PH-112 (2023 Spring): Tutorial Sheet 0

Notes:

1. This assignment is a check of your background knowledge. This will not be evaluated.
2. * marked problems will be solved in the Wednesday tutorial class.
3. Please make sure that you do the assignment by yourself. You can consult your classmates and seniors and ensure you understand the concept. However, do not copy assignments from others.

Photoelectric Effect:

1. *In a photoelectric effect experiment, excited hydrogen atoms are used as light source. The light emitted from this source is directed to a metal of work function Φ. In this experiment, the following data on stopping potentials (V_s), for various Balmer lines of hydrogen, is obtained.

$$
\begin{aligned}
n = 4 \rightarrow n = 2,\ \text{transition line} &: V_S = 0.43\ \text{V} \\
n = 5 \rightarrow n = 2,\ \text{transition line} &: V_S = 0.75\ \text{V} \\
n = 6 \rightarrow n = 2,\ \text{transition line} &: V_S = 0.94\ \text{V}
\end{aligned}
$$

   a) What is the work function Φ of the metal in eV?
   b) What is the stopping potential (in Volts) for Balmer line of the shortest wavelength?
   c) What will be the photocurrent corresponding to Paschen series (ending in $n = 3$) transitions?

2. In an experiment on photoelectric effect of a metal, the stopping potentials were found to be 4.62 V and 0.18 V for $\lambda_1 = 1850$ Å and $\lambda_2 = 5460$ Å, respectively. Find the value of Planck's constant, the threshold frequency and the work function of the metal.

3. *A monochromatic light of intensity 1.0 μW/cm^2 falls on a metal surface of area 1 cm^2 and work function 4.5 eV. Assume that only 3% of the incident light is absorbed by the metal (rest is reflected back) and that the photoemission efficiency is 100 % (i.e. each absorbed photon produces one photo-electron). The measured saturation current is 2.4 nA.

   (a) Calculate the number of photons per second falling on the metal surface.
   (b) What is the energy of the incident photon in eV ?
   (c) What is the stopping potential ?

4. In a photoelectric experiment, a photocathode is illuminated separately by two light sources of same intensity but different wavelengths, 480 nm and 613 nm. The resulting photocurrent is measured as a function of the potential difference (V) between the cathode and the anode. Observed photocurrent for three values of V is given below

V	current (nA)

	480 nm	613 nm
-0.1	76.3097	64.7039
-0.2	67.6194	44.4078
-0.3	58.9291	24.1118

(a) Using this data, obtain the work function of the photocathode and the cut off wavelength.

(b) What is the maximum kinetic energy of the electron for $\lambda = 480$ nm? What should be the wavelength of light to emit electrons half this kinetic energy?

(c) When the photocathode material is changed, it is found that the cut off frequency is 1.2 times the cut off frequency of the old material. What is the work function of the new material?

5. Light of wavelength 2000 Å falls on a metal surface. If the work function of the metal is 4.2 eV, find the kinetic energy of the fastest and the slowest emitted photoelectrons. Also find the stopping potential and cutoff wavelength for the metal.

Bohr Model:

1. If the nucleus in the Bohr atom is assumed to be of finite mass, show that the equation for angular momentum, orbital radius and the energy will be the same except for the replacement of m by a reduced mass μ.

2. Two similar masses (m) are connected by a spring of spring constant k (neglect the natural length and mass of the spring and any other forces). The masses are made to rotate in a circle about their common centre of mass such that the distance between them is R. Apply Bohr's quantization rule to this system and find the allowed value of r and E_r in terms of m, k and fundamental constants.

3. *If the wavelength λ of the hydrogen atom spectra were to be given by the following expression (instead of the usual one),

$$\frac{1}{\lambda} = R\left(\frac{1}{m^3} - \frac{1}{n^3}\right)$$

where R is a constant and m, n are integers ($n > m$). If the condition for angular momentum quantization can be written as $L = a\hbar$, find the values of a so as to obtain the above spectra. Construct a theory similar to Bohr's using this quantization condition and find an expression of the energy and Bohr's radius.

4. *A muon is an elementary particle of charge $-e$ and mass $m = 207m_e$ (m_e is the mass of the electron). A muonic atom consists of a nucleus of charge Ze with the muon moving in circular oribits about the nucleus. For a muonic atom with $Z = 1$,

(a) calculate the radius of the first Bohr orbit.

(b) calculate the binding energy.

