MA 110
Linear Algebra and Differential Equations
Lecture 19

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Recall: In the last lecture, we proved the following theorem and then discussed its applications to classification of conic sections and quadric surfaces.

Theorem (Principal Axis Theorem)

Let Q be a real quadratic form and $\mathbf{A} \in \mathbb{R}^{n\times n}$ be the symmetric matrix associated with Q. If $\mathbf{C}$ is an orthogonal matrix such that the matrix $\mathbf{D} := \mathbf{C}^{\mathsf{T}}\mathbf{A}\,\mathbf{C}$ is diagonal, then

$$Q(\mathbf{x}) = Q_{\mathbf{D}}(\mathbf{y}), \quad \textit{where} \quad \mathbf{y} := \mathbf{C}^{\mathsf{T}}\mathbf{x} \textit{ for all } \mathbf{x} \in \mathbb{R}^{n\times 1}.$$

Clarification about a Degenerate Conic: As discussed earlier, a **conic section** is the locus in $\mathbb{R}^2$ of an equation $f(x,y) = 0$, where $f(x,y) = a\,x^2 + b\,y^2 + c\,xy + a'x + b'y + c'$, where $a, b, c, a', b', c' \in \mathbb{R}$ and at least one among a, b, c is nonzero. We call this **degenerate** if $f(x,y)$ factors as a product of two linear polynomials (with coefficients in $\mathbb{C}$). This corresponds to the locus in $\mathbb{R}^2$ being a pair of lines (including the case of coincident lines), a single point, or the empty set.

Equation	Surface	Eigenvalues of A
$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	ellipsoid	all three positive
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z}{c}=0$	elliptic paraboloid	two positive, one zero
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$	elliptic cone	two positive, one negative
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	1-sheeted hyperboloid	two positive, one negative
$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	2-sheeted hyperboloid	one positive, two negative
$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z}{c}=0$	hyperbolic paraboloid	one positive, one negative, one zero.

We also discussed the above classification of quadric surface (leaving aside the degenrate cases).

Example Consider the quadric surface given by

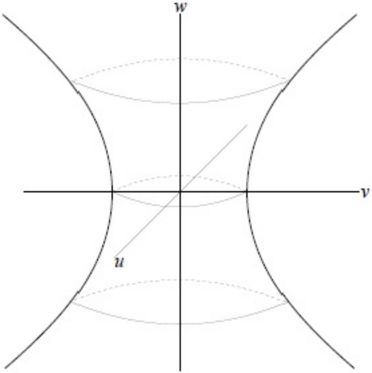
$$x^2 + y^2 + z^2 + 4xy + 4yz - 4zx - 27 = 0,$$

and the associated ternary quadratic form
$Q(x, y, z) := x^2 + y^2 + z^2 + 4xy + 4yz - 4zx$.

We have already transformed the associated symmetric matrix
$\mathbf{A} := \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}$ to a diagonal form, and have obtained
$Q(x, y, z) = Q_{\mathbf{D}}(u, v, w) = 3(u^2 + v^2 - w^2)$ (with x_1, x_2, x_3 and y_1, y_2, y_3 in place of x, y, z and u, v, w),
where $\mathbf{D} := \text{diag}(3, 3, -3)$ and
$u = (x+y)/\sqrt{2}$, $v = (-x+y+2z)/\sqrt{6}$, $w = (x-y+z)/\sqrt{3}$.

Under this change of coordinates, the quadric surface reduces to $u^2 + v^2 - w^2 = 9$.

This is an equation of a one-sheeted hyperboloid in the uvw-space, as shown in the following figure, where the u-axis, the v-axis and the w-axis are determined by the eigenvectors $\mathbf{u}_1 := \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}^\mathsf{T}$, $\mathbf{u}_2 := \begin{bmatrix} -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}^\mathsf{T}$ and $\mathbf{u}_3 := \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}^\mathsf{T}$. (See Lecture 16.)



