MA 110
Linear Algebra and Differential Equations
Lecture 08

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Review of last lecture

We have discussed the following important notions.

- Rank of a matrix
- Vector subspaces (of $\mathbb{R}^{n\times 1}$).
- Basis of a subspace
- Dimension of a subspace
- Span of a subset of a vector subspace.
- Null space and the column space of a matrix.

And we proved several important results such as:

- Characterizations of a basis of a vector subspace
- Rank-Nullity Theorem
- Fundamental Theorem for Linear Systems:

We also saw how a basis of the row space $\mathcal{R}(\mathbf{A})$ and a basis of the column space $\mathcal{C}(\mathbf{A})$ of a matrix $\mathbf{A}$ can be found by looking at a row echelon form of $\mathbf{A}$.

Determinants

You already know formulas for determinants of 1×1, 2×2 and 3×3 matrices. Let us recall them.

$$\det \begin{bmatrix} a_1 \end{bmatrix} = a_1, \quad \det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = a_1 b_2 - a_2 b_1 \quad \text{and}$$

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2),$$

which is also equal to
$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$.

We shall presently give formulas for the determinant of an $n \times n$ matrix, that is, of a matrix of size n, where $n \in \mathbb{N}$, and we shall explore their use in matrix theory.

Let $n \in \mathbb{N}$ and let $\mathbf{A} := \begin{bmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{j1} & \cdots & \boxed{a_{jk}} & \cdots & a_{jn} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nk} & \cdots & a_{nn} \end{bmatrix} \in \mathbb{R}^{n \times n}$.

The determinant of $\mathbf{A}$ is a real number defined inductively as follows. For $n := 1$, define $\det \mathbf{A} := a_{11}$. Let $n \geq 2$, and suppose we have defined the determinant of any $(n-1)\times(n-1)$ matrix. For $j, k = 1, \ldots, n$, let $\mathbf{A}_{jk}$ denote the submatrix of $\mathbf{A}$ obtained by deleting the jth row and the kth column of $\mathbf{A}$, and let $M_{jk} := \det \mathbf{A}_{jk}$, called the (j, k)th **minor** of $\mathbf{A}$. Define

$$\det \mathbf{A} := a_{11} M_{11} - a_{12} M_{12} + \cdots + (-1)^{1+k} a_{1k} M_{1k} + \cdots + (-1)^{1+n} a_{1n} M_{1n}.$$

This is also known as the **expansion for the determinant of A in terms of the first row of A**.

An immediate consequence of our definition is the following.

Proposition

If $\mathbf{A}$ is lower triangular, then the determinant of $\mathbf{A}$ is the product of its diagonal entries.

Proof. $\det \mathbf{A} = a_{11} M_{11}$ since $a_{12} = \cdots = a_{1n} = 0$, etc. $\square$

Next, it can be proved by induction on the size n of a matrix that $\det \mathbf{A}$ is equal to the following expansions in terms of the jth row of $\mathbf{A}$, and also in terms of the kth column of $\mathbf{A}$:

$$\det \mathbf{A} = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad \text{for each } j \in \{1, \ldots, n\}$$

$$\det \mathbf{A} = \sum_{j=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad \text{for each } k \in \{1, \ldots, n\}$$

(For a proof, see Kreyszig, Appendix 4, page A81.)

Note: The signs $(-1)^{j+k}$ follow a zigzag pattern.

Proposition

Let $\mathbf{A}$ be a square matrix. Then $\det \mathbf{A}^{\mathsf{T}} = \det \mathbf{A}$.

Proof. This is obvious if $n = 1$. Let now $n \geq 2$, and assume this property for all $(n-1)\times(n-1)$ matrices. Note that $(\mathbf{A}^{\mathsf{T}})_{jk} = (\mathbf{A}_{kj})^{\mathsf{T}}$ for all $j, k = 1, \ldots, n$, that is, the submatrix obtained by deleting the jth row and the kth column of $\mathbf{A}^{\mathsf{T}}$ is the same as the transpose of the submatrix obtained by deleting the kth row and the jth column of $\mathbf{A}$. (For example,

$$\mathbf{A} := \begin{bmatrix} a_{11} & \boxed{a_{12}} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \implies \mathbf{A}^{\mathsf{T}} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ \boxed{a_{12}} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix},$$

so that $(\mathbf{A}^{\mathsf{T}})_{21} = \begin{bmatrix} a_{21} & a_{31} \\ a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}^{\mathsf{T}} = (\mathbf{A}_{12})^{\mathsf{T}}$.)

