Chapter 3: Work-Energy Theorem

Newton's Law of Motion

- First we define the momentum ($\mathbf{p}$) of a particle (or a more complicated system) as

$$\mathbf{p} = m\mathbf{v},$$

  where m is the mass of the system, and $\mathbf{v}$ is its velocity.
- Newtons' second law of motion states that force $\mathbf{F}$ acting on a system is equal to its rate of change of momentum

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}.$$

- This is the most general definition of Newton's second law of motion and is applicable also to those systems, such as a rocket, whose mass m is not constant.
- However, for a system whose mass does not change with time, we have its more familiar form

$$\mathbf{F} = \frac{d(m\mathbf{v})}{dt} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}.$$

- Next, we explore some consequences of Newton's second law.

Work and Energy

- Consider a 1D system confined to move in x direction
- Let the force acting on a system of mass m, be $F(x)$
- Thus the force can depend on the position
- Then the work done dW in moving the particle by an infinitesimal amount dx is given by

$$dW = F(x)dx$$

- Thus, the work done W_{ab} in moving the particle from position $x = a$ to $x = b$, will be the integral of the expression above

$$W_{ab} = \int_a^b F(x)dx$$

Work-Energy relationship

- Let us manipulate this expression

$$W_{ab} = \int_a^b F(x)dx = \int_a^b madx = m\int_a^b \frac{dv}{dt}dx.$$

- But, we can write

$$dx = \frac{dx}{dt}dt = vdt.$$

- Substituting it above, we have

$$W_{ab} = m\int_a^b \frac{dv}{dt}vdt = m\int_a^b \frac{1}{2}\frac{dv^2}{dt}dt = \int_a^b \frac{d}{dt}\left(\frac{1}{2}mv^2\right)dt$$

$$\implies W_{ab} = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2$$

Work-Energy Theorem

- Thus, we have shown in 1D that work done on a particle in taking it from point A to B, is nothing but change in its kinetic energy during the journey
- This is nothing but the statement of work-energy theorem in 1D
- But most forces are three dimensional in nature, as are most of the displacements
- For a 3D case, the force $\mathbf{F}$, at position $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, can be written in terms of Cartesian components as

$$\mathbf{F}(\mathbf{r}) = \mathbf{F}(x,y,z) = F_x(x,y,z)\hat{\mathbf{i}} + F_y(x,y,z)\hat{\mathbf{j}} + F_z(x,y,z)\hat{\mathbf{k}}$$

- Note that each component of force is a function of all the three Cartesian coordinates
- A position dependent vector quantity such as $\mathbf{F}(\mathbf{r})$, is called a vector field.

Work-Energy Theorem in 3D

- Suppose this force displaces a particle of mass m by an infinitesimal vector $d\mathbf{r} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$, then the total work done will be

$$dW = F_x dx + F_y dy + F_z dz = \mathbf{F}\cdot d\mathbf{r}$$

- Obviously, work done in displacing the particle by a finite amount, starting from $\mathbf{r} = \mathbf{r}_a$ to $\mathbf{r} = \mathbf{r}_b$, will be

$$W_{ab} = \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F}\cdot d\mathbf{r}.$$

  Such three dimensional integrals are called line integrals, which need to be evaluated along a path.

- Similar to the 1D case, we have

$$d\mathbf{r} = \frac{d\mathbf{r}}{dt}dt = \mathbf{v}dt$$

Work-Energy Theorem 3D...

- Then

$$W_{ab} = \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_a}^{\mathbf{r}_b} m\frac{d\mathbf{v}}{dt} \cdot \mathbf{v} dt = \int_{\mathbf{r}_a}^{\mathbf{r}_b} \frac{d}{dt}\left(\frac{1}{2}m\mathbf{v} \cdot \mathbf{v}\right) dt$$
$$= \int_{\mathbf{r}_a}^{\mathbf{r}_b} \frac{d}{dt}\left(\frac{1}{2}mv^2\right) dt$$

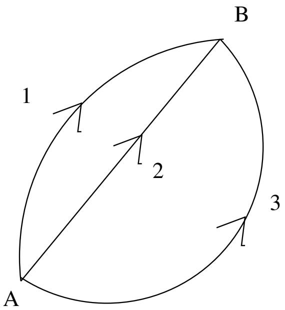
- Finally

$$W_{ab} = \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2.$$

- Thus the form of Work-Energy theorem in 3D is similar to that in 1D

Line Integrals

- Because work done is expressed in terms of a line integral ($W_{ab} = \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F} \cdot d\mathbf{r}$), it will, in principle, depend on the path connecting points A and B.
- For example, for the three paths shown below, the line integral, in general, will have three different values



- Do we have forces $\mathbf{F}(\mathbf{r})$ for which this line integral is path independent?