Orthogonal Projection

Let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Recall that in Lecture 13, we have defined the (perpendicular) projection of $\mathbf{x} \in \mathbb{K}^{n\times 1}$ in the direction of nonzero $\mathbf{y} \in \mathbb{K}^{n\times 1}$ as follows:

$$P_{\mathbf{y}}(\mathbf{x}) := \frac{\langle \mathbf{y},\, \mathbf{x}\rangle}{\langle \mathbf{y},\, \mathbf{y}\rangle}\,\mathbf{y}.$$

In particular, if $\mathbf{y}$ is a unit vector, then $P_{\mathbf{y}}(\mathbf{x}) = \langle \mathbf{y},\, \mathbf{x}\rangle \mathbf{y}$.

We noted that the vector $P_{\mathbf{y}}(\mathbf{x})$ is a scalar multiple of the vector $\mathbf{y}$, and proved the important relation

$$\big(\mathbf{x} - P_{\mathbf{y}}(\mathbf{x})\big) \perp \mathbf{y}.$$

As a consequence,

$$\begin{aligned} \|\mathbf{x} - P_{\mathbf{y}}(\mathbf{x})\|^2 &= \langle \mathbf{x} - P_{\mathbf{y}}(\mathbf{x}),\, \mathbf{x} - P_{\mathbf{y}}(\mathbf{x})\rangle \\ &= \langle \mathbf{x},\, \mathbf{x} - P_{\mathbf{y}}(\mathbf{x})\rangle \\ &= \|\mathbf{x}\|^2 - \langle \mathbf{x},\, P_{\mathbf{y}}(\mathbf{x})\rangle. \end{aligned}$$

More generally, let Y be a nonzero subspace of $\mathbb{K}^{n\times 1}$. We would like to find a (perpendicular) projection of $\mathbf{x} \in \mathbb{K}^{n\times 1}$ into Y, that is, we want to find $\mathbf{y} \in Y$ such that $(\mathbf{x} - \mathbf{y}) \in Y^{\perp}$. (This $\mathbf{y}$ is 'the foot of the perpendicular' from $\mathbf{x}$ into Y.)

If $\mathbf{u}_1, \ldots, \mathbf{u}_k$ is an orthonormal basis for the subspace Y, then a vector belongs to $Y^{\perp}$ if and only if it is orthogonal to each $\mathbf{u}_j$ for $j = 1, \ldots, k$. As we saw while studying G-S OP, the vector

$$\tilde{\mathbf{y}} := \mathbf{x} - P_{\mathbf{u}_1}(\mathbf{x}) - \cdots - P_{\mathbf{u}_k}(\mathbf{x}) = \mathbf{x} - \langle \mathbf{u}_1, \mathbf{x} \rangle \mathbf{u}_1 - \cdots - \langle \mathbf{u}_k, \mathbf{x} \rangle \mathbf{u}_k$$

is orthogonal to each $\mathbf{u}_j$ for $j = 1, \ldots, k$, and so $\tilde{\mathbf{y}} \in Y^{\perp}$.

Since the vector $\mathbf{y} := \langle \mathbf{u}_1, \mathbf{x} \rangle \mathbf{u}_1 + \cdots + \langle \mathbf{u}_k, \mathbf{x} \rangle \mathbf{u}_k$ belongs to Y, it is a (perpendicular) projection of $\mathbf{x}$ in Y.

The following result shows that this is the only vector in Y that works!

Proposition (Projection Theorem)

Let Y be a subspace of $\mathbb{K}^{n\times 1}$. Then for every $\mathbf{x} \in \mathbb{K}^{n\times 1}$, there are unique $\mathbf{y} \in Y$ and $\tilde{\mathbf{y}} \in Y^\perp$ such that $\mathbf{x} = \mathbf{y} + \tilde{\mathbf{y}}$, that is, $\mathbb{K}^{n\times 1} = Y \oplus Y^\perp$. The map $P_Y : \mathbb{K}^{n\times 1} \to \mathbb{K}^{n\times 1}$ given by $P_Y(\mathbf{x}) = \mathbf{y}$ is linear and satisfies $(P_Y)^2 = P_Y$.

