t=1}^{\infty}.$$

$$= \int_0^1 \frac{dt}{t^2+1} + 1 < \infty.$$

$\therefore \displaystyle\int_0^\infty \frac{dt}{t^2+1}$ exists.

(b) $\displaystyle\int_1^\infty t^{-2}e^t\,dt$. Note that

$$e^t \ge \frac{t^4}{4!} \quad \forall\ t \ge 1.$$

$$\therefore \lim_{T\to\infty}\int_1^T t^{-2}e^t\,dt \ge \lim_{T\to\infty}\int_1^T \frac{t^2}{4!}\,dt$$

$$= \infty.$$

so, $\int_1^\infty t^{-2} e^t \, dt$ does not exist.

2. (a) $L(\cosh t \sin t)$

$$= L\left(\left(\frac{e^t + e^{-t}}{2}\right) \sin t \right)$$

$$= \frac{1}{2} \left\{ L(e^t \sin t) + L(e^{-t} \sin t) \right\}$$

Recall: $L(e^{at} f(t)) = F(s-a)$
if $L(f) = F(s)$

$$= \frac{1}{2} \left\{ \frac{1}{(s-1)^2+1} + \frac{1}{(s+1)^2+1} \right\}$$

$$= \frac{1}{2} \left\{ \frac{1}{s^2-2s+2} + \frac{1}{s^2+2s+2} \right\}$$

$$= \frac{s^2+2}{(s^2-2s+2)(s^2+2s+2)}.$$

(b) Note that
$$\cosh^2 t = \frac{1}{4} \left(e^{2t} + e^{-2t} + 2 \right)$$

$$\Rightarrow \quad L(\cosh^2 t) = \frac{1}{4}\left(\frac{1}{s-2} + \frac{1}{s+2} + \frac{2}{s}\right).$$

$$= \frac{s^2-2}{(s^2-4)s}$$

(c) $$\sin(t + \pi/4) = \cos\frac{\pi}{4}\sin t + \sin\frac{\pi}{4}\cos t$$

$$\Rightarrow \quad L(\sin(t+\pi/4)) = \frac{\cos\frac{\pi}{4}}{s^2+1} + \frac{s\cdot(\sin\pi/4)}{s^2+1}$$

$$= \frac{1}{\sqrt{2}}\left(\frac{s+1}{s^2+1}\right).$$

(d) $$f(t) = \begin{cases} e^{-t} & \text{if } 0 \le t < 1 \\ e^{-2t} & \text{if } t \ge 1 \end{cases}$$

$$f(t) = e^{-t} + u(t-1)\left(e^{-2t} - e^{-t}\right)$$

where $u(t) = \begin{cases} 1 & \text{if } t \ge 0 \\ 0 & \text{if } t < 0. \end{cases}$

and we know that

$$L(u(t-a)) = \frac{e^{-sa}}{s}$$

$$\therefore L(f(t)) = L(e^{-t}) + L(u(t-1)e^{-2(t)}) - L(u(t-1)e^{-t})$$

$$= \frac{1}{s+1} + e^{-s}\frac{1}{(s+1)+2} - \frac{e^{-s}}{(s+1)+1}$$

$$= \frac{1}{s+1} + \frac{e^{-s}}{s+3} - \frac{e^{-s}}{s+2}$$

(f)

$$f(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 1 & \text{if } t \ge 1. \end{cases}$$

$$= t + u(t-1)(1-t)$$

$$\Rightarrow L(f(t)) = \frac{1}{s^2} + \frac{e^{-s}}{s} - \frac{e^{-s}}{(s+1)^2}$$

3 ⓐ $L(f(t)) = F(s)$

i.e., $$F(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

$$\Rightarrow F^{(k)}(s) = \int_0^\infty (-t)^k e^{-st} f(t)\,dt$$

$$= (-1)^k \int_0^\infty e^{-st} t^k f(t)\,dt$$

$$= (-1)^k L(t^k f(t))$$

$$\Rightarrow \quad L(t^k f(t)) = (-1)^k F^{(k)}(s).$$

(b) By (a), we get

$$L(t^n) = (-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s}\right)$$

$$= \frac{n!}{s^{n+1}}.$$

(4) Let f is piece wise continuous and of exponential order s_0.

