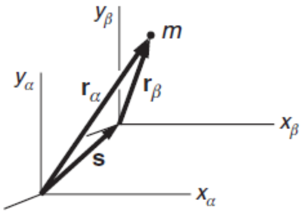
PH111: Introduction to Classical Mechanics
Chapter 4: Non-Inertial Frames and Pseudo Forces

Introduction

- Question: How do laws of physics change, when we change the frame of reference (coordinate system)?
- Are laws of physics same in all inertial frames of reference, i.e., frames moving with constant velocities?
- What if the frames of reference are accelerating?
- Underlying assumption will be that frames are moving with nonrelativistic velocities ($v \ll c$)
- For relativistic velocities, correct theory is Einstein's Special theory of relativity.

Inertial Frames of Reference

- First we consider inertial frames of reference
- We demonstrate that in inertial frames of reference, Newton's second law holds good
- Let α and β be two frames of reference displaced by vector S



- We consider the dynamics of a particle of mass m in both the frames
- Position r_β of the particle in frame β, is related to its position r_α in frame α by
$$\mathrm{r}_\beta = \mathrm{r}_\alpha - \mathrm{S}$$

Inertial Frames of Reference....

- If an observer in frame α measures the acceleration of mass m to be a_α, according to her the force acting on the mass F_α is

$$\mathrm{F}_\alpha = m\mathrm{a}_\alpha$$

- Similarly the observer in β on measuring its acceleration to be a_β will conclude that the force is

$$\mathrm{F}_\beta = m\mathrm{a}_\beta.$$

- Question is: What is the relationship between F_α and F_β?

Inertial frames....

- By taking time derivatives of $r_\beta = r_\alpha - S$, we obtain

$$\begin{aligned} v_\beta &= v_\alpha - V \\ a_\beta &= a_\alpha - A \end{aligned}$$

- Where $V = \dot{S}$ and $A = \ddot{S}$, are the velocity and acceleration, respectively, of frame β w.r.t. frame α.
- If $A = 0$, i.e., β is an inertial frame, then

$$\begin{aligned} a_\beta &= a_\alpha \\ \implies F_\alpha &= ma_\alpha = ma_\beta = F_\beta \end{aligned}$$

- Thus measured force is same in both the frames, as is the equation of motion.

Non-inertial Frames

- Thus, Newton's second law is unchanged when the frames of reference are inertial
- What about non-inertial frames of reference, i.e., when $\mathsf{A} \neq 0$?
- We already have the result that

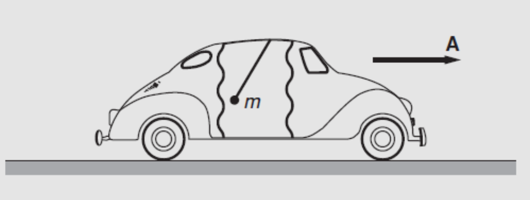
$$
\begin{aligned}
& \mathsf{a}_\beta = \mathsf{a}_\alpha - \mathsf{A} \\
\implies & \mathsf{F}_\beta = m\mathsf{a}_\beta = m\mathsf{a}_\alpha - m\mathsf{A} = \mathsf{F}_\alpha + \mathsf{F}_p \neq \mathsf{F}_\alpha \\
\text{where } & \mathsf{F}_p = -m\mathsf{A}.
\end{aligned}
$$

  Above notation F_p stands for pseudo Force.
- Thus force measured by an observer in a non-inertial frame is different from the one measured by an observer in an inertial frame.

- According to the observer in the non-inertial frame, the object is experiencing an additional force $-mA$, in a direction opposite to that of the acceleration
- Because this force is absent for an observer in the inertial frame, it is called "Pseudo Force".
- To illustrate this, we consider the example of a pendulum in an accelerating car

Example: Pendulum in an Accelerating Car

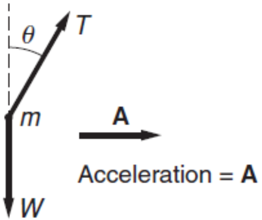
- Consider a car with a pendulum inside, moving with an acceleration A



- We want to find the tension in the string T, and the angle θ the pendulum makes from the vertical
- We will analyze the problem both in the lab frame (static on ground) and the accelerated frame (moving with car)

Analysis in Lab Frame (Attached to the Ground)

- Free body diagram in the lab frame is



- With respect to lab frame, mass m has an acceleration A
- As shown, it experiences two forces, tension T of the string, and its own weight $W = mg$
- We want to find angle of inclination θ, and T
- Application of Newton's law in vertical and horizontal directions, yields

$$T\cos\theta - W = 0 \text{ (vertical)}$$

$$T\sin\theta = mA \text{ (horizontal)}$$

Lab frame contd.

