MA406

General Topology

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Attendance policy : Compulsory attendance

Reference : Willard

Munkres

Continuity of fn^n b/w metric space

A fn^n $f:(X, d_1) \to (Y, d_2)$ is said to be continuous at $a \in X$ if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t $d_2(f(x), f(a)) < \epsilon$ whenever $d_1(x, a) < \delta$

eg :- 1. $(\mathbb{R}, |.|)$

$$d(x,y) = |x-y|$$

2. $(\mathbb{R}^2, \|.\|_2)$

$$d_2(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$$

3. $(\mathbb{R}^2, \|.\|_\infty)$

$$d_\infty(x,y) = \max\{|x_1-y_1|, |x_2-y_2|\}$$

4. $(\mathbb{R}^2, d)$

$$d(x,y) = |x_1-y_1| + |x_2-y_2|$$

- $d(x,y) \geq 0 \quad \forall x, y$ ✓
- $d(x,y) = 0 \Rightarrow |x_1-y_1| = 0$ & $|x_2-y_2| = 0 \Rightarrow x = y$ ✓
- $d(x,y) = d(y,x)$ ✓
- $d(x,y) = |x_1-y_1| + |x_2-y_2|$

$$\leq \underbrace{|x_1-z_1| + |x_2-z_2|}_{d(x,z)} + \underbrace{|z_1-y_1| + |z_2-y_2|}_{d(z,y)} \quad ✓$$

5. X is an set.

$$\rho(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

(Discrete metric)

A fx^n $f: X \to Y$, $(X,d), (Y,d')$ are metric spaces is cont. if $\forall a \in X$, $\forall \epsilon > 0$, $\exists \delta > 0$ s.t

$$f(N(a,\delta)) \subseteq N(f(a), \epsilon)$$

where $N(a,\delta) = \{ x \in X : d(x,a) < \delta \}$

(or $B(a,\delta)$)

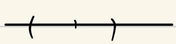
$B(a,\delta)$ has the property that for any $x \in B(a,\delta)$, $\exists \eta > 0$ s.t $B(x,\eta) \subseteq B(a,\delta)$

In particular, $\eta = \dfrac{\delta - |x-a|}{2}$ satisfies this.

__Open set:__

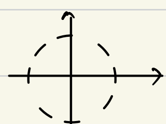
Let (X,d) be a metric sp. A subset $U \subseteq X$ is said to be open if $\forall a \in U$, $\exists \delta > 0$, s.t $B(a,\delta) \subseteq U$

eg: 1.



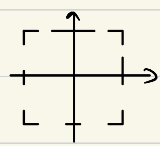
$(\mathbb{R}, |\cdot|)$

2.



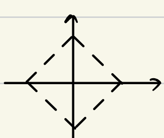
$(\mathbb{R}, \|\cdot\|_2)$

3.



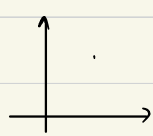
$(\mathbb{R}, \|\cdot\|_\infty)$

4.



$(\mathbb{R}^2, d)$

5.



$(\mathbb{R}^2, \rho)$

If $\delta \leq 1$, isolated pt. Else all pts.

Is every open interval an open neighbourhood of some pt.?

Note: $\emptyset$ & X are open in X.

$\emptyset$ is open vacuously.

X is open since for any $x \in X$, $B_x(1) \subseteq X$

($\because$ everything is in X).

Ppt: Arbitrary union of open sets is open.

Let $\{A_k\}_{k \in \mathcal{I}}$ be a coll. of open sets in X.

Consider $S = \bigcup\limits_{k \in \mathcal{I}} A_k$.

If $S = \emptyset$, it is open vacuously.

Else, consider $x \in S$.

Then $\exists\, k_0 \in \mathcal{I}$ s.t $x \in A_{k_0}$

$\because A_{k_0}$ is open $\Rightarrow \exists\, \delta > 0$ s.t $B(x, \delta) \subseteq A_{k_0}$

$\because A_{k_0} \subseteq S \Rightarrow B(x, \delta) \subseteq S$

Ppt: Finite intersection of open sets is open

Let $\{A_k\}_{k=1}^{n}$ be a coll. of open sets in X.

Consider $S = \bigcap_{k=1}^{n} A_k$

If $S = \emptyset$, it is open vacuously.

Else, consider $x \in S$.

Then $x \in A_k \quad \forall \ 1 \le k \le n$

$\because A_k$ is open $\forall \ 1 \le k \le n \Rightarrow \exists \ \{\delta_k\}_{k=1}^{n}, \ \delta_k > 0 \ \forall \ 1 \le k \le n$

s.t $B(x, \delta_k) \subseteq A_k$

Let $\delta = \min_{1 \le k \le n} \delta_k$.

