# MA 110
# Linear Algebra and Differential Equations
# Lecture 02

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# Linear System

Let $m, n \in \mathbb{N}$. A **linear system** of $m$ equations in the $n$ unknowns $x_1, ..., x_n$ is given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \qquad (1) \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \qquad (2) \\ \vdots \qquad \vdots \qquad \vdots \qquad\quad &\vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \qquad (m) \end{aligned}$$

where $a_{jk} \in \mathbb{R}$ for $j = 1, \ldots, m; k = 1, \ldots, n$ and also $b_j \in \mathbb{R}$ for $j = 1, \ldots, m$ are given.

Let $\mathbf{A} := [a_{jk}] \in \mathbb{R}^{m \times n}$, $\mathbf{x} := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\mathsf{T} \in \mathbb{R}^{n \times 1}$ and $\mathbf{b} := \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix}^\mathsf{T} \in \mathbb{R}^{m \times 1}$. Using matrix multiplication, we write the linear system as

$$\mathbf{Ax} = \mathbf{b}.$$

The $m \times n$ matrix $\mathbf{A}$ is known as the **coefficient matrix** of the linear system.

A column vector $\mathbf{x}_0 \in \mathbb{R}^{n\times 1}$ is called a **solution** of the above linear system if it satisfies $\mathbf{A}\mathbf{x}_0 = \mathbf{b}$.

Case (i) **Homogeneous Linear System**: $\mathbf{b} := \mathbf{0}$, that is, $b_1 = \cdots = b_m = 0$.
A homogenous linear system always has a solution, namely the zero solution $\mathbf{0} := \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}^{\mathsf{T}}$ since $\mathbf{A}\mathbf{0} = \mathbf{0}$.

Also, if $r \in \mathbb{N}$ and $\mathbf{x}_1, \ldots, \mathbf{x}_r$ are solutions of such a system, then so is their linear combination $\alpha_1\mathbf{x}_1 + \cdots + \alpha_r\mathbf{x}_r$, since $\mathbf{A}(\alpha_1\mathbf{x}_1 + \cdots + \alpha_r\mathbf{x}_r) = \alpha_1\mathbf{A}\mathbf{x}_1 + \cdots + \alpha_r\mathbf{A}\mathbf{x}_r = \mathbf{0} + \cdots + \mathbf{0} = \mathbf{0}$.

Case (ii) **General Linear System**: $\mathbf{b} \in \mathbb{R}^{m\times 1}$ is arbitrary.

A nonhomogenous linear system, that is, where $\mathbf{b} \neq \mathbf{0}$, may not have a solution, may have only one solution or may have (infinitely) many solutions.

## Examples

(i) The linear system $x_1 + x_2 = 1$, $2x_1 + 2x_2 = 1$ does not have a solution.

(ii) The linear system $x_1 + x_2 = 1$, $x_1 - x_2 = 0$ has a unique solution, namely $x_1 = 1/2 = x_2$.

(iii) The linear system $x_1 + x_2 = 1$, $2x_1 + 2x_2 = 2$ has (infinitely) many solutions, namely $x_1 = \alpha$, $x_2 = 1 - \alpha$, $\alpha \in \mathbb{R}$.

## Important Note:

Let $S$ denote the set of all solutions of a homogeneous linear system $\mathbf{Ax} = \mathbf{0}$. If $\mathbf{x}_0$ is a particular solution of the general system $\mathbf{Ax} = \mathbf{b}$, then the set of all solutions of the general system $\mathbf{Ax} = \mathbf{b}$ is given by $\{\mathbf{x}_0 + \mathbf{s} : \mathbf{s} \in S\}$ since

$\mathbf{s} \in S \implies \mathbf{A}(\mathbf{x}_0 + \mathbf{s}) = \mathbf{Ax}_0 + \mathbf{As} = \mathbf{b} + \mathbf{0} = \mathbf{b}$, and also

$\mathbf{Ax} = \mathbf{b} \implies \mathbf{A}(\mathbf{x} - \mathbf{x}_0) = \mathbf{Ax} - \mathbf{Ax}_0 = \mathbf{b} - \mathbf{b} = \mathbf{0}$, so that $(\mathbf{x} - \mathbf{x}_0) \in S$, that is, $\mathbf{x} = \mathbf{x_0} + \mathbf{s}$ for some $\mathbf{s} \in S$.