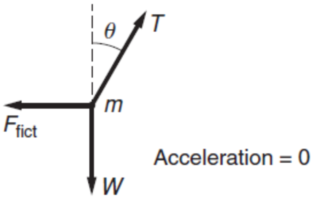
- Leading to the solution

$$\tan\theta = \frac{A}{g}$$
$$T = m\sqrt{g^2 + A^2}$$

- Let us analyze the problem in the non-inertial frame, next

Analysis in non-Inertial Frame (moving with the car)

- In the non-inertial (NI) frame, the free-body diagram is



- In the NI frame, particle is stationary, and in equilibrium
- But it is acted upon by three forces, instead of two
- Additional force is the fictitious (or pseudo) force $\mathrm{F}_{fict} = -m\mathrm{A}$
- Equations of motion are

$$-F_{fict} + T\sin\theta = 0 \text{ (horizontal)}$$
$$T\cos\theta - W = 0 \text{ (vertical)}$$

Non-inertial frame contd.

- Because $F_{fict} = mA$, both these equations are essentially same as in case of inertial frame
- Thus we obtain the same expressions for T and θ.
- Next, we discuss different types of accelerating frames, i.e., the rotating frames of reference

Rotating frames of reference

- Rotating frames of references are non-inertial
- Because any particle executing circular motion experiences centripetal acceleration
- Next, we develop the theory of rotating frames of references
- But, before that, we illustrate the vector nature of angular velocity

Vector nature of angular velocity

- We denoted the position of a particle as a vector

$$\mathrm{r} = x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}}$$

- Can we similarly specify the angular position of a particle

$$\boldsymbol{\theta} = \theta_x\hat{\mathrm{i}} + \theta_y\hat{\mathrm{j}} + \theta_z\hat{\mathrm{k}}\,?$$

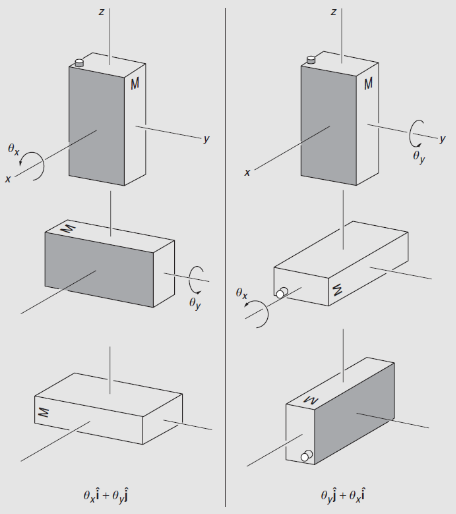
- The answer is no because such an expression does not satisfy commutative law of vector addition

$$\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2 \neq \boldsymbol{\theta}_2 + \boldsymbol{\theta}_1$$

- Let us rotate a block first around the x axis, and then around the y axis. Compare that to the same operations performed in the reverse order

Non-commutative nature of finite rotations

- Consider those two rotations, with each one of them being $\pi/2$



- Clearly $\theta_x \hat{\imath} + \theta_y \hat{\jmath} \neq \theta_y \hat{\jmath} + \theta_x \hat{\imath}$

Vector nature of Angular Velocity

- On the other hand, one can verify that infinitesimal rotations commute to first order terms

$$\Delta\theta_x \hat{\mathrm{i}} + \Delta\theta_y \hat{\mathrm{j}} \approx \Delta\theta_y \hat{\mathrm{j}} + \Delta\theta_x \hat{\mathrm{i}}$$

- Thus, infinitesimal rotations can be represented as vectors
- Because angular velocity is defined in terms of infinitesimal rotations

$$\boldsymbol{\omega} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{\theta}}{\Delta t},$$