Conservative Forces

- Most fundamental forces in nature satisfy this property
- Examples: gravitational force, electrostatic force
- For such forces work done will not depend on the path of displacement
- Rather it will depend only on the positions of the end points (A and B in this case) of the path
- Such forces are called "Conservative Forces"

Potential Energy

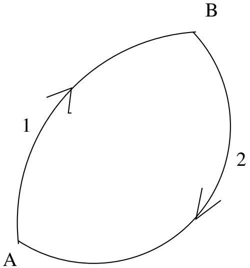
- Thus, for conservative forces, a mathematical function function $V(\mathbf{r})$ exists such that

$$W_{ab} = \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F}\cdot d\mathbf{r} = -\left(V(\mathbf{r}_b) - V(\mathbf{r}_a)\right)$$

  Above negative sign on the RHS is chosen as a matter of convention
- If such a function $V(\mathbf{r})$ did not exist, line integral will always depend on the path connecting A and B
- Thus $V(\mathbf{r})$ guarantees that the work done depends only on the endpoints of the path, and not the path itself
- The function $V(\mathbf{r})$ has dimensions of energy, and is called the potential energy. $V(\mathbf{r})$ is a scalar field, unlike $\mathbf{F}(\mathbf{r})$, which is a vector field.

Potential energy: properties

- It is easier to deal with scalars rather than vectors, because one doesn't have to worry about a direction.
- For conservative forces, work done along a closed path is zero
- Consider the closed path shown below



- Along the closed path shown above

$$\oint \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathbf{r}_b}^{\mathbf{r}_a} \mathbf{F} \cdot d\mathbf{r}$$

$$= -(V(\mathbf{r}_b) - V(\mathbf{r}_a)) - (V(\mathbf{r}_a) - V(\mathbf{r}_b))$$

$$= 0$$

Conservation of Energy

- A consequence of work-energy theorem for conservative forces is that sum of kinetic and potential energies of a system is conserved
- For a conservative force we have

$$W_{ab} = \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2 = V(\mathbf{r}_a) - V(\mathbf{r}_b)$$

$$\Longrightarrow \frac{1}{2}mv_a^2 + V(\mathbf{r}_a) = \frac{1}{2}mv_b^2 + V(\mathbf{r}_b)$$

  which is nothing but conservation of total energy (kinetic + potential)
- That is the reason behind the name "conservative force".

Potential energy at a point

- So far we have computed only the potential energy difference between two points (A and B, say)
- How do we define the potential energy $V(\mathbf{r})$, at a given point $\mathbf{r}$ in space?
- It is defined with respect to a reference point $\mathbf{r}_O$, which is normally taken to be infinity
- It is defined as the work done against the force $\mathbf{F}(\mathbf{r})$, in bringing the particle from the reference point O to point $\mathbf{r}$

$$V(\mathbf{r}) = -\int_{\mathbf{r}_O}^{\mathbf{r}} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

Relation between force and potential energy

- Consider a 1D conservative force, so that

$$V(b) - V(a) = -\int_a^b F(x)dx.$$

- Let points $x = a$ and $x = b$ be infinitesimally close to each other, i.e., $a = x$ and $b = x + \Delta x$, with Δx small

$$V(x+\Delta x) - V(x) = -\int_x^{x+\Delta x} F(x')dx'.$$

- We define $\Delta V(x) = V(x+\Delta x) - V(x)$, and for small Δx, we have

$$\int_x^{x+\Delta x} F(x')dx' \approx F(x)\Delta x + \cdots$$

- Substituting it above, we obtain

$$\Delta V \approx -F(x)\Delta x$$

$$F(x) \approx -\frac{\Delta V}{\Delta x}.$$

Force and Potential Energy....

- In the limit $\Delta x \to 0$, we get

$$F(x) = -\frac{dV}{dx}$$

- This is the required relationship between F and V in 1D.
- How to generalize it to 3D?

