PH110: Tutorial Sheet 3 (Quantum Mechanics)

* marked problems will be solved in the Wednesday tutorial class.

Wave packets: Group and Phase Velocity

1. Consider two wave functions $\psi_1(y,t) = 5y\cos 7t$ and $\psi_2(y,t) = -5y\cos 9t$, where y and t are in meters and seconds, respectively. Show that their superposition generates a wave packet. Plot it and identify the modulated and modulating functions.

2. *Two harmonic waves which travel simultaneously along a wire are represented by

$$y_1 = 0.002\cos(8.0x - 400t) \quad \& \quad y_2 = 0.002\cos(7.6x - 380t)$$

where x, y are in meters and t is in sec.

   (a) Find the resultant wave and its phase and group velocities

   (b) Calculate the range Δx between the zeros of the group wave. Find the product of Δx and Δk ?

3. The angular frequency of the surface waves in a liquid is given in terms of the wave number k by $\omega = \sqrt{gk + Tk^3/\rho}$, where g is the acceleration due to gravity, ρ is the density of the liquid, and T is the surface tension (which gives an upward force on an element of the surface liquid). Find the phase and group velocities for the limiting cases when the surface waves have:
(a) very large wavelengths and
(b) very small wavelengths.

4. *Calculate the group and phase velocities for the wave packet corresponding to a relativistic particle.

5. Consider an electromagnetic (EM) wave of the form $A\exp(i[kx - \omega t])$. Its speed in free space is given by $c = \frac{\omega}{k} = 1/\sqrt{\epsilon_0\mu_0}$, where ϵ_0, μ_0 is the electric permittivity, magnetic permeability of free space, respectively.

   (a) Find an expression for the speed (v) of EM waves in a medium, in terms of its permittivity ϵ and permeability μ.

   (b) Suppose the permittivity of the medium depends on the frequency, given by $\epsilon = \epsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)$ where ω_p is a constant called the plasma frequency, find the dispersion relation for the EM waves in a medium. wp is a constant and is called the plasma frequency of the medium (assume $\mu = \mu_0$).

(c) Consider waves with $\omega = 3\omega_p$. Find the phase and group velocity of the waves. What is the product of group and phase velocities?

6. A wave packet describes a particle having momentum p. Starting with the relativistic relationship $E^2 = p^2c^2 + E_0^2$, show that the group velocity is βc and the phase velocity is c/β (where $\beta = v/c$). How can the phase velocity physically be greater than c?

7. *Consider a squre 2-D system with small balls (each of mass m) connected by springs. The spring constants along the x- and y-directions are β_x and β_y, respectively. The dispersion relation for this system is given by

$$-\omega^2 m + 2\beta_x\left(1 - \cos k_x a_x\right) + 2\beta_y\left(1 - \cos k_y a_y\right) = 0$$

where $\vec{k} = k_x\hat{\imath} + k_y\hat{\jmath}$ is the wave vector and a_x,a_y are the natural distances between the two successive masses along the x-, y-directions, respectively. Find the group velocity and the angle that it makes with the x-axis

Fourier Transform

1. * If $\phi(k) = A(a - |k|), |k| \leq a$, and 0 elsewhere. Where a is a positive parameter and A is a normalization factor to be found.
(a) Find the Fourier transform for $\phi(k)$
(b) Calculate the uncertainties Δx and Δp and check whether they satisfy the uncertainty principle.

2. A wave packet is of the form $f(x) = \cos^2\left(\frac{x}{2}\right)$ (for $-\pi \leq x \leq \pi$) and $f(x) = 0$ elsewhere
(a) Plot $f(x)$ versus x.
(b) Calculate the Fourier transform of $f(x)$, i.e. $g(k) = \int_{-\infty}^{+\infty} f(x)e^{-ikx}dx$?
(c) At what value of k, $|g(k)|$ attains its maximum value?
(d) Calculate the value(s) of k where the function $g(k)$ has its first zero.
(e) Considering the first zero(s) of both the functions $f(x)$ and $g(k)$ to define their spreads (i.e. Δx and Δk), calculate the uncertainty product $\Delta x.\Delta k$.

3. Find the Fourier transform of the following functions: a) $f(x) = \begin{cases} a, & -\ell < x < 0, a > 0 \\ 0, & \text{otherwise} \end{cases}$

b) $f(x) = \begin{cases} a, & -\ell < x < 0 \\ b, & 0 < x < \ell \quad a > 0, b > 0 \\ 0, & \text{otherwise} \end{cases}$

4. A wave packet is of the form $f(x) = \exp(-\alpha|x|) \cdot \exp\left(ik_0 x\right)$ (for $-\infty \leq x \leq \infty$) where α, k_0 are positive constants.

(a) Plot $|f(x)|$ versus x.

(b) At what values of x does $|f(x)|$ attain half of its maximum value? Consider the full width at half maxima (FWHM) as a measure of the spread (uncertainty) in x, find Δx

(c) Calculate the Fourier transform of $f(x)$, i.e. $g(k) = \int_{-\infty}^{+\infty} f(x) e^{ikx} dx$

(d) Plot $g(k)$ versus k.

(e) Find the values of k at which $g(k)$ attains half its maximum value? Using the same concept of FWHM as in part (b), calculate Δk ? Hence calculate the product $\Delta x.\Delta k$

[Given : $\int_0^{\infty} e^{-(\alpha - ik)x} dx = \frac{1}{\alpha - ik}$]