- Angular velocity can be denoted as a vector

$$\boldsymbol{\omega} = \omega_x \hat{\mathrm{i}} + \omega_y \hat{\mathrm{j}} + \omega_z \hat{\mathrm{k}}$$

- And, in general,

$$\boldsymbol{\omega} = \omega \hat{\mathrm{n}},$$

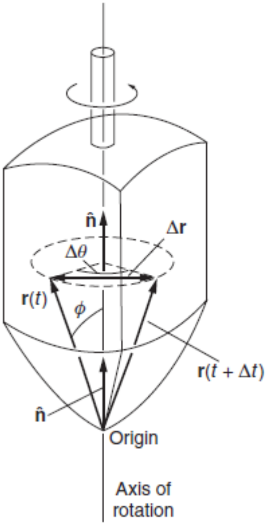
where $\hat{\mathrm{n}}$ is the direction of the axis of rotation, and ω is the magnitude of the angular velocity.

Relation between linear velocity and angular velocity

- It is obvious that angular velocity $\boldsymbol{\omega}$, will give rise to linear velocity v
- What is the mathematical relation between the two?

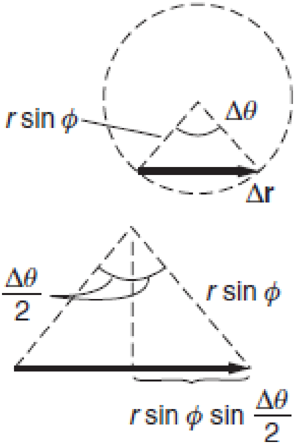
Linear and Angular Velocities

- Consider a rigid body rotating with a uniform angular velocity $\boldsymbol{\omega} = \omega\hat{\mathbf{n}}$ as shown



Linear and angular velocity...

- Note that the position vector r of a particular point in the rigid body, precesses about the axis of rotation, and forms a cone with its tip at the origin
- ϕ is the constant angle between r and n̂
- During precession, r traces a circle of radius $r \sin \phi$
- Let $\Delta\theta$ be the angle by which r rotates in time Δt



Relation between v and $\boldsymbol{\omega}$

- It is obvious from the figure that the magnitude of change in r, i.e., Δr is given by

$$|\Delta \mathrm{r}| = 2r \sin\phi \sin\frac{\Delta\theta}{2}$$

- For $\Delta t \to 0$, $\Delta\theta \to 0$, so that

$$|\Delta \mathrm{r}| = 2r \sin\phi \sin\frac{\Delta\theta}{2} \approx 2r \sin\phi \frac{\Delta\theta}{2} \approx r \sin\phi \Delta\theta$$

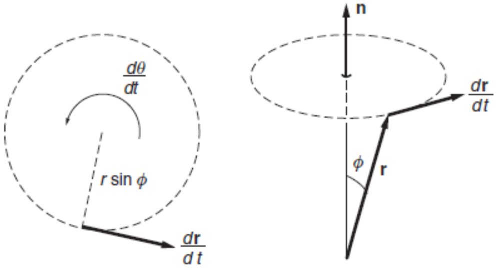
- Leading to (for $\Delta t \to 0$)

$$\left|\frac{\Delta \mathrm{r}}{\Delta t}\right| = r \sin\phi \frac{\Delta\theta}{\Delta t}$$

$$\implies \left|\frac{d\mathrm{r}}{dt}\right| = r \sin\phi \frac{d\theta}{dt} = \omega r \sin\phi$$

$\mathbf{v} - \boldsymbol{\omega}$ relation....

- Obviously, for $\Delta t \to 0$, $\frac{d\mathbf{r}}{dt}$ is in tangential direction



- Keeping the direction and magnitude of $\frac{d\mathbf{r}}{dt}$ in mind, we conclude

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

Rate of change of a general rotating vector

- Previous discussion was about when position vector r was precessing with a constant angular velocity ω, about the axis in direction $\hat{n}$.
- But the same arguments will hold if, instead of r, some general vector A, was precessing with a constant angular velocity ω, about the axis in direction $\hat{n}$.
- Then, we will have

$$\frac{d\mathrm{A}}{dt} = \boldsymbol{\omega} \times \mathrm{A}.$$

- This is a very important general relation about the rate of change of rotating vectors.
- Let $\mathrm{A} = \mathrm{v}$, then using above, we get the expression for acceleration of a rotating particle

$$\frac{d\mathrm{v}}{dt} = \mathrm{a} = \boldsymbol{\omega} \times \mathrm{v} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathrm{r}) \text{ (centripetal acceleration)}$$