Then $\delta > 0$ & $B(x, \delta) \subseteq B(x, \delta_k) \subseteq A_k \quad \forall \ 1 \le k \le n$

$\Rightarrow B(x, \delta) \subseteq S$

$f : X \to Y$ is cont.

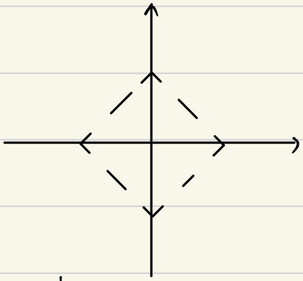
$\forall \epsilon > 0, \exists \delta > 0, \quad f(B(a,\delta)) \subseteq B(f(a),\epsilon))$

$\Leftrightarrow \quad B(a,\delta) \subseteq f^{-1}(B(f(a),\epsilon))$

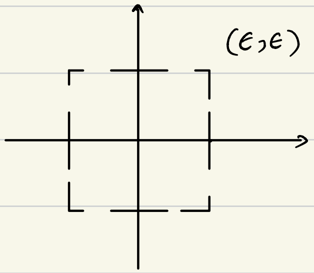
eg: $f : (\mathbb{R}^2, \|.\|_2) \to (\mathbb{R}^2, \| \|_\infty)$

1. $(x,y) \mapsto (x,y)$
2. $(x,y) \mapsto (x+y, x-y)$
3. $(x,y) \mapsto (xy, 0)$

Note: Any f^{x^n} $F : (\mathbb{R}^2, \|.\|_2) \to (\mathbb{R}^2, \|.\|_2)$ in cont.
if $F = (f,g)$, $f,g : (\mathbb{R}^2, \|.\|_2) \to (\mathbb{R}, \|.\|_2)$ are cont.



f



$f^{-1}(B((0,0),\epsilon))$

$B((0,0),\epsilon)$
in $(\mathbb{R}^2, \|.\|_\infty)$

$(x,y) \in f^{-1}(B((0,0),\epsilon))$

$\Rightarrow \quad f((x,y)) \in B((0,0),\epsilon) \quad \Rightarrow \quad -\epsilon \leq x+y \leq \epsilon$

$-\epsilon \leq x-y \leq \epsilon$

Claim: $f:(X,d) \to (Y,d')$ is cont. iff $\forall$ open sets $U \subseteq Y$, $f^{-1}(U)$ is open

Pf: ($\Rightarrow$) f is cont. & U is open in Y.
To show: $f^{-1}(U)$ is open

Consider $a \in f^{-1}(U)$. $\Rightarrow f(a) \in U$
$\because U$ is open, $\exists\, \epsilon > 0$ s.t $B_{f(a)}(\epsilon) \subseteq U$

$\because f$ is cont. $\Rightarrow \exists\, \delta > 0$ s.t $B_a(\delta) \subseteq f^{-1}(B_{f(a)}(\epsilon))$
$\subseteq f^{-1}(U)$

$\therefore f^{-1}(U)$ is open

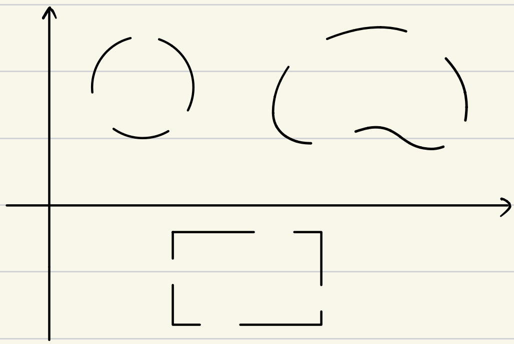
($\Leftarrow$) To show: $\forall a \in X$, $\forall \epsilon > 0$, $\exists\, \delta > 0$ s.t $B_a(\delta) \subseteq f^{-1}(B_{f(a)}(\epsilon))$

Consider $a \in X$, $\epsilon > 0$.
$\because B_{f(a)}(\epsilon)$ is open $\Rightarrow f^{-1}(B_{f(a)}(\epsilon))$ is open.

$\because a \in f^{-1}(B_{f(a)}(\epsilon))$ & it is open, $\exists\, \delta > 0$ s.t
$B_a(\delta) \subseteq f^{-1}(B_{f(a)}(\epsilon))$

eg: Open sets in

$(\mathbb{R}^2, \|\cdot\|_2)$



$(\mathbb{R}^2, \|\cdot\|_\infty)$

Rem: Open sets are same in both metric sp.