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt \quad , \quad s > s_0.$$

$\exists$ M, t_0 s.t

$$|f(t)| \le M e^{s_0 t} \quad \forall \ t \ge t_0.$$

$$\therefore \quad F(s) = \int_0^{t_0} e^{-st} f(t)\,dt + \int_{t_0}^\infty e^{-st} f(t)\,dt$$

$$\Rightarrow \quad |F(s)| \le \int_0^{t_0} e^{-st} |f(t)|\, dt + \int_{t_0}^{\infty} e^{-st} |f(t)|\, dt$$

$$\le C \int_0^{t_0} e^{-st}\, dt + M \int_{t_0}^{\infty} e^{-(s-s_0)t}\, dt$$

where c is some constant depends only on t_0 and f.

$$\therefore \quad |F(s)| \le \underbrace{C \left. \frac{e^{-st}}{-s} \right|_0^{t_0} + M \left. \frac{e^{-(s-s_0)t}}{-(s-s_0)} \right|_{t_0}^{\infty}}_{\to\ 0 \text{ as } s \to \infty}$$

$$\therefore \quad \lim_{s \to \infty} F(s) = 0.$$

5. $L\left(\int_0^t f(\tau)\, d\tau \right)$

Let $g(t) = \int_0^t f(\tau)\, d\tau$

Then $\qquad g'(t) = f(t)$

$$\therefore \quad L(g'(t)) = s\,L(g) - g(0)$$

$$\Rightarrow \quad L(f(t)) = s\,L\left(\int_0^t f(\tau)\,d\tau\right)$$

$\left(\text{since } g(0)=0\right).$

$$\Rightarrow \quad L\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s} L(f(t)).$$

⑥ $\quad f : [0,\infty) \to \mathbb{R}$

piecewise conti and of exponential order s_0.

and $\lim_{t \to 0^+} f(t)$ = exists.

Consider

$$\int_s^\infty F(r)\,dr = \int_s^\infty \left(\int_0^\infty e^{-rt} f(t)\,dt\right) dr$$

$$= \int_0^\infty \left(\int_s^\infty e^{-rt}\,dr\right) f(t)\,dt$$

$$= \int_0^\infty \frac{f(t)}{t} e^{-st}\,dt$$

$$= L\left(\frac{f(t)}{t}\right).$$

7 a Let $g(t) = \int_0^t e^{-s_0 \tau} f(\tau)\, d\tau$

and $|g(t)| \le M \quad \forall\, t \geq 0.$

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \lim_{T\to\infty} \int_0^T e^{-st} f(t)\, dt$$

we want to show that this limit exist if $s > s_0$.

Since $g(t) = \int_0^t e^{-s_0\tau} f(\tau)\, d\tau$

$$\Rightarrow \quad f(t) = g'(t)\, e^{s_0 t}$$

$$\therefore \int_0^T e^{-st} f(t)\, dt = \int_0^T e^{-st} g'(t)\, e^{s_0 t}\, dt$$

$$= \int_0^T e^{(s_0 - s)t} g'(t)\, dt$$

$$= g(T)\, e^{(s_0 - s)T} - (s_0 - s)\int_0^T g(t)\, e^{(s_0 - s)t}\, dt \qquad (*)$$

Note that if $s > s_0$ the right hand side of (*) has a limit exist as $T \to \infty$.

$\therefore$ $F(s)$ exists $\forall\, s > s_0$.

(b) Assume that $F(s_0)$ exists.

That is,

$$F(s_0) = \int_0^\infty e^{-s_0 t} f(t)\, dt \quad \text{exists}$$

$\Rightarrow \exists\, M \geq 0$ s.t

$$\left| \int_0^t e^{-s_0 \tau} f(\tau)\, d\tau \right| \leq M \quad \forall\, t \geq 0$$

$\Rightarrow$ By (a), $F(s)$ exists $\forall\, s > s_0$.

