MA 110
Linear Algebra and Differential Equations
Lecture 05

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Recall:

A matrix $\mathbf{A} \in \mathbb{R}^{\mathbf{m} \times \mathbf{n}}$ is said to be in row canonical form (RCF) if it is in REF, all pivots are equal to 1 and all the entries in the column above (as well as below) each pivot are zero.

We have noted that:

- Every matrix can be transformed by a sequence of EROs to a matrix in RCF. Moreover, the RCF of a matrix is unique.
- A square matrix $\mathbf{A} \in \mathbb{R}^{\mathbf{n} \times \mathbf{n}}$ is invertible, if and only if its RCF is $\mathbf{I}$, or equivalently, if $\mathbf{A}$ can be transformed to $\mathbf{I}$ by EROs.
- (Gauss-Jordan Method): To find the inverse of $\mathbf{A} \in \mathbb{R}^{n \times n}$, if it exists, consider the $n \times 2n$ matrix $[\mathbf{A} \,|\, \mathbf{I}]$. Transform it to $[\mathbf{A}' \,|\, \mathbf{X}]$ by ERO, where $\mathbf{A}'$ is the RCF of $\mathbf{A}$, If $\mathbf{A}' \neq \mathbf{I}$, then $\mathbf{A}$ is not invertible, whereas if $\mathbf{A}' = \mathbf{I}$, then $\mathbf{A}$ is invertible and $\mathbf{X}$ is the inverse of $\mathbf{A}$.

Also recall:

A set S of vectors is called **linearly dependent** if there is $m \in \mathbb{N}$, and there are (distinct) vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$ in S, and scalars $\alpha_1, \ldots, \alpha_m$, not all zero, such that

$$\alpha_1 \mathbf{a}_1 + \cdots + \alpha_m \mathbf{a}_m = \mathbf{0}.$$

We proved the following crucial result.

Proposition

Let S be a set of s vectors, each of which is a linear combination of elements of a (fixed) set of r vectors. If $s > r$, then the set S is linearly dependent.

And then deduced the following useful corollary.

Corollary

Let $n \in \mathbb{N}$ and S be a set of vectors of length n. If S has more than n elements, then S is linearly dependent.

Linear Independence

A set S of vectors is called **linearly independent** if it is not linearly dependent, that is,

$$\alpha_1 \mathbf{a}_1 + \cdots + \alpha_m \mathbf{a}_m = \mathbf{0} \implies \alpha_1 = \cdots = \alpha_m = 0,$$

whenever $\mathbf{a}_1, \ldots, \mathbf{a}_m$ are (distinct) vectors in S and $\alpha_1, \ldots, \alpha_m$ are scalars. We may also say that the vectors in S are linearly independent.

Linearly independent sets are important because each one of them gives us data that we cannot obtain from any linear combination of the others. In this sense, each element of a linearly independent set is indispensable!

Examples

(i) The empty set is linearly independent vacuously.

(ii) Let S be the subset of $\mathbb{R}^{n\times 1}$ consisting of the basic column vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Then S is linearly independent. To see this, let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $\alpha_1\mathbf{e}_1 + \cdots + \alpha_n\mathbf{e}_n = \mathbf{0}$. Then the jth entry α_j of $\alpha_1\mathbf{e}_1 + \cdots + \alpha_n\mathbf{e}_n$ is equal to 0 for $j = 1, \ldots, n$.

(iii) Let S denote the subset of $\mathbb{R}^{1\times 4}$ consisting of the vectors $\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 1 & 1 & 0\end{bmatrix}$ and $\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}$.

Then S is linearly independent. To see this, let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ be such that

$$\alpha_1 \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix} + \alpha_2 \begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix} + \alpha_3 \begin{bmatrix}1 & 1 & 1 & 0\end{bmatrix} + \alpha_4 \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix}.$$

Then $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$, $\alpha_2 + \alpha_3 + \alpha_4 = 0$, $\alpha_3 + \alpha_4 = 0$ and $\alpha_4 = 0$, that is, $\alpha_4 = \alpha_3 = \alpha_2 = \alpha_1 = 0$.

How to Decide Linear Independence of Column Vectors?