In fact, if $\mathbf{u}_1, \ldots, \mathbf{u}_k$ is an orthonormal basis for Y, then

$$P_Y(\mathbf{x}) = \langle \mathbf{u}_1, \mathbf{x}\rangle \mathbf{u}_1 + \cdots + \langle \mathbf{u}_k, \mathbf{x}\rangle \mathbf{u}_k.$$

Proof. If $Y = \{\mathbf{0}\}$, then every $\mathbf{x} \in Y^\perp$, and so $\mathbf{x} = \mathbf{0} + \mathbf{x}$.

Suppose $Y \neq \{\mathbf{0}\}$, and let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ be an orthonormal basis for Y. For $\mathbf{x} \in \mathbb{K}^{n\times 1}$, define

$$P_Y(\mathbf{x}) := \langle \mathbf{u}_1, \mathbf{x}\rangle \mathbf{u}_1 + \cdots + \langle \mathbf{u}_k, \mathbf{x}\rangle \mathbf{u}_k.$$

Clearly, $\mathbf{y} := P_Y(\mathbf{x}) \in Y$. Define $\tilde{\mathbf{y}} := \mathbf{x} - P_Y(\mathbf{x})$. Then

$$
\begin{aligned}
\langle \mathbf{u}_j, \tilde{\mathbf{y}} \rangle &= \langle \mathbf{u}_j, \mathbf{x} - P_Y(\mathbf{x}) \rangle \\
&= \langle \mathbf{u}_j, \mathbf{x} \rangle - \sum_{\ell=1}^{k} \langle \mathbf{u}_\ell, \mathbf{x} \rangle \langle \mathbf{u}_j, \mathbf{u}_\ell \rangle \\
&= \langle \mathbf{u}_j, \mathbf{x} \rangle - \langle \mathbf{u}_j, \mathbf{x} \rangle = 0
\end{aligned}
$$

for $j = 1, \ldots, k$ by orthonormality. Hence $\tilde{\mathbf{y}} \in Y^\perp$. Thus $\mathbf{x} = \mathbf{y} + \tilde{\mathbf{y}}$ with $\mathbf{y} \in Y$ and $\tilde{\mathbf{y}} \in Y^\perp$. This proves existence.

To prove uniqueness, let $\mathbf{z} \in Y$ and $\tilde{\mathbf{z}} \in Y^\perp$ be such that $\mathbf{x} = \mathbf{z} + \tilde{\mathbf{z}}$. Then $(\mathbf{y} - \mathbf{z}) \in Y$ and also $\mathbf{y} - \mathbf{z} = (\tilde{\mathbf{z}} - \tilde{\mathbf{y}}) \in Y^\perp$, so that $(\mathbf{y} - \mathbf{z}) \in Y \cap Y^\perp$. Hence $(\mathbf{y} - \mathbf{z}) \perp (\mathbf{y} - \mathbf{z})$, and so $\mathbf{y} - \mathbf{z} = \mathbf{0}$. Thus $\mathbf{z} = \mathbf{y}$, and in turn, $\tilde{\mathbf{z}} = \tilde{\mathbf{y}}$.

The map P_Y is linear since the inner product is linear in the second variable. Also, $P_Y(\mathbf{u}_j) = \mathbf{u}_j$ for all $j = 1, \ldots, k$. Hence $P_Y(\mathbf{x}) = \mathbf{x}$ if and only if $\mathbf{x} \in Y$. As a consequence, $P_Y^2(\mathbf{x}) = P_Y(P_Y(\mathbf{x})) = P_Y(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{K}^{n \times 1}$. $\square$

Let Y be a subspace of $\mathbb{K}^{n\times 1}$. Then the linear map $P_Y : \mathbb{K}^{n\times 1} \to \mathbb{K}^{n\times 1}$ whose existence and uniqueness is proved in the above result is called the **orthogonal projection map** of $\mathbb{K}^{n\times 1}$ onto the subspace Y.