Force and Potential Energy: Connection in 3D

- In 3D, both $\mathbf{F}(\mathbf{r})$ and $V(\mathbf{r})$ are functions of all three Cartesian coordinates x, y, and z.
- So, we have to be careful with our mathematics
- We know

$$V(\mathbf{r}_b) - V(\mathbf{r}_a) = -\int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F} \cdot d\mathbf{r}.$$

- As before, we choose $\mathbf{r}_a = \mathbf{r}$ and $\mathbf{r}_b = \mathbf{r} + \Delta\mathbf{r}$, to obtain

$$V(\mathbf{r} + \Delta\mathbf{r}) - V(\mathbf{r}) = -\int_{\mathbf{r}}^{\mathbf{r}+\Delta\mathbf{r}} \mathbf{F} \cdot d\mathbf{r}'.$$

- Because $\Delta\mathbf{r} = \Delta x\hat{\mathbf{i}} + \Delta y\hat{\mathbf{j}} + \Delta z\hat{\mathbf{k}}$ is an infinitesimal displacement vector in 3D, so

$$-\int_{\mathbf{r}}^{\mathbf{r}+\Delta\mathbf{r}} \mathbf{F} \cdot d\mathbf{r}' \approx -\mathbf{F}(\mathbf{r}) \cdot \Delta\mathbf{r} = -F_x\Delta x - F_y\Delta y - F_z\Delta z$$

Force and Potential Energy in 3D....

- To compute $V(\mathbf{r}+\Delta\mathbf{r}) - V(\mathbf{r})$, we use Taylor's expansion for multiple variables

$$
\begin{aligned}
V(\mathbf{r}+\Delta\mathbf{r}) &= V(x+\Delta x, y+\Delta y, z+\Delta z) \\
&= V(x,y,z) + \frac{\partial V}{\partial x}\Delta x + \frac{\partial V}{\partial y}\Delta y + \frac{\partial V}{\partial z}\Delta z + O(dr^2) \\
&= V(\mathbf{r}) + \nabla V \cdot \Delta\mathbf{r} + O(dr^2)
\end{aligned}
$$

- Symbol ∇V, stands for "gradient of V", defined as

$$
\nabla V = \frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}
$$

- $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$, and $\frac{\partial V}{\partial z}$ are called "partial derivatives", computed by taking the derivative with respect to the given variable (say x), treating other two variables (say y and z) as constants.
- Note that gradient operator applies on a scalar field, and the result is a vector field.

Potential energy contd...

- With this

$$V(\mathbf{r}+\Delta\mathbf{r}) - V(\mathbf{r}) = \nabla V \cdot \Delta\mathbf{r} = -\mathbf{F}(\mathbf{r}) \cdot \Delta\mathbf{r}.$$

- Because $\Delta\mathbf{r}$ is an arbitrary displacement, therefore,

$$\nabla V \cdot \Delta\mathbf{r} = -\mathbf{F}(\mathbf{r}) \cdot \Delta\mathbf{r}$$
$$\implies \mathbf{F}(\mathbf{r}) = -\nabla V$$

- This is a very important result showing that a conservative force can be written as the gradient of corresponding potential energy.
- Before we proceed further, let us have a bit of mathematical exploration

Calculation of Gradient: Example 1

- First let us compute a few partial derivatives
- Let $f(x,y,z) = r^2 = x^2 + y^2 + z^2$, then

$$\frac{\partial f}{\partial x} = 2x$$
$$\frac{\partial f}{\partial y} = 2y$$
$$\frac{\partial f}{\partial z} = 2z$$

- So that

$$\nabla f = 2(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) = 2\mathbf{r}$$

Calculation of Gradient: Example 2

- Let $g(x,y,z) = xyz$, then

$$\frac{\partial g}{\partial x} = yz$$
$$\frac{\partial g}{\partial y} = xz$$
$$\frac{\partial g}{\partial z} = xy$$

- So that

$$\nabla g = yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}$$

A few suggested exercises

- Compute the gradient of the following scalar functions

$$f(x,y,z) = x^4 + y^4 + z^4$$
$$g(x,y,z) = x^2y^2 + y^2z^2 + z^2x^2$$
$$\Phi(x,y,z) = 3xy^2z^3 + 2xyz + 4x^2y^2$$

- Gradient and other functions can also be computed in other coordinate systems such as plane-polar coordinates