(8) (a) $\mathcal{L}\left(\dfrac{\sin \omega t}{t}\right) = ?$

By problem (6), we have

$$\mathcal{L}\left(\frac{\sin \omega t}{t}\right) = \int_s^\infty \frac{\omega}{r^2 + \omega^2}\, dr$$

let $r = \omega y$

$dr = \omega\, dy$

$$= \frac{1}{\omega} \int_{s/\omega}^{\infty} \frac{1}{y^2+1}\, \omega\, dy$$

$$= \int_{s/\omega}^{\infty} \frac{1}{1+y^2}\, dy$$

$$= \tan^{-1} y \Big|_{y=s/\omega}^{\infty}$$

$$= \pi/2 - \tan^{-1}(s/\omega).$$

(b) $$L\left(\frac{e^{at} - e^{bt}}{t}\right) = \int_s^{\infty} \left(\frac{1}{r-a} - \frac{1}{r-b}\right) dr.$$

$$= \log \frac{r-a}{r-b} \Big|_{r=s}^{\infty}$$

$$= \log \frac{s-b}{s-a}.$$

(c) $$L\left(\frac{\cosh t - 1}{t}\right) = \int_s^{\infty} \left(\frac{r}{r^2-1} - \frac{1}{r}\right) dr$$

$$= \log \frac{\sqrt{r^2-1}}{r} \Bigg|_{r=s}^{\infty}$$

$$= \log \frac{s}{\sqrt{s^2-1}}.$$

(d) $$L\left(\frac{\sin h^2 t}{t}\right) = L\left(\frac{e^{2t}+e^{-2t}-2}{4t}\right)$$

$$= \frac{1}{4}\int_s^{\infty}\left(\frac{1}{r-2}+\frac{1}{r+2}-\frac{2}{r}\right)dr$$

$$= \frac{1}{4}\log \frac{(r-2)(r+2)}{r^2}\Bigg|_{r=s}^{\infty}$$

$$= \frac{1}{4}\log \frac{s^2}{(s-2)(s+2)}$$

$$= \frac{1}{4}\log \frac{s^2}{s^2-4}.$$

(9) (a) Since f is periodic function of period T.

so it is bounded on it is continuous on $[0,T]$

$\therefore \quad \mathcal{L}(f)(s)$ is defined $\forall\ s \geq 0.$

(b) $$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \sum_{r=0}^{\infty} \int_{rT}^{(r+1)T} e^{-st} f(t)\,dt$$

$$= \sum_{r=0}^{\infty} \int_{rT}^{rT+T} e^{-st} f(t)\,dt$$

let $t = rT + t_1$

$dt = dt_1$

$$= \sum_{r=0}^{\infty} \int_0^T e^{-s(rT+t_1)} f(t_1)\,dt_1$$

$$= \int_0^T f(t_1)\, e^{-st_1} \left(\sum_{r=0}^{\infty} e^{-srT} \right) dt_1$$

$$= \frac{1}{1-e^{-sT}} \int_0^T f(t)\, e^{-st}\,dt.$$

$\forall\ s > 0.$

(10) (a) $f(t) = \begin{cases} t & ; \ 0 \le t < 1 \\ 2-t & ; \ 1 \le t < 2 \end{cases}$

$f(t+2) = f(t)$
$\forall \ t \geq 0.$

By (9),

$$F(s) = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t)\,dt.$$

$$= \frac{1}{1-e^{-2s}} \left[\int_0^1 t\, e^{-st}\,dt + \int_1^2 e^{-st} (2-t)\,dt \right]$$

$$\int_0^1 t\, e^{-st}\,dt = \frac{t\, e^{-st}}{-s} \Big|_0^1 + \frac{1}{s} \int_0^1 e^{-st}\,dt$$

$$= \frac{e^{-s}}{-s} + \frac{1}{s} \left[\frac{e^{-st}}{-s} \Big|_{t=0}^{1} \right]$$

$$= \frac{e^{-s}}{-s} + \frac{1}{s} \left[\frac{e^{-s}}{-s} + \frac{1}{s} \right]$$

$$= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}$$

and $\int_1^2 (2-t)\, e^{-st}\, dt$

$$= \left.\frac{e^{-st}(2-t)}{-s}\right|_{t=1}^{2} - \frac{1}{s}\int_1^2 e^{-st}\, dt$$

$$= \frac{e^{-s}}{s} - \frac{1}{s}\left[\left.\frac{e^{-st}}{-s}\right|_{t=1}^{2}\right]$$

$$= \frac{e^{-s}}{s} - \frac{1}{s}\left[\frac{e^{-2s}}{-s} + \frac{e^{-s}}{s}\right]$$

$$= \frac{e^{-s}}{s} + \frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s^2}$$

$$\therefore\ F(s) = \frac{1}{1-e^{-2s}}\left[\frac{1}{s^2} - \frac{e^{-s}}{s^2} - \cancel{\frac{e^{-s}}{s}} + \cancel{\frac{e^{-s}}{s}} + \frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{1}{s^2} + \frac{e^{-2s}}{s^2} - \frac{2e^{-s}}{s^2}\right]$$