Physics in Rotating Reference Frames

- Consider a general vector A which is changing with time
- When observed from an inertial frame, its rate of change is $\left(\frac{dA}{dt}\right)_{in}$.
- Suppose we have a non-inertial frame of reference which is rotating with a constant angular velocity $\boldsymbol{\Omega}$
- What is the rate of change of A, i.e., $\left(\frac{dA}{dt}\right)_{rot}$, with respect to the rotating frame?
- Let $\hat{i}, \hat{j}, \hat{k}$ be the basis vectors of the inertial frame
- And $\hat{i}', \hat{j}', \hat{k}'$ be the basis vectors of the rotating frame

Physics in rotating frames...

- This means that $\hat{i}', \hat{j}', \hat{k}'$ are rotating w.r.t. the inertial frame with angular velocity $\boldsymbol{\Omega}$.
- At a given point in time, A in two frames can be expressed as

$$A = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \text{ (inertial)}$$
$$A = A'_x\hat{i}' + A'_y\hat{j}' + A'_z\hat{k}' \text{ (rotating)}$$

- Therefore, in inertial frame

$$\left(\frac{dA}{dt}\right)_{in} = \frac{dA_x}{dt}\hat{i} + \frac{dA_y}{dt}\hat{j} + \frac{dA_z}{dt}\hat{k}$$

Physics in rotating frames contd.

- We can also compute the time derivative of the second expression of A
- Keeping in mind that not only the components of A, but the basis vectors $\hat{\mathrm{i}}', \hat{\mathrm{j}}', \hat{\mathrm{k}}'$ are changing with time due to rotation
- Therefore, the rate of change of A will be

$$\left(\frac{d\mathrm{A}}{dt}\right)_{in} = \frac{dA'_x}{dt}\hat{\mathrm{i}}' + \frac{dA'_y}{dt}\hat{\mathrm{j}}' + \frac{dA'_z}{dt}\hat{\mathrm{k}}' + A'_x\frac{d\hat{\mathrm{i}}'}{dt} + A'_y\frac{d\hat{\mathrm{j}}'}{dt} + A'_z\frac{d\hat{\mathrm{k}}'}{dt}$$

- Because vectors $\hat{\mathrm{i}}', \hat{\mathrm{j}}', \hat{\mathrm{k}}'$ are rotating with angular velocity $\boldsymbol{\Omega}$, w.r.t. to the inertial frame

$$\frac{d\hat{\mathrm{i}}'}{dt} = \boldsymbol{\Omega} \times \hat{\mathrm{i}}'$$
$$\frac{d\hat{\mathrm{j}}'}{dt} = \boldsymbol{\Omega} \times \hat{\mathrm{j}}'$$
$$\frac{d\hat{\mathrm{k}}'}{dt} = \boldsymbol{\Omega} \times \hat{\mathrm{k}}'$$

Rotating frames

- So that

$$\left(\frac{d\mathrm{A}}{dt}\right)_{in} = \frac{dA'_x}{dt}\hat{\mathrm{i}}' + \frac{dA'_y}{dt}\hat{\mathrm{j}}' + \frac{dA'_z}{dt}\hat{\mathrm{k}}' + \boldsymbol{\Omega} \times \left(A'_x\hat{\mathrm{i}}' + A'_y\hat{\mathrm{j}}' + A'_z\hat{\mathrm{k}}'\right)$$

- First three terms denote the rate of change of A as seen in the rotating frame, i.e.,

$$\frac{dA'_x}{dt}\hat{\mathrm{i}}' + \frac{dA'_y}{dt}\hat{\mathrm{j}}' + \frac{dA'_z}{dt}\hat{\mathrm{k}}' = \left(\frac{d\mathrm{A}}{dt}\right)_{rot}$$

- Leading to

$$\left(\frac{d\mathrm{A}}{dt}\right)_{in} = \left(\frac{d\mathrm{A}}{dt}\right)_{rot} + \boldsymbol{\Omega} \times \mathrm{A}$$