A bit of vector calculus: Gradient of a Scalar Function

- Consider a scalar function $T(x, y, z)$
- We want to compute the change in T, as we move from initial coordinates $\mathbf{r} \equiv (x, y, z)$ infinitesimally to the new position $\mathbf{r} + \mathbf{dr} \equiv (x + dx, y + dy, z + dz)$
- Using Taylor expansion (for multi-variables), and retaining terms up to first order

$$T(\mathbf{r}+\mathbf{dr}) = T(\mathbf{r}) + dx\frac{\partial T}{\partial x} + dy\frac{\partial T}{\partial y} + dz\frac{\partial T}{\partial z} + \text{higher order terms}$$

- Or, to the first order terms,

$$T(\mathbf{r}+\mathbf{dr}) = T(\mathbf{r}) + \mathbf{dr}\cdot\nabla T$$

- Where

$$\mathbf{dr} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$\nabla T = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}$$

Gradient contd.

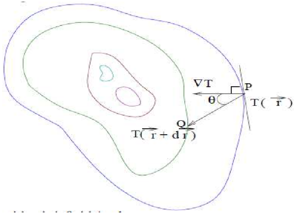
- Defining $T(\mathbf{r}+\mathbf{dr}) = T(\mathbf{r}) + dT$, we conclude

$$dT = \mathbf{dr} \cdot \nabla T,$$

  where the vector ∇T defined above is called the gradient of scalar field T.
- Thus ∇T defines the rate of change of the scalar field with respect to the spatial coordinates, and is itself a vector quantity
- Let us examine ∇T a bit more

Physical Meaning of Gradient

- Let us plot the constant surfaces of a given scalar field T



- As per the figure, we can write the change in the scalar field dT as

$$dT = \mathbf{dr} \cdot \nabla T = |\mathbf{dr}||\nabla T| \cos \theta$$

- Let us consider two possibilities:
  - **dr** is along a constant T surface
  - **dr** is in an arbitrary direction

Gradient, physical meaning...

- If $\mathbf{dr}$ is along a constant T surface then $dT = 0$. This means

$$|\mathbf{dr}||\nabla T|\cos\theta = 0$$
$$\implies \cos\theta = 0$$

- Thus the direction of ∇T at a given point $\mathbf{r}$ is always perpendicular to the constant T surface passing through that point
- Let us consider $\mathbf{dr}$ to be in an arbitrary direction
- Then from $dT = |\mathbf{dr}||\nabla T|\cos\theta$, it is obvious that the magnitude of the maximum possible change in T is

$$dT_{max} = |\mathbf{dr}||\nabla T|,$$

  i.e., when $\cos\theta = 1$.
- Thus the direction of ∇T is also the direction of maximum change in the scalar function T.

Gradient continued...

- Thus, at a given point **r**, if one moves in the direction of ∇T, maximum change in T will take place
- This property of gradient is used in optimization problems involving location of maxima/minima of scalar functions

Examples:

1. Let us consider a scalar function

$$T = r^2 = x^2 + y^2 + z^2$$

It is easy to see that

$$\nabla T == \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k} = 2x\hat{i} + 2y\hat{j} + 2z\hat{k} = 2\mathbf{r}$$

2. Consider $\Phi(x, y, z) = x^2y + y^2z + z^2x + 2xyz$

Gradient calculation

Clearly

$$\nabla\Phi = \frac{\partial\Phi}{\partial x}\hat{i} + \frac{\partial\Phi}{\partial y}\hat{j} + \frac{\partial\Phi}{\partial z}\hat{k}$$
$$= (2xy + 2yz + z^2)\hat{i} + (2yz + 2xz + x^2)\hat{j} + (2zx + 2xy + y^2)\hat{k}$$

- Thus, in Cartesian coordinates, the gradient operator can be denoted as
$$\nabla \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$
- In curvilinear coordinates the gradient operator has more complicated forms

$$\text{Cylindrical } \nabla \equiv \frac{\partial}{\partial\rho}\hat{\rho} + \frac{\partial}{\rho\partial\theta}\hat{\theta} + \frac{\partial}{\partial z}\hat{k}$$
$$\text{Spherical } \nabla \equiv \frac{\partial}{\partial r}\hat{r} + \frac{\partial}{r\partial\theta}\hat{\theta} + \frac{\partial}{r\sin\theta\partial\phi}\hat{\phi}$$