(c) find the wavelength of the first line in the Lymann series.

5. One of the spectral lines in the Hydrogen atom has a wavelength of 4861.320 Å. Along with this, a faint line appears at 4859.975 Ådue to the presence of a small amount of deuterium. Compute the ratio of deuterium mass to the proton mass.

6. *A positronium atom (an electron and a positron revolving about their common centre of mass) is excited from a state with $n = 1$ to $n = 4$. Apply Bohr's theory with suitable modifications and calculate the

   (a) energy that would have been absorbed by the atom

   (b) minimum possible wavelength emitted when such an electron de-excites

   (c) recoil speed and recoil energy of the positronium atom, assumed initially at rest, after the excitation takes place.

7. A hydrogen atom, initially at rest, makes a transition from $n = 4$ to $n = 1$.

   (a) Calculate the recoil speed.

   (b) Assuming the gas to be classical, find the temperature at which the hydrogen atoms can achieve this average speed.

   (c) What is the kinetic energy given to the hydrogen atom?

   (d) What would you expect if the hydrogen atoms were in motion instead of initially at rest?

   (e) Can you think of a simple experiment that can be used for cooling a gas?

PH110: Tutorial Sheet 1 (Quantum Mechanics)

This tutorial sheet deals with problems related to the relativistic energy-momentum conservation.

1. If a particle of rest mass m_0 has total energy $4m_0c^2$, what is its total momentum expressed in terms of m_0c? What will be the energy of this particle when its total momentum is $2m_0c$? Ans ($\sqrt{15}m_0c$, $\sqrt{5}m_0c^2$)

2. If a particle has the rest-mass energy of 100 MeV, what will be its total energy if it moves with the velocity: (a) $0.9c$, (b) $0.99c$, and (c) $0.999c$? (Useful info: 1 MeV $\equiv$ Million electron volts $= 1.0 \times 10^6$ eV)

3. A ρ-meson of rest-mass energy 760 MeV decays into two π-mesons each of rest-mass energy 150 MeV. What is the speed of one π-meson relative to the other? Ans: $0.997c$

4. A pion of rest mass energy 140 MeV decays into a muon of the rest-mass energy 100 MeV, and a neutrino of zero rest mass. In the rest frame of the pion, calculate the momentum and speed of the muon in the units of MeV/c and c, respectively. Ans (24.29 MeV/c, $0.324c$)

5. Let m_0 be the rest mass of an electron and its anti-particle positron. If a hypothetical particle X rest mass $4m_0$ is produced during a collision between an electron and a positron, answer the following:

   (a) if the momentum of the electron is $p\hat{k}$ and that of positron is $-p\hat{k}$, find p. Ans ($\sqrt{3}m_0c$)

   (b) if the electron is initially at rest and the positron has momentum $q\hat{k}$, find q and the speed with which X is produced. Ans: ($\sqrt{48}m_0c$, $\frac{\sqrt{3}}{2}c$)

PH110: Tutorial Sheet 2 (Quantum Mechanics)

* marked problems will be solved in the Wednesday tutorial class.

Compton Scattering

1. Two Compton scattering experiments were performed using x-rays (incident energies E_1 and $E_2 = E_1/2$). In the first experiment, the increase in wavelength of the scattered x-ray, when measured at an angle $\theta = 45°$, is 7×10^{-14} m. In the second experiment, the wavelength of the scattered x-ray, when measured at an angle $\theta = 60°$, is 9.9×10^{-12} m.

   (a) Calculate the Compton wavelength and the mass (m) of the scatterer.

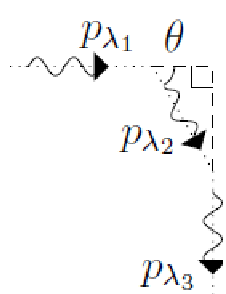
   (b) Find the wavelengths of the incident x-rays in the two experiments.

2. Find the smallest energy that a photon can have and still transfer 50% of its energy to an electron initially at rest.

3. *γ-rays are scattered from electrons initially at rest. Assume the it is back-scattered and its energy is much larger than the electron's rest-mass energy, $E \gg m_e c^2$.
   (a) Calculate the wavelength shift
   (b) Show that the energy of the scattered beam is half the rest mass energy of the electron, regardless of the energy of the incident beam
   (c) Calculate the electron's recoil kinetic energy if the energy of the incident radiation is 150MeV

4. In Compton Scattering, show that the maximum energy of the scattered photon will be $2m_0c^2$, irrespective of the energy of the incident photon. Find the value of θ_0, the angle at which the maximum energy occurs.