We shall, therefore, address the problem of finding all solutions of a homogeneous linear system $\mathbf{Ax} = \mathbf{0}$, and one particular solution of the corresponding general system $\mathbf{Ax} = \mathbf{b}$.

## A Special Case

Suppose the coefficient matrix $\mathbf{A}$ is upper triangular and its diagonal elements are nonzero. Then the linear system

$$
\begin{array}{rcll}
a_{11}x_1 + a_{12}x_2 + \cdots + \cdots + \cdots + \cdots + a_{1n}x_n & = & b_1 & (1) \\
a_{22}x_2 + a_{23}x_3 + \cdots + \cdots + \cdots + a_{2n}x_n & = & b_2 & (2) \\
\ddots \qquad\qquad\qquad\qquad \vdots & \vdots & \vdots & \\
a_{(n-1)(n-1)}x_{n-1} + a_{(n-1)n}x_n & = & b_{n-1} & (n-1) \\
a_{nn}x_n & = & b_n & (n)
\end{array}
$$

can be solved by back substitution as follows.

$$
\begin{aligned}
x_n &= b_n/a_{nn} \\
x_{n-1} &= (b_{n-1} - a_{(n-1)n}x_n)/a_{(n-1)(n-1)}, \text{ where } x_n = b_n/a_{nn} \\
\vdots \;&\;\vdots\; \vdots \\
x_2 &= (b_2 - a_{2n}x_n - \cdots - a_{23}x_3)/a_{22}, \text{where } x_n = \cdots, x_3 = \cdots, \\
x_1 &= (b_1 - a_{1n}x_n - \cdots - \cdots - a_{12}x_2)/a_{11}, \text{where } x_n = \cdots, x_2 = \cdots
\end{aligned}
$$

Here the homogeneous system $\mathbf{Ax} = \mathbf{0}$ has only the zero solution and the general system $\mathbf{Ax} = \mathbf{b}$ has a unique solution.

Taking a cue from this special case of an upper triangular matrix, we shall attempt to transform any $m \times n$ matrix to an upper triangular form. In this process, we successively attempt to eliminate the unknown $x_1$ from the equations $(m), \ldots, (2)$, the unknown $x_2$ from the equations $(m), \ldots, (3)$, and so on.

## Example

Consider the linear system

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ -x_1 + x_2 - x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 20x_1 + 10x_2 &= 80. \end{aligned}$$

Eliminating $x_1$ from the 4th, 3rd and 2nd equations,

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 0 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 30x_2 - 20x_3 &= 80. \end{aligned}$$

Interchanging the 2nd and the 3rd equations,

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 0 &= 0 \\ 30x_2 - 20x_3 &= 80. \end{aligned}$$

Eliminating $x_2$ from the 4th equation, and then interchanging the 3rd and the 4th equations,

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ -95x_3 &= -190 \\ 0 &= 0. \end{aligned}$$

Now back substitution gives $x_3 = 2$, $x_2 = (90 - 25x_3)/10 = 4$ and $x_1 = -x_3 + x_2 = 2$, that is, $\mathbf{x} = \begin{bmatrix} 2 & 4 & 2 \end{bmatrix}^T$.

The above process can be carried out without writing down the entire linear system by considering the **augmented matrix**

$$[\mathbf{A}|\mathbf{b}] := \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{12} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array}\right].$$

This $m \times (n+1)$ matrix completely describes the linear system. In the above example,

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right].$$

Subtracting 20 times the first row from the 4th row, and adding the first row to the second row, we obtain

$$\xrightarrow{R_4-20R_1,\, R_2+R_1,} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array}\right].$$

Interchanging the 2nd and the 3rd rows, we obtain

$$\xrightarrow{R_2\longleftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & 0 & 0 \\ 0 & 30 & -20 & 80 \end{array}\right].$$

Finally, subtracting 3 times the 2nd row from the 4th row and interchanging the 3rd and the 4th rows, we arrive at

$$\xrightarrow{R_4-3R_2,\, R_3\longleftrightarrow R_4} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The upper triangular nature of the $3 \times 3$ matrix on the top left enables back substitution.