Some Properties of Gradient

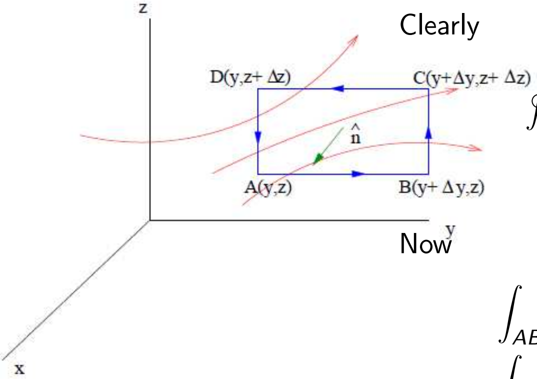
1. $$\nabla(U+V)=\nabla U+\nabla V$$

2. $$\nabla(UV)=U\nabla V+V\nabla U$$

3. $$\nabla(V^n)=nV^{n-1}\nabla V$$

Curl of a Vector Field

Let us consider a vector field $\mathbf{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$, and evaluate its line integral along a infinitesimal rectangular path shown below



Clearly

$$\oint \mathbf{F}.d\mathbf{l} = \int_{AB} \mathbf{F}.d\mathbf{l} + \int_{BC} \mathbf{F}.d\mathbf{l} + \int_{CD} \mathbf{F}.d\mathbf{l} + \int_{DA} \mathbf{F}.d\mathbf{l}$$

Now

$$\int_{AB} \mathbf{F}.d\mathbf{l} = \int \mathbf{F}.dy\hat{j} \approx F_y(y,z)\Delta y$$

$$\int_{BC} \mathbf{F}.d\mathbf{l} = \int \mathbf{F}.dz\hat{k} \approx F_z(y+\Delta y, z)\Delta z$$

Curl contd....

Using first order Taylor expansion

$$F_z(y+\Delta y, z) = F_z(y,z) + \frac{\partial F_z}{\partial y}\Delta y$$

So that

$$\int_{AB}\mathbf{F}.d\mathbf{l} + \int_{BC}\mathbf{F}.d\mathbf{l} = \left(F_y\Delta y + F_z\Delta z + \frac{\partial F_z}{\partial y}\Delta z\Delta y\right)$$

Similarly one can show (by integrating in AD and DC directions)

$$\int_{CD}\mathbf{F}.d\mathbf{l} + \int_{DA}\mathbf{F}.d\mathbf{l} = -\left(F_y\Delta y + F_z\Delta z + \frac{\partial F_y}{\partial z}\Delta z\Delta y\right)$$

By adding all the contributions we obtain

$$\oint \mathbf{F}.\mathbf{dl} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\Delta S_x \tag{1}$$

Curl contd....

Where $\Delta S_x = \Delta y \Delta z$, is the area of the infinitesimal loop, directed along the x axis. Let us define a quantity called curl, denoted as $\nabla\times$

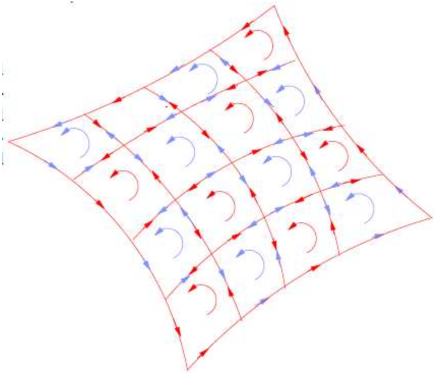
$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}.$$

Using this we can cast Eq. 1 as

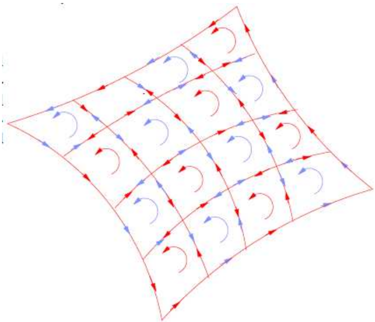
$$\oint \mathbf{F}.d\mathbf{l} = (\nabla \times \mathbf{F})_x \Delta S_x = (\nabla \times \mathbf{F}) \cdot \Delta \mathbf{S} \tag{2}$$

Stokes' Theorem: If a vector field **F** is integrated along a closed loop of an arbitrary shape, then the line integral is equal to the surface integral of the curl of **F**, evaluated over the area enclosed by the loop

$$\oint \mathbf{F}.\mathbf{dl} = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{dS}$$



Outline of the proof:



We can split the area enclosed by the loop into a large number of infinitesimal loops as shown, for each one of which Eq. 2 will hold. Upon adding the contribution of all such loops, we get the desired result

$$\oint \mathbf{F}.d\mathbf{l} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

Note that in the line integral, the contribution only from the boundary of the loop will survive because the contribution from the internal lines gets canceled from adjacent loops.