(b)

$$f(t) = \begin{cases} 1 & \text{if } \quad 0 \le t < 1/2 \\ -1 & \text{if } \quad \frac{1}{2} \le t < 1 \end{cases}$$

$$f(t+1) = f(t) \quad \forall\, t \geq 0.$$

$$F(s) = \frac{1}{1 - e^{-s/2}} \int_0^1 f(t)\, e^{-st}\, dt$$

$$= \frac{1}{1 - e^{-s/2}} \left[\int_0^{1/2} e^{-st}\, dt - \int_{1/2}^{1} e^{-st}\, dt \right]$$

$$= \frac{1}{1 - e^{-s/2}} \left[\frac{e^{-s/2}}{-s} + \frac{1}{s} + \frac{e^{-s}}{s} - \frac{e^{-s/2}}{s} \right]$$

$$= \frac{1}{1 - e^{-s/2}} \left[\frac{-2e^{-s/2}}{s} + \frac{1}{s} + \frac{e^{-s}}{s} \right].$$

(c)

$$f(t) = \begin{cases} \sin t & \text{if } \quad 0 \le t < \pi \\ 0 & \pi \le t < 2\pi \end{cases}$$

$$f(t+\pi) = f(t) \quad \forall\, t \geq 0.$$

$$F(s) = \frac{1}{1-e^{-\pi s}}\left[\int_0^{\pi} \sin t \, e^{-st}\,dt\right]$$

$$I = \int_0^{\pi} \sin t \, e^{-st}\,dt .$$

$$= \frac{e^{-st}}{-s}\sin t \Big|_0^{\pi} + \frac{1}{s}\int_0^{\pi} \cos t \, e^{-st}\,dt$$

$$= \frac{1}{s}\left[\frac{e^{-st}}{-s}\cos t\Big|_0^{\pi} + \frac{1}{s}\int_0^{\pi} \sin \, e^{-st}\,dt\right]$$

$$I = \frac{1}{s}\left\{\frac{e^{-\pi s}}{s} + \frac{1}{s} - \frac{1}{s} I\right\}$$

$$\Rightarrow \quad s I = \frac{e^{-\pi s}}{s} + \frac{1}{s} - \frac{I}{s}$$

$$\Rightarrow \quad I(s^2 + 1) = e^{-\pi s} + 1$$

$$\Rightarrow \quad I = \frac{e^{-\pi s} + 1}{s^2+1} .$$

$$\therefore \quad F(s) = \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{-\pi s}+1}{s^2+1}\right].$$

(d) $f(t) = |\sin t|$.

$$= \begin{cases} \sin t & \text{if } 0 \le t < \pi \\ -\sin t & \text{if } \pi \le t < 2\pi \end{cases}$$

$$f(t+2\pi) = f(t) \quad \forall\, t \geq 0$$

$$F(s) = \frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi} \sin t\, e^{-st}\,dt - \int_{\pi}^{2\pi} \sin t\, e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi} \sin t\, e^{-st}\,dt + e^{-s\pi}\int_0^{\pi} \sin t\, e^{-st}\right.$$

$$= \frac{1+e^{-s\pi}}{1-e^{-2\pi s}}\left[\int_0^{\pi} \sin t\, e^{-st}\,dt\right]$$

$$= \frac{(1+e^{-s\pi})}{1-e^{-2\pi s}}\,\frac{(1+e^{-\pi s})}{1+s^2}$$

$$= \frac{(1+e^{-s\pi})^2}{(1+s^2)(1-e^{-2\pi s})}$$

11 (a) $L^{-1}\left(\frac{3}{(s-7)^4}\right)$

$\frac{1}{2} e^{t} t^3$ $\quad \frac{3}{(s-1)^4}$

Recall:

$L(f(t)) = F(s) \Rightarrow L(e^{at} f(t)) = F(s-a)$

and $L(t^n) = \frac{n!}{s^{n+1}}$.