5. * In a Compton scattering experiment (see figure), X-rays scattered off a free electron initially at rest at an angle $\theta(> \pi/4)$), gets re-scattered by another free electron, also initially at rest.



   (a) If $\lambda_3 - \lambda_1 = 1.538 \times 10^{-12}$ m, find the value of θ.

   (b) If $\lambda_2 = 68 \times 10^{-12}$ m , find the angle at which the first electron recoils due to the collision.

de Broglie hypothesis

1. Calculate the wavelength of the matter waves associated with the following:

   (a) A 2000 kg car moving with a speed of 100 km/h.

   (b) A 0.28 kg cricket ball moving with a speed of 40 m/s.

   (c) An electron moving with a speed of 10^7 m/s.

   Compare in each case the result with the respective dimension of the object. In which case will it be possible to observe the wave nature.

2. Show that the Bohr's angular momentum quantization leads to the formation of standing waves by the electrons along the orbital circumference in hydrogen atom.

3. Determine the de Broglie wavelength of a particle of mass m and kinetic energy K. Do this for both (a) a relativistic and (b) a non-relativistic particle.

4. *Thermal kinetic energy of a hydrogen atom is $\sim k_B T$ and the radius is $\sim r_1$ ($= 0.53$ Å, radius of the $n = 1$ Bohr orbit). Find the temperature at which its de Broglie wavelength has a value of $2r_1$. Take the mass of the hydrogen atom to be that of a proton.

PH110: Tutorial Sheet 3 (Quantum Mechanics)

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* marked problems will be solved in the Wednesday tutorial class.

Wave packets: Group and Phase Velocity

1. Consider two wave functions $\psi_1(y,t) = 5y\cos 7t$ and $\psi_2(y,t) = -5y\cos 9t$, where y and t are in meters and seconds, respectively. Show that their superposition generates a wave packet. Plot it and identify the modulated and modulating functions.

2. *Two harmonic waves which travel simultaneously along a wire are represented by

$$y_1 = 0.002\cos(8.0x - 400t) \quad \& \quad y_2 = 0.002\cos(7.6x - 380t)$$

   where x, y are in meters and t is in sec.

   (a) Find the resultant wave and its phase and group velocities

   (b) Calculate the range Δx between the zeros of the group wave. Find the product of Δx and Δk ?

3. The angular frequency of the surface waves in a liquid is given in terms of the wave number k by $\omega = \sqrt{gk + Tk^3/\rho}$, where g is the acceleration due to gravity, ρ is the density of the liquid, and T is the surface tension (which gives an upward force on an element of the surface liquid). Find the phase and group velocities for the limiting cases when the surface waves have:
   (a) very large wavelengths and
   (b) very small wavelengths.

4. *Calculate the group and phase velocities for the wave packet corresponding to a relativistic particle.

5. Consider an electromagnetic (EM) wave of the form $A\exp(i[kx - \omega t])$. Its speed in free space is given by $c = \frac{\omega}{k} = 1/\sqrt{\epsilon_0\mu_0}$, where ϵ_0, μ_0 is the electric permittivity, magnetic permeability of free space, respectively.

   (a) Find an expression for the speed (v) of EM waves in a medium, in terms of its permittivity ϵ and permeability μ.

   (b) Suppose the permittivity of the medium depends on the frequency, given by $\epsilon = \epsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)$ where ω_p is a constant called the plasma frequency, find the dispersion relation for the EM waves in a medium. wp is a constant and is called the plasma frequency of the medium (assume $\mu = \mu_0$).

(c) Consider waves with $\omega = 3\omega_p$. Find the phase and group velocity of the waves. What is the product of group and phase velocities?

6. A wave packet describes a particle having momentum p. Starting with the relativistic relationship $E^2 = p^2c^2 + E_0^2$, show that the group velocity is βc and the phase velocity is c/β (where $\beta = v/c$). How can the phase velocity physically be greater than c?