Curl in Cylindrical Coordinates: For a vector field $\mathbf{A} = A_\rho \hat{\rho} + A_\theta \hat{\theta} + A_z \hat{\mathbf{z}}$

$$\nabla \times \mathbf{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\theta} + \frac{1}{\rho}\left(\frac{\partial(\rho A_\theta)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right)\hat{\mathbf{z}}$$

Curl in Spherical Polar Coordinates: For a vector field $\mathbf{A} = A_r\hat{\boldsymbol{r}} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$

$$\begin{aligned}\nabla \times \mathbf{A} &= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(A_\phi \sin\theta\right) - \frac{\partial A_\theta}{\partial\phi}\right)\hat{\boldsymbol{r}} \\ &+ \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rA_\phi\right)\right)\hat{\theta} \\ &+ \frac{1}{r}\left(\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right)\hat{\phi}\end{aligned}$$

Examples of Calculation of Curl

1. Calculate the curl of the vector field $\mathbf{F} = -y\hat{i} + z\hat{j} + x^2\hat{k}$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & z & x^2 \end{vmatrix}$$
$$= -\hat{i} + 2x\hat{j} + \hat{k}$$

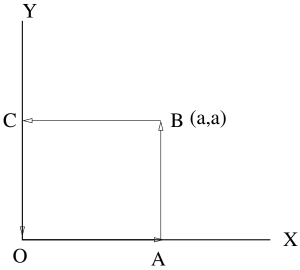
2. Easy to verify that $\nabla \times \mathbf{r} = 0$, where $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Stokes theorem: An example

- Consider a 2D vector field

$$\mathbf{F} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$$

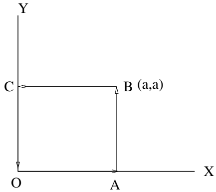
- And a closed loop shaped like a square as shown



- Let us verify Stokes theorem for this case.

$$\oint \mathbf{F}.d\mathbf{l} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

Stokes theorem, example contd.

- 

- LHS of Stokes theorem

$$\oint \mathbf{F}.d\mathbf{l} = \int_{OA} \mathbf{F}.d\mathbf{l} + \int_{AB} \mathbf{F}.d\mathbf{l} + \int_{BC} \mathbf{F}.d\mathbf{l} + \int_{CO} \mathbf{F}.d\mathbf{l}$$

- Now

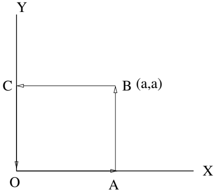
$$\int_{OA} \mathbf{F}.d\mathbf{l} = \int \mathbf{F}.dx\hat{\mathbf{i}} = \int_0^a (-y)dx = 0 \text{ , } \because y = 0$$

$$\int_{AB} \mathbf{F}.d\mathbf{l} = \int \mathbf{F}.dy\hat{\mathbf{j}} = \int_0^a xdy = a\int_0^a dy = a^2$$

$$\int_{BC} \mathbf{F}.d\mathbf{l} = \int \mathbf{F}.dx\hat{\mathbf{i}} = \int_a^0 (-y)dx = -a\int_a^0 dy = a^2$$

$$\int_{CO} \mathbf{F}.d\mathbf{l} = \int \mathbf{F}.dy\hat{\mathbf{j}} = \int_a^0 xdy = 0 \text{ , } \because x = 0$$

Stokes theorem, example...

