PH111 Introduction to Classical Mechanics
Chapter 2: Kinematics

Velocity and Acceleration: General Definitions

- Velocity of a particle is defined as

$$\mathrm{v} = \lim_{\Delta t \to 0} \frac{\mathrm{r}(t+\Delta t) - \mathrm{r}(t)}{\Delta t} = \frac{d\mathrm{r}}{dt}.$$

- Similarly, the acceleration is defined as

$$\mathrm{a} = \lim_{\Delta t \to 0} \frac{\mathrm{v}(t+\Delta t) - \mathrm{v}(t)}{\Delta t} = \frac{d\mathrm{v}}{dt} = \frac{d}{dt}\left(\frac{d\mathrm{r}}{dt}\right) = \frac{d^2\mathrm{r}}{dt^2}.$$

Velocity and Acceleration: Cartesian Coordinates

- In Cartesian coordinates

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

- Therefore

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}}$$

- So that the three Cartesian components can be deduced

$$v_x = \frac{dx}{dt}$$
$$v_y = \frac{dy}{dt}$$
$$v_z = \frac{dz}{dt}$$

- Likewise, for acceleration we have

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2},$$

Cartesian coordinates: Accelaration

- so that

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$$
$$a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}$$
$$a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}$$

Cartesian Coordinates: Solving the kinematic equations

- The question is: suppose we know the acceleration of a particle as a function of time, $\mathrm{a}(t)$
- How can we obtain its velocity $\mathrm{v}(t)$, and the position $\mathrm{r}(t)$, as a function of time?
- This clearly requires integrating the so-called kinematic equation

$$\frac{d^2\mathrm{r}}{dt^2} = \mathrm{a}(t).$$

- How do we achieve that?

Cartesian Coordinates: Integrating the kinematic equations

- Formally speaking, one has to follow a two-step procedure
- First, we integrate the acceleration to obtain velocity

$$
\begin{aligned}
\frac{d\mathrm{v}}{dt} &= \mathrm{a}(t). \\
\implies d\mathrm{v} &= \mathrm{a}dt \\
\implies \int d\mathrm{v} &= \int \mathrm{a}dt
\end{aligned}
$$

- The integral above is an indefinite integral, therefore, we need initial conditions to integrate it fully
- Assuming that at some initial time t_0, the velocity of the particle is v_0, i.e., $\mathrm{v}(t_0) = \mathrm{v}_0$.

Integrating the kinematic equations: velocity

- Then

$$\int_{\mathrm{v}_0}^{\mathrm{v}(t)} d\mathrm{v} = \int_{t_0}^{t} \mathrm{a} dt'$$

$$\mathrm{v}(t) - \mathrm{v}(t_0) = \int_{t_0}^{t} \mathrm{a} dt'$$

$$\mathrm{v}(t) = \mathrm{v}_0 + \int_{t_0}^{t} \mathrm{a}(t') dt'.$$

In the last equation above, we changed the variable of integration to t', to keep it distinct from the general time t.

- The integrated expression above is a vector equation, which has three Cartesian components.

Integrating the kinematic equations: position

- We need to perform one more integration to obtain position r, assuming that now the velocity is known

$$\frac{d\mathrm{r}}{dt} = \mathrm{v}(t)$$

- Upon integrating this similar to the previous case, we obtain

$$\mathrm{r}(t) = \mathrm{r}_0 + \int_{t_0}^{t} \mathrm{v}(t)dt,$$

where $\mathrm{r}(t_0) = \mathrm{r}_0$, is the initial position of the particle.