7. *Consider a squre 2-D system with small balls (each of mass m) connected by springs. The spring constants along the x- and y-directions are β_x and β_y, respectively. The dispersion relation for this system is given by

$$-\omega^2 m + 2\beta_x (1 - \cos k_x a_x) + 2\beta_y (1 - \cos k_y a_y) = 0$$

where $\vec{k} = k_x\hat{\imath} + k_y\hat{\jmath}$ is the wave vector and a_x,a_y are the natural distances between the two successive masses along the x-, y-directions, respectively. Find the group velocity and the angle that it makes with the x-axis

Fourier Transform

1. * If $\phi(k) = A(a - |k|), |k| \leq a$, and 0 elsewhere. Where a is a positive parameter and A is a normalization factor to be found.
(a) Find the Fourier transform for $\phi(k)$
(b) Calculate the uncertainties Δx and Δp and check whether they satisfy the uncertainty principle.

2. A wave packet is of the form $f(x) = \cos^2\left(\frac{x}{2}\right)$ (for $-\pi \leq x \leq \pi$) and $f(x) = 0$ elsewhere
(a) Plot $f(x)$ versus x.
(b) Calculate the Fourier transform of $f(x)$, i.e. $g(k) = \int_{-\infty}^{+\infty} f(x)e^{-ikx}dx$?
(c) At what value of $k, |g(k)|$ attains its maximum value?
(d) Calculate the value(s) of k where the function $g(k)$ has its first zero.
(e) Considering the first zero(s) of both the functions $f(x)$ and $g(k)$ to define their spreads (i.e. Δx and Δk), calculate the uncertainty product $\Delta x.\Delta k$.

3. Find the Fourier transform of the following functions: a) $f(x) = \begin{cases} a, & -\ell < x < 0, a > 0 \\ 0, & \text{otherwise} \end{cases}$

b) $f(x) = \begin{cases} a, & -\ell < x < 0 \\ b, & 0 < x < \ell \quad a > 0, b > 0 \\ 0, & \text{otherwise} \end{cases}$

4. A wave packet is of the form $f(x) = \exp(-\alpha|x|) \cdot \exp(ik_0 x)$ (for $-\infty \leq x \leq \infty$) where α, k_0 are positive constants.

(a) Plot $|f(x)|$ versus x.

(b) At what values of x does $|f(x)|$ attain half of its maximum value? Consider the full width at half maxima (FWHM) as a measure of the spread (uncertainty) in x, find Δx

(c) Calculate the Fourier transform of $f(x)$, i.e. $g(k) = \int_{-\infty}^{+\infty} f(x)e^{ikx}dx$

(d) Plot $g(k)$ versus k.

(e) Find the values of k at which $g(k)$ attains half its maximum value? Using the same concept of FWHM as in part (b), calculate Δk ? Hence calculate the product $\Delta x.\Delta k$

[Given : $\int_0^{\infty} e^{-(\alpha - ik)x}dx = \frac{1}{\alpha - ik}$]

PH110: Tutorial Sheet 4 (Quantum Mechanics)

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* marked problems will be solved in the Wednesday tutorial class.

Fourier Transform

1. * If $\phi(k) = A(a - |k|), |k| \leq a$, and 0 elsewhere. Where a is a positive parameter and A is a normalization factor to be found.
   (a) Find the Fourier transform for $\phi(k)$
   (b) Calculate the uncertainties Δx and Δp and check whether they satisfy the uncertainty principle.

2. A wave packet is of the form

$$f(x) = \begin{cases} \cos^2\left(\frac{x}{2}\right), & -\pi \leq x \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

   (a) Plot $f(x)$ versus x.
   (b) Calculate the Fourier transform of $f(x)$, i.e. $g(k) = \int_{-\infty}^{+\infty} f(x)e^{-ikx}dx$?
   (c) At what value of $k, |g(k)|$ attains its maximum value?
   (d) Calculate the value(s) of k where the function $g(k)$ has its first zero.
   (e) Considering the first zero(s) of both the functions $f(x)$ and $g(k)$ to define their spreads (i.e. Δx and Δk), calculate the uncertainty product $\Delta x.\Delta k$.

3. A wave function $\psi(x)$ is defined such that $\psi(x) = \sqrt{2/L}\sin(\pi x/L)$ for $0 \leq x \leq L$ and $\psi(x) = 0$ otherwise.
   (a) Writing $\psi(x) = \int_{-\infty}^{\infty} a(k)e^{ikx}dk$, find $a(k)$.
   (b) What is the amplitude of the plane wave of wavelength L constituting $\psi(x)$?