# Row Echelon Form

We shall now consider a general form of a matrix for which the method of back substitution works.

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$, that is, $\mathbf{A}$ is an $m \times n$ matrix with real entries.

A row of $\mathbf{A}$ is said to be **zero** if all its entries are zero.

If a row is not zero, then its first nonzero entry (from the left) is called the **pivot**.Thus all entries to the left of a pivot equal 0.

Suppose $\mathbf{A}$ has $r$ nonzero rows and $m - r$ zero rows.
Then $0 \leq r \leq m$. Clearly, $r = 0 \iff \mathbf{A} = \mathbf{O}$.

Example

If $\mathbf{A} = \begin{bmatrix} 0 & \boxed{1} & 4 \\ 0 & 0 & 0 \\ \boxed{5} & 6 & 7 \end{bmatrix}$, then $m = n = 3$ and $r = 2$.

The pivot in the 1st row is 1 and the pivot in the 3rd row is 5.

A matrix **A** is said to be in a row echelon form (REF)[1] if the following conditions are satisfied.

(i) The nonzero rows of **A** precede the zero rows of **A**.

(ii) If **A** has $r$ nonzero rows, where $r \in \mathbb{N}$, and the pivot in row 1 appears in the column $k_1$, the pivot in row 2 appears in the column $k_2$, and so on the pivot in row $r$ appears in the column $k_r$, then $k_1 < k_2 < \cdots < k_r$.

Examples

The matrices $\begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 0 \\ 5 & 6 & 7 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 4 \\ 5 & 6 & 7 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 4 \\ 0 & 5 & 7 \\ 0 & 0 & 0 \end{bmatrix}$

are not in REF. The matrix $\begin{bmatrix} 5 & 6 & 7 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ is in a REF.

[1] In French, echelon means level.

## Pivotal Columns

(i) Suppose a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is in a REF. If $\mathbf{A}$ has exactly $r$ nonzero rows, then there are exactly $r$ pivots. A column of $\mathbf{A}$ containing a pivot, is called a **pivotal column**. Thus there are exactly $r$ pivotal columns, and so $0 \le r \le n$.
(We have already noted that $0 \le r \le m$.)

(ii) In a pivotal column, all entries below the pivot equal 0.

Here is a typical example of how back substitution works when a matrix $\mathbf{A}$ is in a REF. Let

$$\mathbf{A} := \begin{bmatrix} 0 & \boxed{a_{12}} & a_{13} & a_{14} & a_{15} & a_{16} \\ 0 & 0 & 0 & \boxed{a_{24}} & a_{25} & a_{26} \\ 0 & 0 & 0 & 0 & \boxed{a_{35}} & a_{36} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} := \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

where $a_{12}, a_{24}, a_{35}$ are nonzero. They are the pivots.

Here $m = 4$, $n = 6$, $r = 3$, pivotal columns: $\boxed{2}$, $\boxed{4}$ and $\boxed{5}$.

Suppose there is $\mathbf{x} := \begin{bmatrix} x_1 & \cdots & x_6 \end{bmatrix}^{\mathsf{T}} \in \mathbb{R}^{6\times 1}$ such that $\mathbf{Ax} = \mathbf{b}$. Then $0x_1 + \cdots + 0x_6 = b_4$, that is, $b_4$ must be equal to 0.

Next, $a_{35}x_5 + a_{36}x_6 = b_3$, that is, $x_5 = (b_3 - a_{36}x_6)/a_{35}$, where we can assign an arbitrary value to the unknown $\boxed{x_6}$.

Next, $a_{24}x_4 + a_{25}x_5 + a_{26}x_6 = b_2$, that is,
$x_4 = (b_2 - a_{25}x_5 - a_{26}x_6)/a_{24}$, where we back substitute the values of $x_5$ and $x_6$.