Full form of integrated equations in Cartesian coordinates

- Three Cartesian components of the acceleration equation

$$v_x(t) = v_{0x} + \int_{t_0}^{t} a_x(t')dt'$$

$$v_y(t) = v_{0y} + \int_{t_0}^{t} a_y(t')dt'$$

$$v_z(t) = v_{0z} + \int_{t_0}^{t} a_z(t')dt'$$

- And, three Cartesian components of the velocity equation

$$x(t) = x_0 + \int_{t_0}^{t} v_x(t')dt'$$

$$y(t) = y_0 + \int_{t_0}^{t} v_y(t')dt'$$

$$z(t) = z_0 + \int_{t_0}^{t} v_z(t')dt'$$

- **Example 1:** A particle moving with constant acceleration $\mathrm{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}$, i.e., a_x, a_y, and a_z are constants..
- Assume, initial velocity $\mathrm{v}(t=0) = \mathrm{u} = u_x\hat{\mathrm{i}} + u_y\hat{\mathrm{j}} + u_z\hat{\mathrm{k}}$.
- And, initial position $\mathrm{r}(t=0) = \mathrm{r}_0 = x_0\hat{\mathrm{i}} + y_0\hat{\mathrm{j}} + z_0\hat{\mathrm{k}}$.
- Equation of motion is

$$\frac{d\mathrm{v}}{dt} = \mathrm{a}(t)$$

which leads to

$$\frac{dv_x}{dt} = a_x$$
$$\frac{dv_y}{dt} = a_y$$
$$\frac{dv_z}{dt} = a_z$$

Example 1 contd.

- The form of the equations is identical for the three directions.
- So we integrate the x equation, and same result will apply to other directions
- We have

$$dv_x = a_x dt$$
$$\implies \int_{u_x}^{v_x} dv_x = a_x \int_{t=0}^{t} dt'$$
$$v_x(t) = u_x + a_x t$$

Similarly for y and z directions

$$v_y(t) = u_y + a_y t$$
$$v_z(t) = u_z + a_z t$$

- These can be combined in a single vector equation

$$\mathrm{v} = \mathrm{u} + \mathrm{a}t$$

Example 1 contd....

- To obtain the trajectory $\mathrm{r}(t)$, we must integrate the velocity equation

$$\mathrm{v} = \frac{d\mathrm{r}}{dt} = \mathrm{u} + \mathrm{a}t$$

- We consider the x component of the equation

$$\frac{dx}{dt} = u_x + a_x t,$$

which can be integrated as

$$\begin{aligned} dx &= u_x dt + a_x t dt \\ \Longrightarrow \int_{x_0}^{x} dx' &= u_x \int_0^t dt' + a_x \int_0^t t' dt' \\ \Longrightarrow x - x_0 &= u_x t + \frac{1}{2} a_x t^2 \\ x(t) &= x_0 + u_x t + \frac{1}{2} a_x t^2 \end{aligned}$$

Example 1 ....

- Similarly, we have for y and z components

$$y(t) = y_0 + u_y t + \frac{1}{2} a_y t^2$$
$$z(t) = z_0 + u_z t + \frac{1}{2} a_z t^2$$

- These yield the vector equation

$$\mathrm{r}(t) = \mathrm{r}_0 + \mathrm{u} t + \frac{1}{2} \mathrm{a} t^2$$

- Define the net displacement $\mathrm{s} = \mathrm{r} - \mathrm{r}_0$, so that

$$\mathrm{s}(t) = \mathrm{u} t + \frac{1}{2} \mathrm{a} t^2.$$

Note the similarity of this equation with the 1D equation $s = ut + \frac{1}{2} at^2$.

Example 2: Electron in an oscillating electric field

- Imagine that an electron of charge $-e$ and mass m is exposed to an electric field
$$\mathrm{E} = E_0 \sin \omega t \hat{\mathrm{i}}$$
- Force F acting on the electron is $\mathrm{F} = -e\mathrm{E}$, so that
$$\mathrm{a}(t) = \frac{\mathrm{F}}{m} = -\frac{e}{m} E_0 \sin \omega t \hat{\mathrm{i}}$$
- Initial conditions: at $t = 0$, electron is at rest, at the origin,
- Effectively, we have 1D motion in x direction, so that
$$\frac{dv}{dt} = -\frac{e}{m} E_0 \sin \omega t.$$