4. A wave packet is of the form $f(x) = \exp(-\alpha|x|) \cdot \exp(ik_0x)$ (for $-\infty \leq x \leq \infty$) where α, k_0 are positive constants.
   (a) Plot $|f(x)|$ versus x.
   (b) At what values of x does $|f(x)|$ attain half of its maximum value? Consider the full width at half maxima (FWHM) as a measure of the spread (uncertainty) in x, find Δx
   (c) Calculate the Fourier transform of $f(x)$, i.e. $g(k) = \int_{-\infty}^{+\infty} f(x)e^{ikx}dx$
   (d) Plot $g(k)$ versus k.
   (e) Find the values of k at which $g(k)$ attains half its maximum value? Using the same concept of FWHM as in part (b), calculate Δk ? Hence calculate the product $\Delta x.\Delta k$
   [Given : $\int_0^{\infty} e^{-(\alpha - ik)x}dx = \frac{1}{\alpha - ik}$]

Heisenberg Uncertainty Principle

1. Estimate the uncertainty in the position of (a) a neutron moving at 5×10^6 m s^{-1} and (b) a 50 kg person moving at 2 m s^{-1}. The error in the measurement of the velocity is 1%.

2. A lead nucleus has a radius 7×10^{-15} m. Consider a proton bound within nucleus. Using the uncertainty relation Δp.Δr $\geq \hbar/2$, estimate the root mean square speed of the proton, assuming it to be non-relativistic. (You can assume that the average value of p^2 is square of the uncertainty in momentum.)

3. * A π^0 meson is an unstable particle produced in highenergy particle collisions. It has a mass-energy equivalent of about 135MeV, and it exists for an average lifetime of only 8.7×10^{-17} s before decaying into two γ rays. Using the uncertainty principle, estimate the fractional uncertainty $\Delta m/m$ in its mass determination.

4. * For a non-relativistic electron, using the uncertainty relation $\Delta x \Delta p_x = \hbar/2$
   (a) Derive the expression for the minimum kinetic energy of the electron localized in a region of size ' a '.
   (b) If the uncertainty in the location of a particle is equal to its de Broglie wavelength, show that the uncertainty in the measurement of its velocity is same as the particle velocity.
   (c) Using the expression in (b), calculate the uncertainty in the velocity of an electron having energy 0.2keV
   (d) An electron of energy 0.2keV is passed through a circular hole of radius 10^{-6} m. What is the uncertainty introduced in the angle of emergence in radians? (Given $\tan\theta \cong \theta$)

5. An atom in an excited state 1.8eV above the ground state remains in that excited state 2.0μs before moving to the ground state. Find (a) the frequency of the emitted photon, (b) its wavelength, and (c) its approximate uncertainty in energy.

6. * An electron microscope is designed to resolve objects as small as 0.14 nm. What energy electrons must be used in this instrument?

7. * Show that the uncertainty principle can be expressed in the form $\Delta L \Delta\theta \geq \hbar/2$, where θ is the angle and L the angular momentum. For what uncertainty in L will the angular position of a particle be completely undetermined?

   For circular motion $L = rp$ and so $\Delta L = r\Delta p$. Along the circle $x = r\theta$ and $\Delta x = r\Delta\theta$. Thus $\Delta p \Delta x = \frac{\Delta L}{r}(r\Delta\theta) = \Delta L \Delta\theta \geq \frac{\hbar}{2}$. For complete uncertainty $\Delta\theta = 2\pi$ and $\Delta L = \frac{\hbar/2}{2\pi} = \frac{\hbar}{4\pi}$

PH110: Tutorial Sheet 5 (Quantum Mechanics)

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* marked problems will be solved in the Wednesday tutorial class.

Free particle

1. *Show that

$$\psi(x) = A\,\sin(kx) + B\,\cos(kx)$$

and

$$\psi(x) = Ce^{ikx} + De^{-ikx}$$

are equivalent solutions of TISE of a free particle. A, B, C and D can be complex numbers.

2. Show that

$$\Psi(x,t) = A\sin(kx - \omega t) + B\cos(kx - \omega t)$$

does not obey the time-dependant Schroedinger's equation for a free particle.

3. The wave function for a particle is given by,

$$\phi(x) = Ae^{ikx} + Be^{-ikx}$$

where A and B are real constants. Show that $\phi(x)^*\phi(x)$ is always a positive quantity.

4. * A free proton has a wave function given by

$$\Psi(x,t) = Ae^{i(5.02*10^{11}x - 8{:}00*10^{15}t)}$$

The coefficient of x is inverse meters, and the coeffcient of t is inverse seconds. Find its momentum and energy.