Given $\mathbf{x} \in \mathbb{K}^{n\times 1}$, we shall show that its orthogonal projection $P_Y(\mathbf{x})$ is the unique vector in Y which is closest to $\mathbf{x}$.

Definition

Let E be a nonempty subset of $\mathbb{K}^{n\times 1}$ and let $\mathbf{x} \in \mathbb{K}^{n\times 1}$. A **best approximation** *to $\mathbf{x}$ from E is an element $\mathbf{y}_0 \in E$ such that $\|\mathbf{x} - \mathbf{y}_0\| \le \|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{y} \in E$.*

Simple examples show that a best approximation to a vector $\mathbf{x}$ from a nonempty subset E of $\mathbb{K}^{n\times 1}$ may not exist, and if it exists, it may not be unique. However, if E is in fact a subspace of $\mathbb{K}^{n\times 1}$, then the following noteworthy result holds.

Proposition

Let Y be a subspace of $\mathbb{K}^{n\times 1}$ and let $\mathbf{x} \in \mathbb{K}^{n\times 1}$. Then there is a unique best approximation to $\mathbf{x}$ from Y, namely, $P_Y(\mathbf{x})$.

Further, $P_Y(\mathbf{x})$ is the unique element of Y such that $\mathbf{x} - P_Y(\mathbf{x})$ is orthogonal to Y. Also, the square of the distance from $\mathbf{x}$ to its best approximation from Y is

$$\|\mathbf{x} - P_Y(\mathbf{x})\|^2 = \|\mathbf{x}\|^2 - \langle \mathbf{x}, P_Y(\mathbf{x})\rangle.$$

Proof. We note that $P_Y(\mathbf{x}) \in Y$ and $\big(\mathbf{x} - P_Y(\mathbf{x})\big) \in Y^{\perp}$.

Let $\mathbf{y} \in Y$. Then $P_Y(\mathbf{x}) - \mathbf{y}$ also belongs to Y. Hence by the Pythagorus theorem,

$$\begin{aligned}
\|\mathbf{x} - \mathbf{y}\|^2 &= \big\|\big(\mathbf{x} - P_Y(\mathbf{x})\big) + \big(P_Y(\mathbf{x}) - \mathbf{y}\big)\big\|^2 \\
&= \|\mathbf{x} - P_Y(\mathbf{x})\|^2 + \|P_Y(\mathbf{x}) - \mathbf{y}\|^2 \\
&\geq \|\mathbf{x} - P_Y(\mathbf{x})\|^2,
\end{aligned}$$

where equality holds if and only if $\mathbf{y} = P_Y(\mathbf{x})$. This shows that

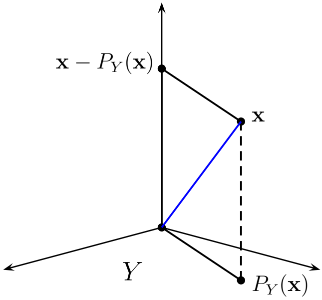
$P_Y(\mathbf{x})$ is the unique best approximation to $\mathbf{x}$ from Y.

Further, let $\mathbf{y} \in Y$ be such that $(\mathbf{x} - \mathbf{y}) \in Y^{\perp}$. Then $\mathbf{x} = \mathbf{y} + (\mathbf{x} - \mathbf{y})$, and since $\mathbb{K}^{n \times 1} = Y \oplus Y^{\perp}$, it follows that $\mathbf{y} = P_Y(\mathbf{x})$.