Let $\mathbf{A} := [a_{jk}]$ and $\mathbf{A}^{\mathsf{T}} := [a'_{jk}]$. Then $a'_{jk} = a_{kj}$ and $M'_{jk} := \det(\mathbf{A}^{\mathsf{T}})_{jk} = \det(\mathbf{A}_{kj})^{\mathsf{T}} = \det \mathbf{A}_{kj} = M_{kj}$ by the inductive hypothesis for $j, k = 1, \ldots, n$. Expanding $\det \mathbf{A}^{\mathsf{T}}$ in terms of its first row, and $\det \mathbf{A}$ in terms of its first column,

$$\begin{aligned}
\det \mathbf{A}^{\mathsf{T}} &= a'_{11}M'_{11} - a'_{12}M'_{12} + \cdots + (-1)^{1+n}a'_{1n}M'_{1n} \\
&= a_{11}M_{11} - a_{21}M_{21} + \cdots + (-1)^{n+1}a_{n1}M_{n1} \\
&= \det \mathbf{A}. \quad \square
\end{aligned}$$

Corollary

If $\mathbf{A}$ is upper triangular, then the determinant of $\mathbf{A}$ is the product of its diagonal entries.

Proof. Let $\mathbf{A}$ be upper triangular. Then $\mathbf{A}^{\mathsf{T}}$ is lower triangular and has the same diagonal entries as those of $\mathbf{A}$. $\square$

Let us write $\boxed{\mathbf{A} := \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_k & \cdots & \mathbf{c}_n \end{bmatrix}}$ in terms of its n columns.

Crucial Properties of the **determinant function** $\mathbf{A} \longmapsto \det \mathbf{A}$ from $\mathbb{R}^{n\times n}$ to $\mathbb{R}$:

1. [Multilinearity] If $k \in \{1, \ldots, n\}$ and $\boxed{\mathbf{c}_k = \alpha\, \mathbf{c}'_k + \beta\, \mathbf{c}''_k}$ for some $\alpha, \beta \in \mathbb{R}$ and column vectors $\mathbf{c}'_k$, $\mathbf{c}''_k \in \mathbb{R}^{n\times 1}$, then $\det \mathbf{A} = \det \begin{bmatrix} \mathbf{c}_1 & \cdots & \alpha\, \mathbf{c}'_k + \beta\, \mathbf{c}''_k & \cdots & \mathbf{c}_n \end{bmatrix}$ is equal to $\alpha \det \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}'_k & \cdots & \mathbf{c}_n \end{bmatrix} + \beta \det \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}''_k & \cdots & \mathbf{c}_n \end{bmatrix}$.

This is proved by expanding $\det \mathbf{A}$ in terms of its kth column. Note that the multilinearity implies that $\det(\alpha \mathbf{A}) = \alpha^n \det \mathbf{A}$.

2. [Alternating Property] Suppose $n \geq 2$. If $\boxed{\mathbf{c}_k = \mathbf{c}_\ell}$ for some $k \neq \ell$, then $\det \mathbf{A} = 0$, that is, if 2 columns of a matrix are identical, then its determinant is 0. This is clear for $n = 2$, and for $n \geq 3$, this is proved using induction on n, by expanding $\det \mathbf{A}$ in terms of a column $\mathbf{c}_p$ of $\mathbf{A}$, where $p \neq k$ and $p \neq \ell$.

3. [Normalization Property] $\boxed{\det \mathbf{I} = 1}$, that is, the determinant of the identity matrix is equal to 1.
This is obvious.

Proposition

Let $\mathbf{A}$ be a square matrix.

(i) If two columns of $\mathbf{A}$ are interchanged, then $\det \mathbf{A}$ gets multiplied by -1.

(ii) Addition of a multiple of a column to another column of $\mathbf{A}$ does not alter $\det \mathbf{A}$.

(iii) Multiplication of a column of $\mathbf{A}$ by a scalar α results in the multiplication of $\det \mathbf{A}$ by α.

Proof: Let $\mathbf{A} := \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_k & \cdots & \mathbf{c}_\ell & \cdots \mathbf{c}_n \end{bmatrix}$, where $k \neq \ell$.