- Because A is a general vector, the previous formula can be symbolically expressed as

$$\left(\frac{d}{dt}\right)_{in} = \left(\frac{d}{dt}\right)_{rot} + \boldsymbol{\Omega}\times$$

Velocity and acceleration in a rotating frame

- Taking $\mathbf{A} = \mathbf{r}$, we have

$$\left(\frac{d\mathbf{r}}{dt}\right)_{in} = \left(\frac{d\mathbf{r}}{dt}\right)_{rot} + \boldsymbol{\Omega} \times \mathbf{r}$$
$$\implies \mathbf{v}_{in} = \mathbf{v}_{rot} + \boldsymbol{\Omega} \times \mathbf{r}$$

- On taking $\mathbf{A} = \mathbf{v}_{in}$, we get

$$\left(\frac{d\mathbf{v}_{in}}{dt}\right)_{in} = \left(\frac{d\mathbf{v}_{in}}{dt}\right)_{rot} + \boldsymbol{\Omega} \times \mathbf{v}_{in}$$
$$\implies \left(\frac{d\mathbf{v}_{in}}{dt}\right)_{in} = \left(\frac{d}{dt}\right)_{rot} (\mathbf{v}_{rot} + \boldsymbol{\Omega} \times \mathbf{r}) + \boldsymbol{\Omega} \times (\mathbf{v}_{rot} + \boldsymbol{\Omega} \times \mathbf{r})$$

Acceleration in Rotating Frame

- Or

$$\left(\frac{d\mathbf{v}_{in}}{dt}\right)_{in} = \left(\frac{d\mathbf{v}_{rot}}{dt}\right)_{rot} + \left(\frac{d(\mathbf{\Omega}\times\mathbf{r})}{dt}\right)_{rot} + \mathbf{\Omega}\times\mathbf{v}_{rot} + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})$$

$$= \left(\frac{d\mathbf{v}_{rot}}{dt}\right)_{rot} + \mathbf{\Omega}\times\mathbf{v}_{rot} + \mathbf{\Omega}\times\mathbf{v}_{rot} + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})$$

$$\implies \mathbf{a}_{in} = \mathbf{a}_{rot} + 2\mathbf{\Omega}\times\mathbf{v}_{rot} + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})$$

  Above, we used that condition that $\mathbf{\Omega} =$ constant, so that $\dot{\mathbf{\Omega}} = 0$.

- Thus, the acceleration as seen in the rotating frame is

$$\mathbf{a}_{rot} = \mathbf{a}_{in} - 2\mathbf{\Omega}\times\mathbf{v}_{rot} - \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})$$

Pseudo Forces in Rotating Frames

- Multiplying the previous equation by m on both sides, and using notations $\mathbf{F}_{rot} = m\mathbf{a}_{rot}$ and $\mathbf{F} = m\mathbf{a}_{in}$, we have

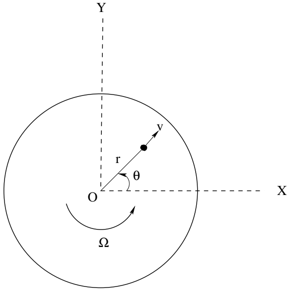
$$\begin{aligned}\mathbf{F}_{rot} &= \mathbf{F} - 2m\boldsymbol{\Omega} \times \mathbf{v}_{rot} - m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \\ &= \mathbf{F} + \mathbf{F}_{coriolis} + \mathbf{F}_{centrifugal} \\ &= \mathbf{F} + \mathbf{F}_{fict}\end{aligned}$$

- Where $\mathbf{F}$ is the real force acting on the particle, while $\mathbf{F}_{coriolis}$ and $\mathbf{F}_{centrifugal}$ are pseudo (fictitious) forces

$$\begin{aligned}\mathbf{F}_{coriolis} &= -2m\boldsymbol{\Omega} \times \mathbf{v}_{rot} \\ \mathbf{F}_{centrifugal} &= -m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})\end{aligned}$$

Probing Centrifugal and Coriolis Forces

- Consider a circular plank (say a merry-go-round) rotating with an angular velocity $\mathbf{\Omega} = \Omega\hat{k}$, with a mass m as shown



- Mass m is moving with a velocity v which is in the radial direction w.r.t. plank
- Location of the particle at a given instant is r, w.r.t. to rotating coordinate system
- What are the magnitudes and directions of pseudo forces?