- First integration of the equation yields

$$v(t) = v_0 + \int_0^t a(t')dt'$$
$$v(t) = v_0 - \frac{eE_0}{m}\int_0^t \sin\omega t' dt'$$
$$v(t) = v_0 + \frac{eE_0}{m\omega}(\cos\omega t - 1)$$

because $v_0 = 0$, we have

$$v(t) = \frac{eE_0}{m\omega}(\cos\omega t - 1)$$

- Integration of velocity equation yields the trajectory

$$x(t) = x_0 + \int_0^t v(t')dt'$$
$$x(t) = x_0 + \frac{eE_0}{m\omega}\int_0^t (\cos\omega t' - 1)\, dt'$$

- On integration, because $v_0 = 0$, we have

$$x(t) = x_0 + \frac{eE_0}{m\omega^2}(\sin\omega t - \omega t)$$

- Using the fact $x_0 = 0$, we finally have

$$x(t) = \frac{eE_0}{m\omega^2}(\sin\omega t - \omega t).$$

Note that besides an oscillating term, we also have a term which denotes drift of the electron with a constant velocity!

Kinematics in plane polar coordinates

- Computing quantities such as velocity and acceleration is a bit more complicated in plane polar coordinates
- The reason: $\hat{\mathrm{r}}$ and $\hat{\theta}$ are direction dependent
- Let us compute the velocity

$$\mathrm{v} = \frac{d\mathrm{r}}{dt} = \frac{d}{dt}(r\hat{\mathrm{r}}).$$

- Using the chain rule, we have

$$\mathrm{v} = \frac{dr}{dt}\hat{\mathrm{r}} + r\frac{d\hat{\mathrm{r}}}{dt}.$$

- Note that as the particle moves, vector $\hat{\mathrm{r}}$ also changes so that $\frac{d\hat{\mathrm{r}}}{dt} \neq 0$.
- But, how to compute $\frac{d\hat{\mathrm{r}}}{dt}$?
- A geometric calculation is possible, but let us take a different approach.

Velocity in plane polar coordinates

- Let us use Cartesian coordinates for the purpose

$$\hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$$

- Because Cartesian basis vectors $\hat{i}$ and $\hat{j}$ have fixed directions in space, so they don't change with time
- Therefore

$$\frac{d\hat{r}}{dt} = \frac{d\cos\theta}{dt}\hat{i} + \frac{d\sin\theta}{dt}\hat{j}$$

- Now

$$\frac{d\cos\theta}{dt} = -\sin\theta\frac{d\theta}{dt} = -\sin\theta\,\dot{\theta}$$
$$\frac{d\sin\theta}{dt} = \cos\theta\frac{d\theta}{dt} = \cos\theta\,\dot{\theta}$$

- So that

$$\frac{d\hat{r}}{dt} = -\sin\theta\,\dot{\theta}\hat{i} + \cos\theta\,\dot{\theta}\hat{j} = \dot{\theta}\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right) = \dot{\theta}\,\hat{\theta}$$

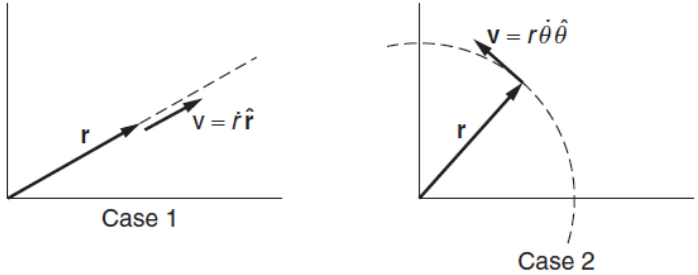
- Therefore, finally we have

$$\mathrm{v} = \frac{dr}{dt}\hat{\mathrm{r}} + r\frac{d\hat{\mathrm{r}}}{dt} = \dot{r}\hat{\mathrm{r}} + r\dot{\theta}\hat{\theta}$$
$$= v_r\hat{\mathrm{r}} + v_\theta\hat{\theta}$$

- Thus, we have obtained an expression for velocity in terms of its radial and angular (also called tangential) components
- What is the physical significance of v_r and v_θ?