(i) Define $\alpha := \det \begin{bmatrix} \mathbf{c}_1 & \cdots & (\mathbf{c}_k + \mathbf{c}_\ell) & \cdots & (\mathbf{c}_k + \mathbf{c}_\ell) & \cdots \mathbf{c}_n \end{bmatrix}$.
Then $\alpha = 0$ since the matrix has two identical columns.

On the other hand, $\alpha = \beta + \gamma$, where
$\beta := \det \begin{bmatrix} \mathbf{c}_1 & \cdots & (\mathbf{c}_k + \mathbf{c}_\ell) & \cdots & \mathbf{c}_k & \cdots \mathbf{c}_n \end{bmatrix}$ and
$\gamma := \det \begin{bmatrix} \mathbf{c}_1 & \cdots & (\mathbf{c}_k + \mathbf{c}_\ell) & \cdots & \mathbf{c}_\ell & \cdots \mathbf{c}_n \end{bmatrix}$.

In turn, $\beta := \beta_1 + \beta_2$ and $\gamma = \gamma_1 + \gamma_2$, where

$$\begin{aligned}
\beta_1 &= \det \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_k & \cdots & \mathbf{c}_k & \cdots \mathbf{c}_n \end{bmatrix}, \\
\beta_2 &= \det \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_\ell & \cdots & \mathbf{c}_k & \cdots \mathbf{c}_n \end{bmatrix}, \\
\gamma_1 &= \det \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_k & \cdots & \mathbf{c}_\ell & \cdots \mathbf{c}_n \end{bmatrix}, \\
\gamma_2 &= \det \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_\ell & \cdots & \mathbf{c}_\ell & \cdots \mathbf{c}_n \end{bmatrix}.
\end{aligned}$$

But $\beta_1 = 0 = \gamma_2$ since two columns are identical. Since $0 = \alpha = \beta + \gamma = \beta_2 + \gamma_1$, we see that $\gamma_1 = -\beta_2$, that is, $\det \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_k & \cdots & \mathbf{c}_\ell & \cdots \mathbf{c}_n \end{bmatrix}$ is equal to $-\det \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_\ell & \cdots & \mathbf{c}_k & \cdots \mathbf{c}_n \end{bmatrix}$, as desired.

(ii) Suppose α times the ℓth column of $\mathbf{A}$ is added to the kth column of $\mathbf{A}$. Then $\det \begin{bmatrix} \mathbf{c}_1 & \cdots & (\mathbf{c}_k + \alpha\, \mathbf{c}_\ell) & \cdots & \mathbf{c}_\ell & \cdots \mathbf{c}_n \end{bmatrix}$ is equal to $\det \mathbf{A} + \alpha \det \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_\ell & \cdots & \mathbf{c}_\ell & \cdots \mathbf{c}_n \end{bmatrix} = \det \mathbf{A}$.

(iii) Suppose the kth column of $\mathbf{A}$ is multiplied by α. Then $\det \begin{bmatrix} \mathbf{c}_1 & \cdots & \alpha\, \mathbf{c}_k & \cdots \mathbf{c}_n \end{bmatrix} = \alpha \det \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_k & \cdots \mathbf{c}_n \end{bmatrix} = \alpha \det \mathbf{A}$.

Corollary

Let $\mathbf{A}$ be a square matrix.

(i) If two rows of $\mathbf{A}$ are interchanged, then $\det \mathbf{A}$ gets multiplied by -1.

(ii) Addition of a multiple of a row to another row of $\mathbf{A}$ does not alter $\det \mathbf{A}$.

(iii) Multiplication of a row of $\mathbf{A}$ by a scalar α results in the multiplication of $\det \mathbf{A}$ by α.

Proof. Since the columns of $\mathbf{A}^\mathsf{T}$ are the rows of $\mathbf{A}$, and since $\det \mathbf{A} = \det \mathbf{A}^\mathsf{T}$, these results follow from the previous proposition. $\square$

The above corollary can be used to find $\det \mathbf{A}$ as follows. Transform $\mathbf{A}$ to $\mathbf{A}'$ by EROs of type I and type II, where $\mathbf{A}'$ is in REF, keeping track of the number of row interchanges.

Now $\mathbf{A}'$ is an upper triangular matrix. Let p be the number of row interchanges, and let $a'_{11}, \ldots, a'_{nn}$ be the diagonal entries of $\mathbf{A}'$. Then $\det \mathbf{A} = (-1)^p \det \mathbf{A}' = (-1)^p a'_{11} \cdots a'_{nn}$.