5. A particle moving in one dimension is in a stationary state whose wave function,

$$\Psi(x) = \begin{cases} 0, & x < -a \\ A\left(1 + \cos\frac{\pi x}{a}\right), & -a \le x \le a \\ 0, & x > a \end{cases}$$

where A and a are real constants.
(a) Is this a physically acceptable wave function? Explain.
(b) Find the magnitude of A so that $\psi(x)$ is normalized.

(c) Evaluate Δx and Δp. Verify that $\Delta x \Delta p \geq \hbar/2$.
(d) Find the classically allowed region.

6. * Consider the 1-dimensional wave function of a particle of mass m, given by

$$\psi(x) = A \left(\frac{x}{x_0}\right)^n e^{-\frac{x}{x_0}}$$

where, A, n and x_0 are real constants.
(a)Find the potential $V(x)$ for which $\psi(x)$ is a stationary state (It is known that $V(x) \to 0$ as $x \to \infty$).
(b)What is the energy of the particle in the state $\psi(x)$?

Particle in a Box:

1. * For a particle in a 1-D box of side L, show that the probability of finding the particle between $x = a$ and $x = a + b$ approaches the classical value b/L, if the energy of the particle is very high.

2. Consider a particle confined to a 1-D box. Find the probability that the particle in its ground state will be in the central one-third region of the box.

3. Consider a particle of mass m moving freely between $x = 0$ and $x = a$ inside an infinite square well potential.
(a) Calculate the expectation values $\langle \hat{X} \rangle_n, \langle \hat{P} \rangle_n, \left\langle \hat{X}^2 \right\rangle_n$, and $\left\langle \hat{P}^2 \right\rangle_n$, and compare them with their classical counterparts.
(b) Calculate the uncertainties product $\Delta x_n \Delta p_n$.
(c) Use the result of (b) to estimate the zero-point energy.

4. Consider a one dimensional infinite square well potential of length L. A particle is in $n = 3$ state of this potential well. Find the probability that this particle will be observed between $x = 0$ and $x = L/6$. Can you guess the answer without solving the integral? Explain how.

5. * Consider a one-dimensional particle which is confined within the region $0 \leq x \leq a$ and whose wave function is $\Psi(x,t) = \sin(\pi x/a)\exp(-i\omega t)$.
(a) Find the potential $V(x)$.
(b) Calculate the probability of finding the particle in the interval $a/4 \leq x \leq 3a/4$.

6. An electron is moving freely inside a one-dimensional infinite potential box with walls at $x = 0$ and $x = a$. If the electron is initially in the ground state ($n = 1$) of the box and if we suddenly quadruple the size of the box (i.e., the right-hand side wall is moved instantaneously from $x = a$ to $x = 4a$), calculate the probability of finding the electron

in:

(a) the ground state of the new box and

(b) the first excited state of the new box.

7. Solve the time independent Schrodinger equation for a particle in a 1-D box, taking the origin at the centre of the box and the ends at $\pm L/2$, where L is the length of the box.

8. * Consider a particle of mass m in an infinite potential well extending from $x = 0$ to $x = L$. Wave function of the particle is given by

$$\psi(x) = A\left[\sin\left(\frac{\pi x}{L}\right) + \sin\left(\frac{2\pi x}{L}\right)\right]$$

where A is the normalization constant

(a) Calculate A

(b) Calculate the expectation values of x and x^2 and hence the uncertainty Δx.

(c) Calculate the expectation values of p and p^2 and hence the uncertaintyΔp.

(d) What is the probability of finding the particle in the first excited state, if an energy measurement is made?

(given, $\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx = 0$, $\int_0^L x^2\cos\left(\frac{n\pi x}{L}\right)dx = 0$, for all n)

PH110: Tutorial Sheet 6 (Quantum Mechanics)

* marked problems will be solved in the Wednesday tutorial class.

Scattering problems:

1. * A potential barrier is defined by $V = 0$ for $x < 0$ and $V = V_0$ for $x > 0$. Particles with energy E ($< V_0$) approaches the barrier from left.

   (a) Find the value of $x = x_0$ ($x_0 > 0$), for which the probability density is $1/e$ times the probability density at $x = 0$.

   (b) Take the maximum allowed uncertainty Δx for the particle to be localized in the classically forbidden region as x_0. Find the uncertainty this would cause in the energy of the particle. Can then one be sure that its energy E is less than V_0.