Velocity in polar coordinates: Physical Significance

- Consider the figure



- **Case 1:** This case corresponds to motion along the radial direction, with θ held fixed ($\dot{\theta} = 0$), so that $v = \dot{r}\hat{r}$.
- **Case 2:** Here there is no radial motion ($\dot{r} = 0$), so velocity will be along the arc of a circle with $v = r\dot{\theta}\hat{\theta}$

Acceleration in polar coordinates

- Acceleration can be computed as

$$
\begin{aligned}
\mathrm{a} &= \frac{d\mathrm{v}}{dt} \\
&= \frac{d}{dt}\left(\dot{r}\hat{\mathrm{r}} + r\dot{\theta}\hat{\theta}\right) \\
&= \frac{d\dot{r}}{dt}\hat{\mathrm{r}} + \dot{r}\frac{d\hat{\mathrm{r}}}{dt} + \frac{d(r\dot{\theta})}{dt}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt} \\
&= \ddot{r}\hat{\mathrm{r}} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}
\end{aligned}
$$

- We compute $\frac{d\hat{\theta}}{dt}$, by expressing $\hat{\theta}$ in Cartesian coordinates

$$
\begin{aligned}
\frac{d\hat{\theta}}{dt} &= \frac{d}{dt}\left(-\sin\theta\hat{\mathrm{i}} + \cos\theta\hat{\mathrm{j}}\right) \\
&= -\cos\theta\,\dot{\theta}\hat{\mathrm{i}} - \sin\theta\,\dot{\theta}\hat{\mathrm{j}} \\
&= -\dot{\theta}\hat{\mathrm{r}}
\end{aligned}
$$

Acceleration in polar coordinates....

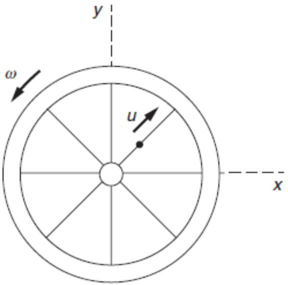
- On substituting the expression of $\frac{d\hat{\theta}}{dt}$, we obtain

$$\mathrm{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathrm{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta}$$

- Different terms have the following interpretations
  - $\ddot{r}$ due to change of radial speed, points in radial direction
  - $-r\dot{\theta}^2$ centripetal acceleration, pointing radially inwards
  - $2\dot{r}\dot{\theta}$ Coriolis acceleration, present whenever both radial and angular velocities are nonzero, and points in the tangential direction
  - $r\ddot{\theta}$ tangential angular acceleration, due to changing angular velocity, points in tangential direction

Example: Bead moving along the spoke of a rotating wheel

- Consider a bead moving along the spoke of a rotating wheel



- Both u and ω are constant
- Let us calculate the velocity and acceleration of the bead in plane polar coordinates

Bead on a spoke...

- Here, clearly

$$\dot{r} = u$$
$$\dot{\theta} = \omega$$
$$\ddot{r} = 0$$
$$\ddot{\theta} = 0$$

- Therefore, velocity in polar coordinates is

$$\mathrm{v} = u\hat{\mathrm{r}} + r\omega\hat{\theta}$$

- But, for this case, clearly $r = ut$, if $r_0 = 0$. Therefore

$$\mathrm{v} = u\hat{\mathrm{r}} + ut\omega\hat{\theta}$$

- And the acceleration

$$\begin{aligned} \mathrm{a} &= -\omega^2 r\hat{\mathrm{r}} + 2u\omega\hat{\theta} \\ &= -\omega^2 ut\hat{\mathrm{r}} + 2u\omega\hat{\theta} \end{aligned}$$