$$\therefore L^{-1}\left(\frac{3}{(s-7)^4}\right) = \frac{1}{2} e^{7t} t^3.$$

(b) $$\frac{2s-4}{s^2-4s+13} = \frac{2s-4}{(s-2)^2+9} = \frac{2(s-2)}{(s-2)^2+9}$$

$$\therefore L^{-1}\left(\frac{2(s-2)}{(s-2)^2+9}\right) = 2e^{2t} L^{-1}\left(\frac{s}{s^2+9}\right)$$

$$= 2e^{2t} \cos 3t.$$

(6) $$\frac{s^2-1}{(s^2+1)^2} = \frac{s^2+1-2}{(s^2+1)^2} = \frac{1}{s^2+1} - \frac{2}{(s^2+1)^2}$$

Let $$f(t) = \int_0^t \sin(t-\tau)\sin\tau \, d\tau$$

$$f'(t) = \int_0^t \cos(t-\tau)\sin\tau \, d\tau$$

$$f''(t) = \sin t - \int_0^t \sin(t-\tau)\sin\tau \, d\tau$$

$$\therefore \quad f''(t) = \sin t - f(t)$$

$$f''(t) + f(t) = \sin t.$$

$$u_1 \underbrace{\cos t}_{y_1} + u_2 \underbrace{\sin t}_{y_2}$$

$$W = 1$$

$$u_1 = \int -\frac{y_2 \sin t}{} \, dt$$

$$= -\int \sin^2 t \, dt$$

$$= \frac{1}{4}\sin 2t + \frac{t}{2}$$

$$v_2 = \int y_1 \sin t \, dt$$

$$= \int \cos t \sin t \, dt$$

$$= \frac{1}{2} \sin^2 t$$

$$\therefore \quad f(t) = \left(\frac{1}{4} \sin 2t + \frac{t}{2}\right) \cos t$$

$$+ \frac{1}{2} \sin^2 t \sin t$$

$$= \frac{1}{4} \sin 2t \cdot \cos t + \frac{t \cos t}{2}$$

$$+ \frac{1}{2} \sin^3 t .$$

$$\therefore \quad L^{-1}\left(\frac{s^2-1}{(s^2+1)^2}\right) = \sin t - 2 f(t)$$

$$= \sin t - \frac{1}{2} \sin 2t \cos t$$

$$- t \cos t - \sin^3 t .$$

(d) $$\frac{s^2-4s+3}{(s^2-4s+5)^2} = \frac{(s^2-4s+5)-2}{(s^2-4s+5)^2}$$

$$= \frac{(s-2)^2 - 1}{\left((s-2)^2+1\right)^2}$$

$$\therefore \quad L^{-1}\left(\frac{(s-2)^2-1}{\left((s-2)^2+1\right)^2}\right) = e^{2t} L^{-1}\left(\frac{s^2-1}{(s^2+1)^2}\right)$$

by previous problem.

$$= e^{2t}\left(\sin t - \frac{1}{2}\sin 2t \cos t - t\cos t - \sin^3 t\right)$$

(e) $$\frac{s^3+2s^2-s-3}{(s+1)^4} = \frac{(s+1)^3 - (s+1)^2 + 6(s+1) - 9}{(s+1)^4}$$

$$= \frac{1}{(s+1)} - \frac{1}{(s+1)^2} + \frac{6}{(s+1)^3} - \frac{9}{(s+1)^4}$$

$$\Rightarrow \quad L^{-1}\left(\frac{s^3+2s^2-s-3}{(s+1)^4}\right) = e^{-t} L^{-1}\left(\frac{1}{s} - \frac{1}{s^2} + \frac{6}{s^3} - \frac{9}{s^4}\right)$$

$$= e^{-t}\left(1 - t + 3t^2 - \frac{3}{2}t^3\right)$$

(f) $$\frac{3-(s+1)(s-2)}{(s+1)(s+2)(s-2)} = \frac{3}{(s+1)(s+2)(s-2)} - \frac{1}{s+2}$$

$$= \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-2}$$

$$\Rightarrow \quad A = -\frac{1}{3}, \quad B = \frac{1}{5}, \quad C = \frac{1}{12}$$

$$\frac{3}{(s+1)(s+2)(s-2)} = -\frac{1}{s+1} + \frac{3}{5(s+2)} + \frac{1}{4(s-2)}$$

$$\therefore \quad L^{-1}\left(\frac{3-(s+1)(s-2)}{(s+1)(s+2)(s-2)}\right) = -e^{-t} + \frac{3}{5}e^{-2t} + \frac{1}{4}e^{2t} - e^{-2t}.$$