Finally, $a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + a_{15}x_5 + a_{16}x_6 = b_1$, that is,
$x_2 = (b_1 - a_{13}x_3 - a_{14}x_4 - a_{15}x_5 - a_{16}x_6)/a_{12}$, where we can assign an arbitrary value to the variable $\boxed{x_3}$, and back substitute the values of $x_4$, $x_5$ and $x_6$.
Also, we can assign an arbitrary value to the variable $\boxed{x_1}$.

The variables $x_1, x_3$ and $x_6$ to which we can assign arbitrary values correspond to the nonpivotal columns $\boxed{1}$, $\boxed{3}$ and $\boxed{6}$.

Suppose an $m \times n$ matrix $\mathbf{A}$ is in a REF, and there are $r$ nonzero rows. Let the $r$ pivots be in the columns $k_1, \ldots, k_r$ with $k_1 < \cdots < k_r$, and let the columns $\ell_1, \ldots, \ell_{n-r}$ be nonpivotal. Then $x_{k_1}, \ldots, x_{k_r}$ are called the **pivotal variables** and $x_{\ell_1}, \ldots, x_{\ell_{n-r}}$ are called the **free variables**.

Let $\mathbf{b} := [b_1 \cdots b_r \; b_{r+1} \cdots b_m]^{\mathsf{T}}$, and consider the linear system

$$\mathbf{A}\mathbf{x} = \mathbf{b}.$$

Important Observations

1. The linear system has a solution $\iff b_{r+1} = \cdots = b_m = 0$. This is known as the **consistency condition**.
2. Let the consistency condition $b_{r+1} = \cdots = b_m = 0$ be satisfied. Then we obtain a **particular solution** $\mathbf{x}_0 := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^{\mathsf{T}}$ of the linear system by letting $x_k := 0$ if $k \in \{\ell_1, \ldots, \ell_{n-r}\}$, and then by determining the pivotal variables $x_{k_1}, \ldots, x_{k_r}$ by back substitution.

3. We obtain $n-r$ **basic solutions** of the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ as follows. Fix $\ell \in \{\ell_1, \ldots, \ell_{n-r}\}$. Define $\mathbf{s}_\ell := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\mathsf{T}$ by $x_k := 1$ if $k = \ell$, while $x_k := 0$ if $k \in \{\ell_1, \ldots, \ell_{n-r}\}$ but $k \neq \ell$. Then determine the pivotal variables $x_{k_1}, \ldots, x_{k_r}$ by back substitution.
4. Let $\mathbf{s} := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\mathsf{T} \in \mathbb{R}^{n\times 1}$ be any solution of the homogeneous system, that is, $\mathbf{As} = \mathbf{0}$. Then $\mathbf{s}$ is a linear combination of the $n-r$ basic solutions $\mathbf{s}_{\ell_1}, \ldots, \mathbf{s}_{\ell_{n-r}}$. To see this, let $\mathbf{y} := \mathbf{s} - x_{\ell_1}\mathbf{s}_{\ell_1} - \cdots - x_{\ell_{n-r}}\mathbf{s}_{\ell_{n-r}}$. Then $\mathbf{Ay} = \mathbf{As} - x_{\ell_1}\mathbf{As}_{\ell_1} - \cdots - x_{\ell_{n-r}}\mathbf{As}_{\ell_{n-r}} = \mathbf{0}$, and moreover, the $k$th entry of $\mathbf{y}$ is 0 for each $k \in \{\ell_1, \ldots, \ell_{n-r}\}$. It then follows that $\mathbf{y} = \mathbf{0}$, that is, $\mathbf{s} = x_{\ell_1}\mathbf{s}_{\ell_1} + \cdots + x_{\ell_{n-r}}\mathbf{s}_{\ell_{n-r}}$. Thus we find that the general solution of the homogeneous system is given by

$$\boxed{\mathbf{s} = \alpha_1\mathbf{s}_{\ell_1} + \cdots + \alpha_{n-r}\mathbf{s}_{\ell_{n-r}}, \text{ where } \alpha_1, \ldots, \alpha_{n-r} \in \mathbb{R}}.$$

5. The general solution of $\mathbf{Ax} = \mathbf{b}$ is given by

$$\boxed{\mathbf{x} = \mathbf{x}_0 + \alpha_1 \mathbf{s}_{\ell_1} + \cdots + \alpha_{n-r} \mathbf{s}_{\ell_{n-r}}, \text{ where } \alpha_1, \ldots, \alpha_{n-r} \in \mathbb{R},}$$

provided the consistency condition is satisfied.

