# PH110: Tutorial Sheet 4 (Quantum Mechanics)

---

* marked problems will be solved in the Wednesday tutorial class.

---

## Fourier Transform

1. * If $\phi(k) = A(a - |k|), |k| \leq a$, and 0 elsewhere. Where $a$ is a positive parameter and $A$ is a normalization factor to be found.
   (a) Find the Fourier transform for $\phi(k)$
   (b) Calculate the uncertainties $\Delta x$ and $\Delta p$ and check whether they satisfy the uncertainty principle.

2. A wave packet is of the form

$$f(x) = \begin{cases} \cos^2\left(\frac{x}{2}\right), & -\pi \leq x \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

   (a) Plot $f(x)$ versus $x$.
   (b) Calculate the Fourier transform of $f(x)$, i.e. $g(k) = \int_{-\infty}^{+\infty} f(x)e^{-ikx}dx$ ?
   (c) At what value of $k, |g(k)|$ attains its maximum value?
   (d) Calculate the value(s) of $k$ where the function $g(k)$ has its first zero.
   (e) Considering the first zero(s) of both the functions $f(x)$ and $g(k)$ to define their spreads (i.e. $\Delta x$ and $\Delta k$), calculate the uncertainty product $\Delta x.\Delta k$.

3. A wave function $\psi(x)$ is defined such that $\psi(x) = \sqrt{2/L}\sin(\pi x/L)$ for $0 \leq x \leq L$ and $\psi(x) = 0$ otherwise.
   (a) Writing $\psi(x) = \int_{-\infty}^{\infty} a(k)e^{ikx}dk$, find $a(k)$.
   (b) What is the amplitude of the plane wave of wavelength $L$ constituting $\psi(x)$ ?

4. A wave packet is of the form $f(x) = \exp(-\alpha|x|) \cdot \exp(ik_0x)$ ( for $-\infty \leq x \leq \infty$) where $\alpha, k_0$ are positive constants.
   (a) Plot $|f(x)|$ versus $x$.
   (b) At what values of $x$ does $|f(x)|$ attain half of its maximum value? Consider the full width at half maxima (FWHM) as a measure of the spread (uncertainty) in $x$, find $\Delta x$
   (c) Calculate the Fourier transform of $f(x)$, i.e. $g(k) = \int_{-\infty}^{+\infty} f(x)e^{ikx}dx$
   (d) Plot $g(k)$ versus $k$.
   (e) Find the values of $k$ at which $g(k)$ attains half its maximum value? Using the same concept of FWHM as in part (b), calculate $\Delta k$ ? Hence calculate the product $\Delta x.\Delta k$
   [ Given : $\int_0^{\infty} e^{-(\alpha-ik)x}dx = \frac{1}{\alpha-ik}$]

## Heisenberg Uncertainty Principle

1. Estimate the uncertainty in the position of (a) a neutron moving at $5 \times 10^6$ m s$^{-1}$ and (b) a 50 kg person moving at 2 m s$^{-1}$. The error in the measurement of the velocity is 1%.

2. A lead nucleus has a radius $7 \times 10^{-15}$ m. Consider a proton bound within nucleus. Using the uncertainty relation $\Delta$p.$\Delta$r $\geq \hbar/2$, estimate the root mean square speed of the proton, assuming it to be non-relativistic. (You can assume that the average value of $p^2$ is square of the uncertainty in momentum.)

3. * A $\pi^0$ meson is an unstable particle produced in highenergy particle collisions. It has a mass-energy equivalent of about 135MeV, and it exists for an average lifetime of only $8.7 \times 10^{-17}$ s before decaying into two $\gamma$ rays. Using the uncertainty principle, estimate the fractional uncertainty $\Delta m/m$ in its mass determination.

4. * For a non-relativistic electron, using the uncertainty relation $\Delta x \Delta p_x = \hbar/2$
   (a) Derive the expression for the minimum kinetic energy of the electron localized in a region of size ' $a$ '.
   (b) If the uncertainty in the location of a particle is equal to its de Broglie wavelength, show that the uncertainty in the measurement of its velocity is same as the particle velocity.
   (c) Using the expression in (b), calculate the uncertainty in the velocity of an electron having energy 0.2keV
   (d) An electron of energy 0.2keV is passed through a circular hole of radius $10^{-6}$ m. What is the uncertainty introduced in the angle of emergence in radians? (Given $\tan\theta \cong \theta$ )

5. An atom in an excited state 1.8eV above the ground state remains in that excited state 2.0$\mu$s before moving to the ground state. Find (a) the frequency of the emitted photon, (b) its wavelength, and (c) its approximate uncertainty in energy.

6. * An electron microscope is designed to resolve objects as small as 0.14 $nm$. What energy electrons must be used in this instrument?

7. * Show that the uncertainty principle can be expressed in the form $\Delta L \Delta\theta \geq \hbar/2$, where $\theta$ is the angle and $L$ the angular momentum. For what uncertainty in $L$ will the angular position of a particle be completely undetermined?

   For circular motion $L = rp$ and so $\Delta L = r\Delta p$. Along the circle $x = r\theta$ and $\Delta x = r\Delta\theta$. Thus $\Delta p \Delta x = \frac{\Delta L}{r}(r\Delta\theta) = \Delta L \Delta\theta \geq \frac{\hbar}{2}$. For complete uncertainty $\Delta\theta = 2\pi$ and $\Delta L = \frac{\hbar/2}{2\pi} = \frac{\hbar}{4\pi}$