$$= -e^{-t} - \frac{2}{5}e^{-2t} + \frac{1}{4}e^{2t}.$$

(g) $$\frac{3+(s-2)(10-2s-s^2)}{(s-2)(s+2)(s-1)(s+3)}$$

$$= \frac{A}{s-2} + \frac{B}{s+2} + \frac{C}{s-1} + \frac{D}{s+3}$$

$$\Rightarrow \quad A = \frac{3}{20}, \quad B = \frac{37}{12}, \quad C = \frac{1}{3}, \quad D = \frac{8}{5}$$

$$\therefore \; L^{-1}(\;) = \frac{3}{20} e^{2t} + \frac{37}{12} e^{-2t} + \frac{1}{3} e^{t} + \frac{8}{5} e^{-3t}.$$

(h) $$\frac{2+3s}{(s^2+1)(s+2)(s+1)} = \frac{As+B}{s^2+1} + \frac{C}{s+2} + \frac{D}{s+1}$$

$$= \frac{As+\frac{11}{10}}{s^2+1} + \frac{4}{5(s+2)} - \frac{1}{2(s+1)}$$

$$L^{-1}(\;) = A\cos t + \frac{11}{10}\sin t + \frac{4}{5}e^{-2t} - \frac{1}{2}e^{-t}.$$

(find out value of A).

(i) $$\frac{3s+2}{(s^2+4)(s^2+9)} = \frac{1}{5}\left(\frac{3s+2}{s^2+4} - \frac{3s+4}{s^2+9}\right)$$

$$= \frac{1}{5}\left(\frac{3s}{s^2+4} + \frac{2}{s^2+4} - \frac{3s}{s^2+9} - \frac{4}{s^2+9}\right)$$

$$L^{-1}(\;) = \frac{3}{5}\cos 2t + \frac{1}{5}\sin 2t - \frac{3}{5}\cos 3t$$

$$- \frac{4}{15}\sin 3t.$$

12. We will look at :

$$y'' + a y' + b y = f(t) \, , \quad y(0) = \alpha$$
$$y'(0) = \beta$$

apply Laplace transform both sides.

$$s^2 Y(s) - \alpha s - \beta + a\left(s Y(s) - \alpha\right) + b Y(s)$$
$$= F(s)$$

$$Y(s)\left[s^2 + as + b \right] - \alpha s - \beta - a\alpha = F(s)$$

$$Y(s) = \frac{\alpha s + \beta + a\alpha + F(s)}{s^2 + as + b}$$

$$= \frac{\alpha s + (\beta + a\alpha)}{s^2 + as + b} + \frac{F(s)}{s^2 + as + b}.$$

$$\Rightarrow \quad y(t) = L^{-1}\left(\frac{\alpha s + (\beta + a\alpha)}{s^2 + as + b} + \frac{F(s)}{s^2 + as + b} \right).$$

(a) $\quad y'' + 3y' + 2y = e^{t} \, , \; y(0) = 1, \; y'(0) = -6$

$$\alpha = 1, \quad \beta = -6$$

$a = 3,\ b = 2$

$f(t) = e^t$

$$y(t) = \mathcal{L}^{-1}\left(\frac{s-3}{s^2+3s+2}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s-1)(s^2+3s+2)}\right)$$

$$\frac{s-3}{s^2+3s+2} = \frac{s-3}{(s+1)(s+2)} = \frac{-4}{s+1} + \frac{5}{s+2}$$

and $$\frac{1}{(s-1)(s+1)(s+2)} = \frac{1/6}{s-1} + \frac{-1/2}{s+1} + \frac{1/3}{s+2}$$

$$\therefore\ y(t) = -4e^{-t} + 5e^{-2t} + \frac{1}{6}e^{t} - \frac{1}{2}e^{-t} + \frac{1}{3}e^{-2t}$$

$$y(t) = -\frac{9}{2}e^{-t} + \frac{16}{3}e^{-2t} + \frac{1}{6}e^{t}.$$

(c) $y'' + y = \sin 2t$, $y(0) = 0$, $y'(0) = 1$.