Example

$$\text{Let} \quad \mathbf{A} := \begin{bmatrix} 0 & \boxed{2} & 1 & 0 & 2 & 5 \\ 0 & 0 & 0 & \boxed{3} & 5 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} := \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}.$$

As we have seen earlier, here $m = 4$, $n = 6$, $r = 3$, pivotal columns: $\boxed{2}$, $\boxed{4}$ and $\boxed{5}$, and nonpivotal columns: $\boxed{1}$, $\boxed{3}$, $\boxed{6}$.

Since $b_4 = 0$, the linear system $\mathbf{Ax} = \mathbf{b}$ is consistent.

For a particular solution of $\mathbf{Ax} = \mathbf{b}$, let $x_1 = x_3 = x_6 = 0$.
Then $x_5 + 2x_6 = 2 \implies x_5 = 2$,
$3x_4 + 5x_5 + 0x_6 = 1 \implies x_4 = -3$,
$2x_2 + x_3 + 0x_4 + 2x_5 + 5x_6 = 0 \implies x_2 = -2$.
Thus $\mathbf{x}_0 := \begin{bmatrix} 0 & -2 & 0 & -3 & 2 & 0 \end{bmatrix}^\mathsf{T}$ is a particular solution.

Basic solutions of $\mathbf{Ax} = \mathbf{0}$:
$x_1 = 1, x_3 = x_6 = 0$ gives $\mathbf{s}_1 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^\mathsf{T}$,
$x_3 = 1, x_1 = x_6 = 0$ gives $\mathbf{s}_3 := \begin{bmatrix} 0 & -1/2 & 1 & 0 & 0 & 0 \end{bmatrix}^\mathsf{T}$,
$x_6 = 1, x_1 = x_3 = 0$ gives $\mathbf{s}_6 := \begin{bmatrix} 0 & -1/2 & 0 & 10/3 & -2 & 1 \end{bmatrix}^\mathsf{T}$.

The general solution of $\mathbf{Ax} = \mathbf{b}$ is given by
$\mathbf{x} = \mathbf{x}_0 + \alpha_1\mathbf{s}_1 + \alpha_3\mathbf{s}_3 + \alpha_6\mathbf{s}_6$, that is,
$x_1 = \alpha_1,\ x_2 = -2 - (\alpha_3 + \alpha_6)/2,\ x_3 = \alpha_3,\ x_4 = -3 + 10\alpha_6/3,$
$x_5 = 2(1 - \alpha_6),\ x_6 = \alpha_6$, where $\alpha_1, \alpha_3, \alpha_6 \in \mathbb{R}$.

## Conclusion

Suppose an $m \times n$ matrix $\mathbf{A}$ is in a REF, and let $r$ be the number of nonzero rows of $\mathbf{A}$. If $\mathbf{b} \in \mathbb{R}^{m\times 1}$, then the linear system $\mathbf{Ax} = \mathbf{b}$ has

(i) no solution if one of $b_{r+1}, \ldots, b_m$ is nonzero.

(ii) a unique solution if $b_{r+1} = \cdots = b_m = 0$ and $r = n$.

(iii) infinitely many solutions if $b_{r+1} = \cdots = b_m = 0$ and $r < n$. (In this case, there are $n - r$ free variables which give $n - r$ degrees of freedom.)

Considering the case $\mathbf{b} = \mathbf{0} \in \mathbb{R}^{m\times 1}$ and recalling that $r \le m$, we obtain the following important results.

**Proposition**

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ be in REF with $r$ nonzero rows. Then the linear system $\mathbf{Ax} = \mathbf{0}$ has only the zero solution if and only if $r = n$. In particular, if $m < n$, then $\mathbf{Ax} = \mathbf{0}$ has a nonzero solution.