Suppose $\mathbf{c}_1, \ldots, \mathbf{c}_n$ are column vectors each of length m. Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be the matrix having $\mathbf{c}_1, \ldots, \mathbf{c}_n$ as its n columns. Then for $x_1, \ldots, x_n \in \mathbb{R}$,

$$x_1\mathbf{c}_1 + \cdots + x_n\mathbf{c}_n = \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{A}\mathbf{x}.$$

Hence the subset $S := \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ of $\mathbb{R}^{m \times 1}$ is linearly independent if and only if the linear system $\mathbf{A}\mathbf{x} = \mathbf{0}$ has only the zero solution. This is the case if and only if in any REF $\mathbf{A}'$ of $\mathbf{A}$, there are n nonzero rows, as we have seen in Lecture 3.

Hence if $\mathbf{A} \in \mathbb{R}^{m \times n}$ is transformed to a REF $\mathbf{A}'$, and $\mathbf{A}'$ has r nonzero rows, then the columns of $\mathbf{A}$ form a linearly independent subset of $\mathbb{R}^{m \times 1}$ if $r = n$, and they form a linearly dependent subset $\mathbb{R}^{m \times 1}$ if $r < n$.

Row Rank of a Matrix

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. The **row rank** of $\mathbf{A}$ is the maximum number of linearly independent row vectors of $\mathbf{A}$. Thus the row rank of $\mathbf{A}$ is equal to r if and only if there is a linearly independent set of r rows of $\mathbf{A}$ and any set of $r + 1$ rows of $\mathbf{A}$ is linearly dependent.

Let r be the row rank of $\mathbf{A}$. Then $r = 0$ if and only if $\mathbf{A} = \mathbf{O}$. Since the total number of rows of $\mathbf{A}$ is m, we see that $r \leq m$. Also, since the row vectors of $\mathbf{A}$ form a subset of $\mathbb{R}^{1 \times n}$, no more than n of them can be linearly independent. Thus $r \leq n$.

Let $\mathbf{a}_1, \ldots, \mathbf{a}_m$ be any m vectors in $\mathbb{R}^{1 \times n}$. Clearly, they are linearly independent if and only if the matrix $\mathbf{A}$ formed with these vectors as row vectors has row rank m, and they are linearly dependent if the row rank of $\mathbf{A}$ is less than m.

Examples

(i) Let $\mathbf{A} := \begin{bmatrix} 3 & 0 & 2 & 2 \\ -3 & 0 & 12 & 27 \\ -21 & 21 & 0 & 15 \end{bmatrix}$.

The row vectors of $\mathbf{A}$ are $\mathbf{a}_1 := \begin{bmatrix} 3 & 0 & 2 & 2 \end{bmatrix}$, $\mathbf{a}_2 := \begin{bmatrix} -3 & 0 & 12 & 27 \end{bmatrix}$ and $\mathbf{a}_3 := \begin{bmatrix} -21 & 21 & 0 & 15 \end{bmatrix}$.

Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ be such that $\alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \alpha_3 \mathbf{a}_3 = \mathbf{0}$.

This means

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 2 & 2 \\ -3 & 0 & 12 & 27 \\ -21 & 21 & 0 & 15 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix},$$

that is, $3\alpha_1 - 3\alpha_2 - 21\alpha_3 = 0$, $21\alpha_3 = 0, 2\alpha_1 + 12\alpha_2 = 0$ and $2\alpha_1 + 27\alpha_2 + 15\alpha_3 = 0$. Clearly, $\alpha_3 = 0$, and the two equations $3\alpha_1 - 3\alpha_2 = 0, 2\alpha_1 + 12\alpha_2 = 0$ show that $\alpha_1 = \alpha_2 = 0$ as well. Thus the 3 rows of $\mathbf{A}$ are linearly independent. Hence the row rank of $\mathbf{A}$ is 3.

(ii)

Let $\mathbf{A} := \begin{bmatrix} 3 & 0 & 2 & 2 \\ -3 & 21 & 12 & 27 \\ -21 & 21 & 0 & 15 \end{bmatrix}$.