- 

- Thus, LHS of Stokes theorem yields

$$\oint \mathbf{F}.d\mathbf{l} = a^2 + a^2 = 2a^2$$

- Let us calculate the RHS

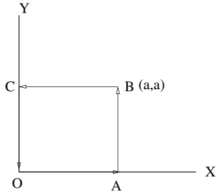
$$\int (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

- It is obvious that

$$\nabla \times \mathbf{F} = 2\hat{\mathbf{k}}$$

$$d\mathbf{S} = dxdy\hat{\mathbf{k}}$$

Stokes theorem verified

- 

- So that

$$\int (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 2 \int dx \int dy = 2a^2$$

- Thus

$$\text{LHS} = \text{RHS}$$

- Stokes theorem stands verified for this case

Stokes theorem and Conservative Forces

- For a general vector field **F**, Stokes theorem states

$$\oint \mathbf{F}.\mathbf{dl} = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{dS}.$$

- If **F** is a conservative force, then we know

$$\oint \mathbf{F}.\mathbf{dl} = 0.$$

- The surface area enclosed by a closed loop, in general, is nonzero
- Therefore, for a conservative force **F**, Stokes theorem implies

$$\nabla \times \mathbf{F} = 0.$$

- Thus, all conservative forces have vanishing curl.

Curl of Conservative Forces

- We also saw that a conservative force can be expressed as

$$\mathbf{F} = -\nabla V(\mathbf{r}) = -\frac{\partial V}{\partial x}\hat{\mathbf{i}} - \frac{\partial V}{\partial y}\hat{\mathbf{j}} - \frac{\partial V}{\partial z}\hat{\mathbf{k}}$$

- Let us calculate the curl of **F**

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{\partial V}{\partial x} & -\frac{\partial V}{\partial y} & -\frac{\partial V}{\partial z} \end{vmatrix}$$

- We obtain

$$\begin{aligned} \nabla \times \mathbf{F} &= \left(-\frac{\partial^2 V}{\partial y \partial z} + \frac{\partial^2 V}{\partial z \partial y}\right)\hat{\mathbf{i}} + \left(-\frac{\partial^2 V}{\partial z \partial x} + \frac{\partial^2 V}{\partial x \partial z}\right)\hat{\mathbf{j}} \\ &+ \left(-\frac{\partial^2 V}{\partial x \partial y} + \frac{\partial^2 V}{\partial y \partial x}\right)\hat{\mathbf{k}} \\ &= 0 \end{aligned}$$

Curl of Force...

- Right hand side vanishes term by term because

$$\frac{\partial^2 V}{\partial y \partial z} = \frac{\partial^2 V}{\partial z \partial y} \text{ etc.}$$

- The result obtained above: Curl of a conservative force vanishes, is due to the general result: curl of gradient of any scalar function vanishes

$$\nabla \times (\nabla \Phi) = 0,$$

where $\Phi(\mathbf{r})$ is any scalar function.

Example: Obtaining potential from the force

- This can be done by integrating the partial differential equation (PDE)

$$-\nabla V = \mathbf{F},$$

- For 3D case, this amounts to integrating three PDEs

$$\frac{\partial V}{\partial x} = -F_x$$
$$\frac{\partial V}{\partial y} = -F_y$$
$$\frac{\partial V}{\partial z} = -F_z$$

- We illustrate the method by a 2D case, where $\mathbf{F}$ is

$$\mathbf{F} = A(x^2\hat{\mathbf{i}} + y\hat{\mathbf{j}})$$

- First we check whether $\nabla \times \mathbf{F} = 0$, or not?
- If $\nabla \times \mathbf{F} \neq 0$, then one cannot find a $V(\mathbf{r})$ which satisfies $-\nabla V = \mathbf{F}$.

Obtaining potential from force

- $$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Ax^2 & Ay & 0 \end{vmatrix} = (0)\hat{\mathbf{i}} + (0)\hat{\mathbf{j}} + (0)\hat{\mathbf{k}} = 0$$
- Thus, **F** is a conservative force, and will satisfy
$$\frac{\partial V}{\partial x} = -Ax^2$$
$$\frac{\partial V}{\partial y} = -Ay$$
- On integrating the x equation, we have
$$V(x,y) = -\frac{Ax^3}{3} + f(y),$$
where $f(y)$ is an unknown function of y.
- On substituting this in y equation we have
$$\frac{\partial}{\partial y}\left(-\frac{Ax^3}{3} + f(y)\right) = -Ay$$

Obtaining potential from force

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Potential from force...

- We have

$$\frac{\partial f}{\partial y} = \frac{df}{dy} = -Ay$$
$$\implies f(y) = -\frac{Ay^2}{2} + C$$

- Leading to the final result

$$V(x, y) = -\frac{Ax^3}{3} - \frac{Ay^2}{2} + C$$