2. Consider a potential

$$\begin{aligned} V(x) &= 0 \text{ for } x < 0, \\ &= -V_0 \text{ for } x > 0 \end{aligned}$$

   Consider a beam of non-relativistic particles of energy $E > 0$ coming from $x \to -\infty$ and being incident on the potential. Calculate the reflection and transmission coefficients.

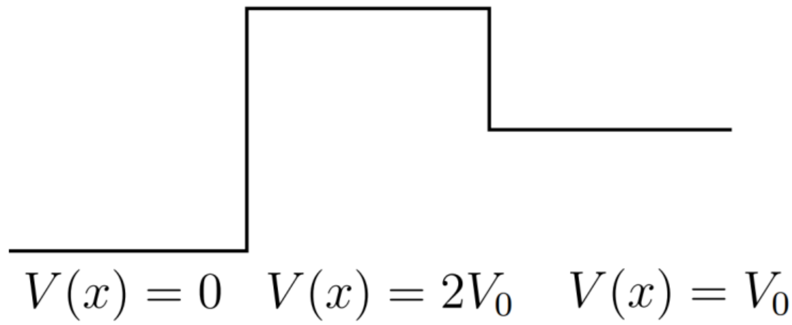
3. A potential barrier is defined by $V = 0$ eV for $x < 0$ and $V = 7$ eV for $x > 0$. A beam of electrons with energy 3 eV collides with this barrier from left. Find the value of x for which the probability of detecting the electron will be half the probability of detecting it at $x = 0$.

4. * A beam of particles of energy E and de Broglie wavelength λ, traveling along the positive x-axis in a potential free region, encounters a one-dimensional potential barrier of height $V = E$ and width L.

   (a) Obtain an expression for the transmission coefficient.

   (b) Find the value of L (in terms of λ) for which the reflection coefficient will be half.

5. A beam of particles of energy $E < V_0$ is incident on a barrier (see figure below) of height $V = 2V_0$. It is claimed that the solution is $\psi_I = A\exp(-k_1 x)$ for region I ($0 < x < L$) and $\psi_{II} = B\exp(-k_2 x)$ for region II ($x > L$), where $k_1 = \sqrt{\frac{2m(2V_0-E)}{\hbar^2}}$ and $k_2 = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$. Is this claim correct? Justify your answer.

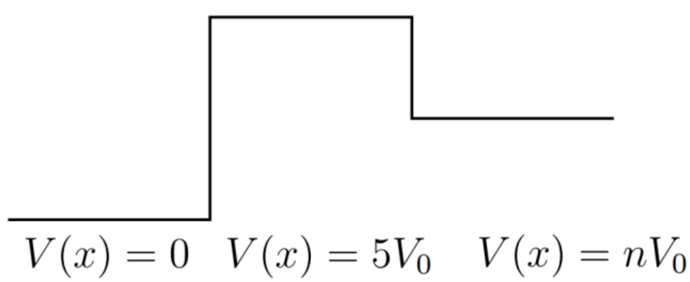


6. * A beam of particles of mass m and energy $9V_0$ (V_0 is a positive constant with the dimension of energy) is incident from left on a barrier, as shown in figure below. $V = 0$ for $x < 0$, $V = 5V_0$ for $x \leq d$ and $V = nV_0$ for $x > d$. Here n is a number, positive or negative and $d = \pi h/\sqrt{8mV_0}$. It is found that the transmission coefficient from $x < 0$ region to $x > d$ region is 0.75.

   (a) Find n. Are there more than one possible values for n?

   (b) Find the un-normalized wave function in all the regions in terms of the amplitude of the incident wave for each possible value of n.

   (c) Is there a phase change between the incident and the reflected beam at $x = 0$? If yes, determine the phase change for each possible value of n. Give your answers by explaining all the steps and clearly writing the boundary conditions used



7. A scanning tunneling microscope (STM) can be approximated as an electron tunneling into a step potential [$V(x) = 0$ for $x \leq 0$, $V(x) = V_0$ for $x > 0$]. The tunneling current (or probability) in an STM reduces exponentially as a function of the distance from the sample. Considering only a single electron-electron interaction, an applied voltage of 5 V and the sample work function of 7 eV, calculate the amplification in the tunneling current if the separation is reduced from 2 atoms to 1 atom thickness (take approximate size of an atom to be 3 Å).