The row vectors of $\mathbf{A}$ are $\mathbf{a}_1 := \begin{bmatrix} 3 & 0 & 2 & 2 \end{bmatrix}$, $\mathbf{a}_2 := \begin{bmatrix} -3 & 21 & 12 & 27 \end{bmatrix}$ and $\mathbf{a}_3 := \begin{bmatrix} -21 & 21 & 0 & 15 \end{bmatrix}$. Observe that $6\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 = \mathbf{0}$. Hence the three row vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are not linearly independent. But the set $\{\mathbf{a}_1, \mathbf{a}_2\}$ is linearly independent since $\mathbf{a}_1 \neq \mathbf{0}, \mathbf{a}_2 \neq \mathbf{0}$, and they are not scalar multiples of each other. (The same holds for the sets $\{\mathbf{a}_2, \mathbf{a}_3\}$ and $\{\mathbf{a}_3, \mathbf{a}_1\}$.) Hence the row rank of $\mathbf{A}$ is 2.

We used the relation $6\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 = \mathbf{0}$ above to determine the row rank of $\mathbf{A}$. It is difficult to think of such a relation out of nowhere. We shall develop a systematic approach to find the row rank of a matrix.

First we prove the following preliminary results.

Proposition

If a matrix $\mathbf{A}$ is transformed to a matrix $\mathbf{A}'$ by elementary row operations, then the row ranks of $\mathbf{A}$ and $\mathbf{A}'$ are equal, that is, EROs do not alter the row rank of a matrix.

Proof. ERO of type I: $R_i \longleftrightarrow R_j$ with $i \neq j$: $\mathbf{A}$ and $\mathbf{A}'$ have the same set of row vectors. So there is nothing to prove.

ERO of type II: $R_i + \alpha R_j$ with $i \neq j$: Suppose the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_i, \ldots, \mathbf{a}_j, \ldots, \mathbf{a}_m\}$ of all row vectors of $\mathbf{A}$ contains a linearly independent subset $S := \{\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_s}\}$ having s elements. We claim that the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_i + \alpha\, \mathbf{a}_j, \ldots, \mathbf{a}_j, \ldots \mathbf{a}_m\}$ of all row vectors of $\mathbf{A}'$ also contains a linearly independent set S' containing s elements. If $\mathbf{a}_i \notin S$, then we let $S' := S$. Next, suppose $\mathbf{a}_i \in S$. Then we may replace $\mathbf{a}_i$ suitably either by $\mathbf{a}_i + \alpha\mathbf{a}_j$ or by $\mathbf{a}_j$ in the set S to form a linearly independent set S'. The last statement follows by considering the cases $\mathbf{a}_i + \alpha\mathbf{a}_j = \mathbf{0}$, $\mathbf{a}_j = \mathbf{0}$, and by observing that if $\mathbf{a}_i + \alpha\,\mathbf{a}_j$ as well as $\mathbf{a}_j$ were linear combinations of vectors in $S \setminus \{\mathbf{a}_i\}$, then so would be $\mathbf{a}_i = (\mathbf{a}_i + \alpha\,\mathbf{a}_j) - \alpha\,\mathbf{a}_j$, and this would contradict the linear independence of S. Note that the converse claim also holds.

ERO of type III: $\alpha\,R_j$ with $\alpha \neq 0$: $\{\mathbf{a}_j, \mathbf{a}_{j_1}, \ldots \mathbf{a}_{j_s}\}$ is linearly independent $\iff$ $\{\alpha\,\mathbf{a}_j, \mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_s}\}$ is linearly independent.

Thus the maximum number of linearly independent rows of $\mathbf{A}$ is the same as the maximum number of linearly independent rows of $\mathbf{A}'$, that is, the row ranks of $\mathbf{A}$ and $\mathbf{A}'$ are equal. $\square$

Proposition

Let a matrix $\mathbf{A}'$ be in REF. Then the nonzero rows of $\mathbf{A}'$ are linearly independent, and so the row rank of $\mathbf{A}'$ is equal to the number of nonzero rows of $\mathbf{A}'$.

Proof. Let the number of nonzero rows of $\mathbf{A}'$ be r. Let the pivots $p_1, \ldots, p_r$ in these rows be in columns $k_1, \ldots, k_r$, where $1 \leq k_1 < \cdots < k_r \leq n$. Suppose $\alpha_1 \mathbf{a}_1 + \cdots + \alpha_r \mathbf{a}_r = \mathbf{0}$, where $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$.