$\alpha = 0,\ \beta = 1$

$a = 0,\ b = 1$

$f(t) = \sin 2t$

$$y(t) = L^{-1}\left(\frac{1}{s^2+1} + \frac{2}{(s^2+4)(s^2+1)} \right)$$

$$= \frac{5}{3} \sin t - \frac{1}{3} \sin 2t$$

(h) $$y'' + 4y' + 5y = e^{-t}(\cos t + 3\sin t)$$

$$y(0) = 0, \quad y'(0) = 4.$$

$$\alpha = 0, \quad \beta = 4$$

$$a = 4, \quad b = 5$$

$$y(t) = L^{-1}\left(\frac{4s+4}{s^2+4s+5} \right) +$$

$$L^{-1}\left(\left(\frac{s+1}{(s+1)^2+1} + \frac{3}{(s+1)^2+1} \right) \frac{1}{s^2+4s+5} \right)$$

$$= 4 e^{-2t}(\cos t - \sin t) +$$

$$e^{-t} L^{-1}\left(\left(\frac{s}{s^2+1} + \frac{3}{s^2+1} \right) \frac{1}{s^2+2s+3} \right)$$

$$L^{-1}\left(\frac{1}{s^2+2s+3} \right) = L^{-1}\left(\frac{1}{(s+1)^2+2} \right) = e^{-t} L^{-1}\left(\frac{1}{s^2+2} \right)$$

$$= \frac{e^{-t}}{\sqrt{2}} \sin \sqrt{2}\, t$$

$$L^{-1}\left(\left(\frac{s}{s^2+1}+\frac{3}{s^2+1}\right)\frac{1}{s^2+2s+3}\right)$$

$$= \frac{1}{\sqrt{2}}\left[(\cos t + 3\sin t) * e^{-t}\sin(\sqrt{2}\,t)\right]$$

convolution.

$$\therefore\ y(t) = 4e^{-2t}(\cos t - \sin t)$$

$$+ \frac{e^{-t}}{\sqrt{2}}\left[(\cos t + 3\sin t) * (e^{-t}\sin\sqrt{2}\,t)\right].$$

Tutorial - 6

1 (b). $f(t) = \begin{cases} t & 0 \le t < 1 \\ t^2 & 1 \le t < 2 \\ 0 & t \geq 2 \end{cases}$

$$= t + u(t-1)\,(t^2 - t) - u(t-2)\, t^2$$

$$L(f(t)) = \frac{1}{s^2} + e^{-s}\, L\left((t+1)^2 - (t+1)\right) - e^{-2s}\, L\left((t+2)^2\right)$$

$$= \frac{1}{s^2} + e^{-s}\left\{\frac{2}{s^3} + \frac{1}{s^2}\right\} - e^{-2s}\left\{\frac{2}{s^3} + \frac{2}{s^2} + \frac{4}{s}\right\}$$

2. (b) $L^{-1}(H(s)) = 1 - t^2 + u(t-2)\left\{3 - (t-2)\right\}$

$$+ u(t-3)\left(4 + 3(t-3)\right)$$

$$= 1 - t^2 + u(t-2)\,(5 - t)$$

$$+ u(t-3)\,(3t - 5).$$

(3) (a) $y'' - y = \begin{cases} e^{2t}, & 0 \le t < 2 \\ 1, & t \geq 2 \end{cases}$

$$y(0) = 3, \quad y'(0) = -1.$$

$$a = 0, \; b = -1$$

$$\alpha = 3, \; \beta = -1$$

$$F(s) = L\left(e^{2t} + u(\cancel{t}-2)\left(1 - e^{2t}\right)\right)$$

$$= \frac{1}{s-2} + e^{-2s}\left(\frac{1}{s} - \frac{e^2}{s-2}\right)$$

$$y(t) = L^{-1}\left(\frac{\alpha s + (\beta + a\alpha)}{s^2 + as + b} + \frac{F(s)}{s^2+as+b}\right)$$

$$= L^{-1}\left(\frac{3s-1}{s^2-1} + \frac{1}{(s^2-1)(s-2)} + e^{-2s}\frac{1}{s(s^2-1)} - \frac{e^{2-2s}}{(s-2)(s^2-1)}\right)$$

$$\frac{3s-1}{s^2-1} = \frac{1}{s-1} + \frac{2}{s+1}$$

$$\frac{1}{(s^2-1)(s-2)} = \frac{(-1/2)}{s-1} + \frac{(\frac{1}{6})}{s+1} + \frac{(1/3)}{s-2}$$