Simple Harmonic Oscillator and 2D/3D Systems

1. Using the uncertainty principle, show that the lowest energy of an oscillator is $\hbar\omega/2$.

2. Determine the expectation value of the potential energy for a quantum harmonic oscillator (with mass m and frequency ω) in the ground state. Use this to calculate the expectation value of the kinetic energy. The ground state wavefunction of quantum harmonic oscillator is:

$$\psi_0(x) = C_0 \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \quad C_0 \text{ is constant;} \tag{1}$$

3. A diatomic molecule behaves like a quantum harmonic oscillator with the force constant $k = 12Nm^{-1}$ and mass $m = 5.6 * 10^{-26}kg$

   (a) What is the wavelength of the emitted photon when the molecule makes the transition from the third excited state to the second excited state ?

   (b) Find the ground state energy of vibrations for this diatomic molecule.

4. Vibrations of the hydrogen molecule can be modeled as a simple harmonic oscillator with the spring constant $k = 1.13 * 10^3 Nm^{-2}$ and mass $m = 1.67 * 10^{27}$ kg.

   (a) What is the vibrational frequency of this molecule ?

   (b) What are the energy and the wavelength of the emitted photon when the molecule makes transition between its third and second excited states ?

5. * A two-dimensional isotropic harmonic oscillator has the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{2}k(x^2 + y^2)$$

   (a) Show that the energy levels are given by

$$E_{n_x,n_y} = \hbar\omega(n_x + n_y + 1) \quad \text{where} \quad n_x, n_y \in (0, 1, 2...) \quad \omega = \sqrt{\frac{k}{m}}$$

   (b) What is the degeneracy of each level?

6. Consider the Hamiltonian of a two-dimensional anisotropic harmonic oscillator $(\omega_1 \neq \omega_2)$

$$H = \frac{{p_1}^2}{2m} + \frac{{p_2}^2}{2m} + \frac{1}{2}m{\omega_1}^2{q_1}^2 + \frac{1}{2}m{\omega_2}^2{q_2}^2$$

   (a) Exploit the fact that the Schrödinger eigenvalue equation can be solved by separating the variables and find a complete set of eigenfunctions of H and the corresponding eigenvalues.

(b) Assume that $\frac{\omega_1}{\omega_2} = \frac{3}{4}$ .Find the first two degenerate energy levels. What can one say about the degeneracy of energy levels when the ratio between ω_1 and ω_2 is not a rational number.

7. A particle of mass m is confined to move in the potential $\left(m\omega^2 x^2\right)/2$. Its normalized wave function is

$$\psi(x) = \left(\frac{2\beta}{\sqrt{3}}\right)\left(\frac{\beta}{\pi}\right)^{1/4} x^2 e^{-\left(\beta x^2/2\right)}$$

where β is a constant of appropriate dimension.
(a) Obtain a dimensional expression for β in terms of m, ω and $\hbar$.
(b) It can be shown that the above wave function is the linear combination

$$\psi(x) = a\psi_0(x) + b\psi_2(x)$$

where $\psi_0(x)$ is the normalized ground state wave function and $\psi_2(x)$ is the normalized second excited state wave function of the potential. Evaluate b and hence calculate the expectation value of the energy of the particle in this state $\psi(x)$.
Given: $I_0(\beta) = \int_{-\infty}^{+\infty} e^{-\beta x^2} dx = \sqrt{\frac{\pi}{\beta}}$, $\quad I_n(\beta) = \int_{-\infty}^{+\infty} \left(x^2\right)^n e^{-\beta x^2} dx = (-1)^n \frac{\partial^n}{\partial \beta^n}\left(I_0(\beta)\right)$,
$\psi_0(x) = \left(\frac{\beta}{\pi}\right)^{1/4} e^{\frac{-\beta x^2}{2}}$

8. Consider an 3D isotropic harmonic oscillator show that the degeneracy g_n of the nth excited state, which is equal the number of ways the non negative integers n_x, n_y, n_z may be chosen to total to n, is given by

$$g_n = \frac{1}{2}(n+1)(n+2)$$

9. * A charged particle of mass ' m ' and charge ' q ' is bound in a 1-dimensional simple harmonic oscillator potential of angular frequency ' ω '. An electric field E_0 is turned on.
(a) What is the total potential V(x) experienced by the charge ?
(b) Express the total potential in the form of an effective harmonic oscillator potential.
(c) Sketch V (x) versus x.
(d) What is the ground state energy of the particle in this potential?
(e) What is the expectation value of the position (x) if the charge is in its ground state ?