Centrifugal Force

- Centrifugal force can be computed as

$$\begin{aligned}
\mathrm{F}_{centrifugal} &= -m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathrm{r}) \\
&= -m\Omega^2 r\hat{\mathrm{k}} \times (\hat{\mathrm{k}} \times \hat{\mathrm{r}}) \\
&= -m\Omega^2 r\hat{\mathrm{k}} \times \hat{\boldsymbol{\theta}} \\
&= -m\Omega^2 r(-\hat{\mathrm{r}}) \\
&= m\Omega^2 \mathrm{r}.
\end{aligned}$$

Thus centrifugal force has the same magnitude as the centripetal force, but opposite direction, as expected of a pseudo force.

- Coriolis force exists only when the particle moves with respect to the rotating frame. Here

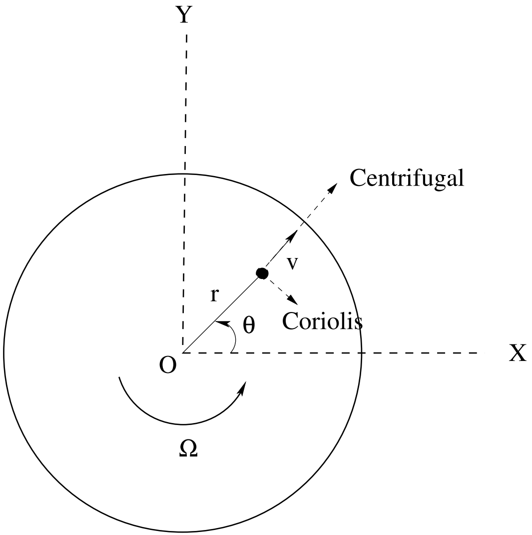
$$\mathrm{v}_{rot} = v\hat{\mathrm{r}}.$$

- Therefore,

$$\begin{aligned}\mathrm{F}_{coriolis} &= -2m\boldsymbol{\Omega} \times \mathrm{v}_{rot} \\ &= -2m\Omega v(\hat{\mathrm{k}} \times \hat{\mathrm{r}}) \\ &= -2m\Omega v\hat{\boldsymbol{\theta}}\end{aligned}$$

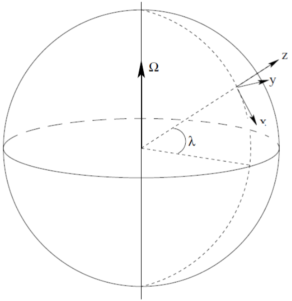
Coriolis and Centrifugal Forces

Thus, finally the direction of the forces



Coriolis Force due to Rotation of Earth

- Earth's Angular Velocity in a Non-Intertial Frame



- Here x points to south, y to east, and z is radially outwards (vertically above from earth), and λ is latitude angle
- In this frame

$$\boldsymbol{\Omega} = -\Omega\cos\lambda\hat{\mathrm{i}} + \Omega\sin\lambda\hat{\mathrm{k}}$$

Coriolis Force on a Falling Object

- If a particle of mass m is falling vertically down, at a given instant with velocity v, then

$$\mathbf{v} = -v\hat{\mathbf{k}}$$

- Then Coriolis force on it due to Earth's rotation is

$$\mathbf{F}_c = -2m(\boldsymbol{\Omega} \times \mathbf{v}) = -2m\Omega \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\cos\lambda & 0 & \sin\lambda \\ 0 & 0 & -v \end{vmatrix} = 2mv\Omega\cos\lambda\hat{\mathbf{j}}$$

- Thus, the object will experience a force towards east, and will get deviated in that direction
- Another example: away from equator, wind flow becomes circular due to Coriolis force
- Note $\mathbf{F}_c \perp \mathbf{v}_{rot}$, so it will lead to a circular motion