Finally, since $P_Y(\mathbf{x}) \in Y$ and $\big(\mathbf{x} - P_Y(\mathbf{x})\big) \in Y^{\perp}$,

$$\begin{aligned} \|\mathbf{x} - P_Y(\mathbf{x})\|^2 &= \langle \mathbf{x} - P_Y(\mathbf{x}),\, \mathbf{x} - P_Y(\mathbf{x}) \rangle \\ &= \langle \mathbf{x},\, \mathbf{x} - P_Y(\mathbf{x}) \rangle \\ &= \|\mathbf{x}\|^2 - \langle \mathbf{x},\, P_Y(\mathbf{x}) \rangle. \end{aligned}$$

□



Consider the system of linear equations

$$\mathbf{Ax} = \mathbf{b} \quad \text{where} \quad \mathbf{A} \in \mathbb{K}^{m\times n} \text{ and } \mathbf{b} \in \mathbb{K}^{m\times 1}.$$

Clearly, if $\mathbf{A} = \begin{bmatrix}\mathbf{c}_1 & \cdots & \mathbf{c}_n\end{bmatrix}$ in terms of its n columns and if $\mathbf{x} := \begin{bmatrix}x_1 & \cdots & x_n\end{bmatrix}^{\mathsf{T}}$, then $\mathbf{Ax} = x_1\mathbf{c}_1 + \cdots + x_n\mathbf{c}_n = \mathbf{b}$. Thus

$$\text{the system } \mathbf{Ax} = \mathbf{b} \text{ has a solution} \iff \mathbf{b} \in \mathcal{C}(\mathbf{A}).$$

Now what if $\mathbf{b} \notin \mathcal{C}(\mathbf{A})$? Then we can find an approximate solution to the linear system $\mathbf{Ax} = \mathbf{b}$ by finding the best approximation to $\mathbf{b}$ from the subspace $\mathcal{C}(\mathbf{A})$ of $\mathbb{K}^{m\times 1}$. For this, we first find an ordered orthonormal basis $(\mathbf{u}_1, \ldots, \mathbf{u}_k)$ of $\mathcal{C}(\mathbf{A})$, where $k \le n$. [This can be done using Gram-Schmidt OP.] Then the best approximation to $\mathbf{b}$ from $\mathcal{C}(\mathbf{A})$ is

$$P_{\mathcal{C}(\mathbf{A})}(\mathbf{b}) = \sum_{j=1}^{k} \langle \mathbf{u}_j, \mathbf{b}\rangle \mathbf{u}_j = \sum_{j=1}^{k} (\mathbf{u}_j^* \mathbf{b})\mathbf{u}_j.$$

Example: Let $\mathbf{A} := \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{b} := \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$. Then $\mathcal{C}(\mathbf{A})$ is the span of the 2 column vectors $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^{\mathsf{T}}$ and $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^{\mathsf{T}}$ of $\mathbf{A}$. Then $\mathbf{u}_1 := \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}^{\mathsf{T}}$ and $\mathbf{u}_2 := \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}^{\mathsf{T}}$ form an orthonormal basis for $\mathcal{C}(\mathbf{A})$. Hence the best approximation to $\mathbf{b}$ from $\mathcal{C}(\mathbf{A})$ is $\langle \mathbf{u}_1, \mathbf{b} \rangle \mathbf{u}_1 + \langle \mathbf{u}_2, \mathbf{b} \rangle \mathbf{u}_2$, which is

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^{\mathsf{T}} - \frac{3}{2}\begin{bmatrix} 1 & -1 & 2 \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} -1 & 2 & -3 \end{bmatrix}^{\mathsf{T}}_{.} = \mathbf{a} \text{ (say)}.$$

The distance from $\mathbf{b}$ to its best approximation $\mathbf{a}$ from $\mathcal{C}(\mathbf{A})$ is

$$\|\mathbf{b} - \mathbf{a}\| = \| \begin{bmatrix} 2 & -2 & -2 \end{bmatrix}^{\mathsf{T}} \| = 2\sqrt{3}.$$

The square of this distance could also have been computed directly using the above proposition as follows:

$$\|\mathbf{b}\|^2 - |\langle \mathbf{u}_1, \mathbf{b} \rangle|^2 - |\langle \mathbf{u}_2, \mathbf{b} \rangle|^2 = 26 - \frac{1}{2} - \frac{27}{2} = 12,$$