Assume for a moment that not all $\alpha_1, \ldots, \alpha_r$ are zero, and let α_j be the first nonzero number among them. Now the k_jth component of $\alpha_1 \mathbf{a}_1 + \cdots + \alpha_r \mathbf{a}_r = \alpha_j \mathbf{a}_j + \cdots + \alpha_r \mathbf{a}_r$ is equal to $\alpha_j p_j$ since all entries in the k_jth column below the pivot p_j are equal to 0. Hence $\alpha_j p_j = 0$. But $p_j \neq 0$, and so $\alpha_j = 0$, contrary to our assumption. Thus $\alpha_1 = \cdots = \alpha_r = 0$. This shows that the first r rows of $\mathbf{A}'$ are linearly independent.

Also, since the last $m - r$ rows of $\mathbf{A}'$ are zero rows, any $r + 1$ row vectors of $\mathbf{A}'$ will contain the vector $\mathbf{0}$, and so they will not be linearly independent. Hence the row rank of $\mathbf{A}'$ is r. $\square$

We have now obtained an important result which tells us how to find the row rank of a matrix.

Proposition

The row rank of a matrix is equal to the number of nonzero rows in any row echelon form of the matrix.

Proof. Let $\mathbf{A}$ be a $m \times n$ matrix. By using EROs of type I and type II, we transform $\mathbf{A}$ to a row echelon form $\mathbf{A}'$. Then the row rank of $\mathbf{A}$ is equal to the row rank of $\mathbf{A}'$, and it is equal to the number of nonzero rows of $\mathbf{A}'$. □

The above proposition implies that if $\mathbf{A}'$ and $\mathbf{A}''$ are two row echelon forms of a matrix $\mathbf{A}$, then they have the same number of nonzero rows; this number is equal to the row rank of $\mathbf{A}$.

Since each nonzero row of a matrix in a REF has exactly one pivot, we see that the row rank of a matrix is equal to the number of pivots in any row echelon form of the matrix.

How to Decide Linear Independence of Row Vectors

Suppose $\mathbf{a}_1, \ldots, \mathbf{a}_m$ are row vectors, each of length n. Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be the matrix having $\mathbf{a}_1, \ldots, \mathbf{a}_m$ as its m rows. Then $\mathbf{a}_1, \ldots, \mathbf{a}_m$ are linearly independent if and only if the row rank of $\mathbf{A}$ is equal to m. This is the case if in any REF $\mathbf{A}'$ of $\mathbf{A}$, there are m nonzero rows, that is, all rows of $\mathbf{A}'$ are nonzero.

Thus if $\mathbf{A} \in \mathbb{R}^{m \times n}$ is transformed to a REF $\mathbf{A}'$, and $\mathbf{A}'$ has r nonzero rows, then the rows of $\mathbf{A}$ form a linearly independent subset of $\mathbb{R}^{1 \times n}$ if $r = m$, and they form a linearly dependent subset $\mathbb{R}^{1 \times n}$ if $r < m$.

Compare this criterion with the criterion for the linear independence of column vectors given in Lecture 4:
If $\mathbf{A} \in \mathbb{R}^{m \times n}$ is transformed to a REF $\mathbf{A}'$, and $\mathbf{A}'$ has r nonzero rows, then the columns of $\mathbf{A}$ form a linearly independent subset of $\mathbb{R}^{m \times 1}$ if $r = n$, and they form a linearly dependent subset $\mathbb{R}^{m \times 1}$ if $r < n$.

Example

In Lecture 3, we had seen that the matrix

$$\mathbf{A} := \begin{bmatrix} 3 & 2 & 2 & -5 \\ 0.6 & 1.5 & 1.5 & -5.4 \\ 1.2 & -0.3 & -0.3 & 2.4 \end{bmatrix}$$

can be transformed to a row echelon form

$$\mathbf{A}' := \begin{bmatrix} \boxed{3} & 2 & 2 & -5 \\ 0 & \boxed{1.1} & 1.1 & -4.4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

by elementary row transformations of type I and type II. Since the number of nonzero rows of $\mathbf{A}'$ is 2, we see that the row rank of $\mathbf{A}$ is 2. This shows that the 3 row vectors of $\mathbf{A}$ are linearly dependent.