The Foucault Pendulum

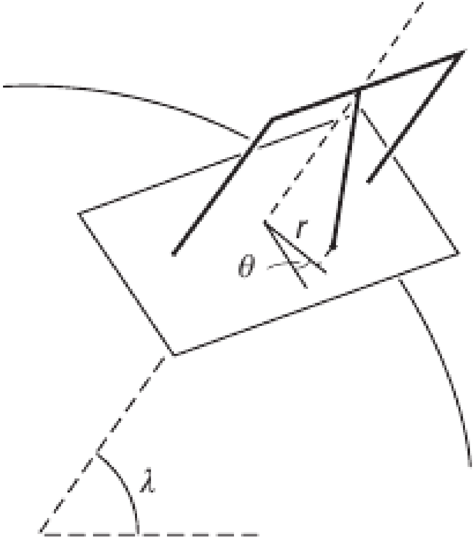
- The Foucault pendulum (FP) is a fine example of Coriolis Force
- It clearly demonstrates that we on earth are located in a rotating frame
- That is, a non-inertial frame
- Very good information on FP is available on Wikipedia with all the history
- Check out the Wikipedia page here
- The following simplified treatment is based on a solved example in Kleppner and Kolenkow.

The Foucault Pendulum...

- If consider a large enough pendulum, we will see that its plane of oscillations rotates
- That is the pendulum doesn't keep oscillating in the same plane
- This rotation and its period can be explained mathematically in terms of the Coriolis force

The Foucault Pendulum...

- Consider the following figure showing a pendulum on Earth's surface



The Foucault Pendulum...

- In the figure, λ denotes the latitude angle
- (r, θ) denote the plane polar coordinates of the bob of the pendulum
- the coordinate system is supposed to be attached to the earth
- the earth is supposed to be flat
- A good approximation given that bob doesn't move over large distances
- We further assume that the length of the pendulum is l
- Let us setup the equations of motion of the bob

The Foucault Pendulum...

- Clearly, the bob is moving along the $\hat{r}$ direction
- If its motion were in a plane, θ will not change with time
- Then $\dot{\theta} = 0$
- The frequency of oscillations ω of the pendulum will be

$$\omega = \sqrt{\frac{l}{g}}$$

- Therefore, $r(t)$ will be given by

$$r(t) = r_0 \sin \omega t,$$

where r_0 is the amplitude

The Foucault Pendulum...

- Assume, as before, the angular velocity of the rotation of earth is $\mathbf{\Omega}$
$$\mathbf{\Omega} = \Omega \sin\lambda \hat{k} - \Omega \cos\lambda \hat{\mathrm{r}},$$
where $\hat{k}$ denotes the perpendicular to earth's surface at the lattitude λ
- If the mass of the bob is m, the Coriolis force acting on the bob is
$$\mathrm{F}_c = -2m(\mathbf{\Omega} \times \mathrm{v})$$
- Given that to a good approximation $\mathrm{v} = \dot{r}\hat{\mathrm{r}}$, we obtain from above
$$\begin{aligned}\mathrm{F}_c &= -2m(\Omega \sin\lambda \hat{k} + \Omega \cos\lambda \hat{\mathrm{r}}) \times (\dot{r}\hat{\mathrm{r}}) \\ &= -2m\dot{r}\Omega \sin\lambda \hat{\boldsymbol{\theta}}\end{aligned}$$

- Thus Coriolis force is strictly in the tangential ($\hat{\boldsymbol{\theta}}$) direction
- So we set up the equation of motion in the tangential direction

$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = -2m\dot{r}\Omega\sin\lambda$$

- Leading to

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = -2\Omega\sin\lambda\,\dot{r}$$

- Although, this equation can be solved quite precisely, but we will use an approximate approach
- Reasonable to assume $\dot{\theta}$ (angular speed of precession) to be constant
- Implying that $\ddot{\theta} = 0$

The Foucault Pendulum...

- This leads to the equation

$$2\dot{r}\dot{\theta} = -2\Omega \sin\lambda\, \dot{r}$$

- Thus we obtain the solution of the problem

$$\dot{\theta} = -\Omega \sin\lambda$$

- With $|\dot{\theta}| = \Omega \sin\lambda$, we obtain the period of the precession to be

$$T = \frac{2\pi}{|\dot{\theta}|} = \frac{2\pi}{\Omega \sin\lambda} = \frac{24 \text{ hr}}{\sin\lambda}$$

- For Paris, $\lambda \approx 49^{\circ}$, so that $T_{Paris} \approx 32$ hr which is very close to the observed value of 31 hr 50 minutes!
- Clearly, at the north pole $T_{north-pole} \approx 24$ hr