$$\frac{1}{s(s^2-1)} = \frac{-1}{s} + \frac{(1/2)}{s-1} + \frac{(-1/2)}{s+1}$$

$$\therefore\ y(t) = \frac{1}{2}e^{t} + \frac{13}{6}e^{-t} + \frac{1}{3}e^{2t}$$

$$+ u(t-2)\left(-1 + \frac{1}{2}e^{(t-2)} - \frac{1}{2}e^{-t+2}\right)$$

$$- e^{2}\, u(t-2)\left(-\frac{1}{2}e^{t-2} + \frac{1}{6}e^{-t+2} + \frac{1}{3}e^{2t-4}\right)$$

(b)

$$y(t) = \mathcal{L}^{-1}\left(\frac{3s-20}{s^2-5s+4} + \frac{F(s)}{s^2-5s+4}\right)$$

where

$$F(s) = \mathcal{L}\big(1-2u(t-1) + u(t-2)\big)$$

$$= \frac{1}{s} - 2\frac{e^{-s}}{s} + \frac{e^{-2s}}{s}$$

$$\therefore\ \frac{3s-20}{s^2-5s+4} = \frac{(-17/3)}{(s-1)} + \frac{(-8/3)}{s-4}$$

$$\frac{1}{s(s-1)(s-4)} = \frac{\frac{1}{4}}{s} + \frac{(-\frac{1}{3})}{s-1} + \frac{\frac{1}{12}}{s-4}$$

$$\therefore\ y(t) = -6e^{t} - \frac{23}{12}e^{4t} + \frac{1}{4}$$
$$-2\ u(t-1)\left\{\frac{1}{4} - \frac{1}{3}e^{t-1} + \frac{1}{12}e^{4t-4}\right\}$$
$$+\ u(t-2)\left\{\frac{1}{4} - \frac{1}{3}e^{t-2} + \frac{1}{12}e^{4t-8}\right\}.$$

5. (a) $\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+4)}\right)$

$= f * g$

whe $f(t) = t,\ g(t) = \frac{1}{2}\sin 2t.$

(c) $\mathcal{L}^{-1}\left(\frac{s}{(s+2)(s^2+9)}\right) = f * g$

When $f(t) = e^{-2t}$

$g(t) = \cos 3t$.

⑥ ⓐ $L\left(\int_0^t \sin a\tau \cos b(t-\tau)\, d\tau \right)$

$$= \frac{as}{(s^2+a^2)(s^2+b^2)}$$

ⓑ $L\left(\int_0^t \sinh a\tau \cosh b(t-\tau)\, d\tau \right)$

$$= \frac{as}{(s^2-a^2)(s^2-b^2)}$$

ⓗ $L\left(\int_0^t (t-\tau)^6 \tau^7\, d\tau \right)$

$$= \frac{6!\,7!}{s^{15}}.$$

(i) $L\left(\int_0^t e^{-\tau} \sin(t-\tau)\, d\tau \right)$

$$= \frac{1}{(s+1)(s^2+1)}.$$

⑦ ⓐ $y'' + 3y' + y = f(t)$, $y(0) = 0$, $y'(0) = 0$.

$$y(t) = \mathcal{L}^{-1}\left(\frac{F(s)}{s^2+3s+1} \right)$$

$$\frac{1}{s^2+3s+1} = \frac{1}{(s-r_1)(s-r_2)} = \frac{(1/\sqrt{5})}{s-r_1} + \frac{(-1/\sqrt{5})}{s-r_2}$$

$$r_1 = -\frac{3}{2} + \frac{\sqrt{5}}{2}$$

$$r_2 = -\frac{3}{2} - \frac{\sqrt{5}}{2}$$

$$y(t) = f(t) * g(t)$$

where $g(t) = \frac{1}{\sqrt{5}}\left(e^{r_1 t} - e^{r_2 t}\right).$

8 a

$$y(t) = t - \int_0^t (t-\tau)\, y(\tau)\, d\tau$$

$$\Rightarrow Y(s) = \frac{1}{s^2} - \frac{Y(s)}{s^2}$$

$$\Rightarrow \left(1 + \frac{1}{s^2}\right) Y(s) = \frac{1}{s^2}$$

$$\Rightarrow Y(s) = \frac{1}{s^2+1}$$

$$\Rightarrow y(t) = \sin t.$$