Example

$$\text{Let } \mathbf{A} := \begin{bmatrix} 0 & 2 & 0 & -1 \\ 1 & 2 & 1 & -1 \\ 0 & 0 & 3 & 2 \\ 1 & -2 & 1 & -2 \end{bmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 3 & 2 \\ 1 & -2 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{R_4 - R_1} \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 3 & 2 \\ 0 & -4 & 0 & -1 \end{bmatrix} \xrightarrow{R_4 + 2R_2} \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & -3 \end{bmatrix} = \mathbf{A}'.$$

Since there is only one row interchange while transforming $\mathbf{A}$ to $\mathbf{A}'$, since $\mathbf{A}'$ is in REF, and since $\det \mathbf{A}' = 1 \cdot 2 \cdot 3 \cdot (-3) = -18$, we see that $\det \mathbf{A} = (-1)(-18) = 18$.

We give a criterion for the invertibility of a square matrix in terms of its determinant.

Proposition

A square matrix $\mathbf{A}$ is invertible if and only if $\det \mathbf{A} \neq 0$.

Proof. Suppose $\mathbf{A}$ is invertible. We have seen that $\mathbf{A}$ can be transformed to its RCF, namely to $\mathbf{I}$, by EROs. Suppose this process involves p row interchanges (that is, EROs of type I) and multiplications of rows by the nonzero scalars $\alpha_1, \ldots, \alpha_q$ (that is, EROs of type III). Then

$$\det(\mathbf{A}) = (-1)^p(\alpha_1 \cdots \alpha_q)^{-1} \det \mathbf{I} = (-1)^p(\alpha_1 \cdots \alpha_q)^{-1} \neq 0.$$

Conversely, suppose $\mathbf{A}$ is not invertible. Then the column rank of $\mathbf{A}$ is less than the number of columns of $\mathbf{A}$. Hence one of its columns is a linear combination of the other columns. WLOG, we suppose that it is the first column, that is, $\mathbf{c}_1 = \alpha_2\mathbf{c}_2 + \cdots + \alpha_k\mathbf{c}_k + \cdots + \alpha_n\mathbf{c}_n$, where $\alpha_2, \ldots, \alpha_n \in \mathbb{R}$.

By the first two crucial properties of the determinant function,

$$
\begin{aligned}
\det \mathbf{A} &= \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_k & \cdots & \mathbf{c}_n \end{bmatrix} \\
&= \begin{bmatrix} \alpha_2\mathbf{c}_2 + \cdots + \alpha_k\mathbf{c}_k + \cdots + \alpha_n\mathbf{c}_n & \mathbf{c}_2 & \cdots & \mathbf{c}_k & \cdots & \mathbf{c}_n \end{bmatrix} \\
&= \alpha_2 \begin{bmatrix} \mathbf{c}_2 & \mathbf{c}_2 & \cdots & \mathbf{c}_k & \cdots & \mathbf{c}_n \end{bmatrix} + \cdots \\
&\quad \alpha_k \begin{bmatrix} \mathbf{c}_k & \mathbf{c}_2 & \cdots & \mathbf{c}_k & \cdots & \mathbf{c}_n \end{bmatrix} + \cdots \\
&\quad \alpha_n \begin{bmatrix} \mathbf{c}_n & \mathbf{c}_2 & \cdots & \mathbf{c}_k & \cdots & \mathbf{c}_n \end{bmatrix} \\
&= 0 + \cdots + 0 + \cdots + 0 = 0. \qquad \square
\end{aligned}
$$

Remark We have given several criteria for the invertibility of an $n \times n$ matrix $\mathbf{A}$. We list them below.

(i) The linear system $\mathbf{Ax} = \mathbf{0}$ has $\mathbf{0}$ as the only solution.
(ii) There is a matrix $\mathbf{B}$ such that $\mathbf{BA} = \mathbf{I}$ or $\mathbf{AB} = \mathbf{I}$.
(iii) The RCF of $\mathbf{A}$ is $\mathbf{I}$.
(iv) $\operatorname{rank} \mathbf{A} = n$.
(v) $\operatorname{nullity}(\mathbf{A}) = 0$,
(vi) $\det \mathbf{A} \neq 0$.