Topology

Let X be a set & $\mathcal{T} \subseteq P(X)$. The $\mathcal{T}$ is said to be a topology on X if

1. $\emptyset, X \in \mathcal{T}$

2. for any $\{A_i\}_{i \in \mathcal{I}} \subseteq \mathcal{T}$, $\bigcup_{i \in \mathcal{I}} A_i \in \mathcal{T}$

3. for any $\{A_i\}_{i=1}^{n} \subseteq \mathcal{T}$, $\bigcap_{i=1}^{n} A_i \in \mathcal{T}$

i.e it is closed under arbitrary unions & finite intersections.

eg - 1. Metric topology: In any metric sp. (X, d), the coll. of open sets defines a topology on X.

2. Indiscrete topology: On any set X, $\mathcal{T} = \{\emptyset, X\}$

3. $X = \{a, b\}$

All topologies on X.

$\mathcal{T}_1 = \{\emptyset, \{a,b\}\}$

$\mathcal{T}_2 = \{\emptyset, \{a\}, \{a,b\}\}$

$\mathcal{T}_3 = \{\emptyset, \{b\}, \{a,b\}\}$

$\mathcal{T}_4 = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$

4. Cofinite topology : Let X be an infinite set.

$$\mathcal{T} = \{ A \in P(X) : X - A \text{ is finite} \} \cup \{\emptyset\}$$

(1) By def^n $\emptyset, X \in \mathcal{T}$

(2) Consider $\{A_n\}_{n \in \mathcal{I}} \in \mathcal{T}$.

Clearly, $X - \bigcup_{n \in \mathcal{I}} A_n \subseteq \underbrace{X - A_n}_{\text{finite}} \quad \forall n \in \mathcal{I}$

$\therefore$ $X - \bigcup_{n \in \mathcal{I}} A_n$ is finite

$\Rightarrow$ $\bigcup_{n \in \mathcal{I}} A_n \in \mathcal{T}$

(3) Consider $\{A_k\}_{k=1}^{n} \in \mathcal{T}$

If $\bigcap_{k=1}^{n} A_k \neq \emptyset$,

Clearly, $X - \bigcap_{k=1}^{n} A_k = \bigcup_{k=1}^{n} \underbrace{(X - A_k)}_{\text{finite}}$

$\because$ Finite union of finite sets is finite $\Rightarrow \bigcap_{k=1}^{n} A_k \in \mathcal{T}$

Else, $\bigcap_{k=1}^{n} A_k = \emptyset$, it is in $\mathcal{T}$ by def^n.

Let $(X, \mathcal{T})$ be a topological sp.
Then all sets in $\mathcal{T}$ are called open sets.

Closed sets in metric sp. are complements of open sets.

A set B is closed in $(X, \mathcal{T})$ if $X - B \in \mathcal{T}$

eg: Let $\mathcal{F} = \{ B : X - B \in \mathcal{T} \}$

(1) $\emptyset, X \in \mathcal{F}$
(2) $\mathcal{F}$ is closed under arbitrary intersections
(3) $\mathcal{F}$ is closed under finite unions.

Zariski topology:

$\mathcal{F} = \{ V(S) : S$ is the set of real polynomials in 2 variables $\}$

$$V(S) = \{ (a, b) \in \mathbb{R}^2 : f(a, b) = 0, \ f \in S \}$$

This defines a top. of closed sets.

Limit pt: Let (X, d) be a metric sp.
Let $A \subseteq X$, then $w \in X$ is a limit pt. of A if
$\exists$ a seq. $(a_n) \in A$ s.t $a_n \to w$ as $n \to \infty$

In (X, d), A is closed iff A contains all its limit pts.

Closure of A
$\bar{A}$ = all lim. pts. of A.

Claim: Let (X, d) be a metric sp. & let $A \subseteq X$.
(1) $A \subseteq \bar{A}$
(2) If B is a closed set s.t $A \subseteq B$ then $\bar{A} \subseteq B$.

Pf: 1. Consider $a \in A$.
Since $\exists$ a seq. $(a_n) \in A$ s.t $a_n \to a$ as $n \to \infty$
where $a_n = a \ \forall n \geq 1$
$\therefore$ $a \in \bar{A}$

2. Let $a \in \bar{A}$. Then $\exists$ a seq. $(a_n) \in A$ s.t $a_n \to a$ as $n \to \infty$
$\because$ $A \subseteq B \Rightarrow (a_n)$ is a seq. in B & a is lim. pt. of B.
$\because$ B is closed, $a \in B$
$\therefore$ $\bar{A} \subseteq B$

In a metric sp. (X,d), $\bar{A}$ can be defined as the smallest closed set containing A.

i.e if $C \subseteq X$ is a closed set s.t

1. $A \subseteq C$
2. $\forall$ closed $B \supseteq A$, $C \subseteq B$

then $C = \bar{A}$

OR

$$\bar{A} = \bigcap_{K \in \Omega} K, \qquad \Omega = \{ K \subseteq X \text{ closed} : A \subseteq K \}$$

<u>Closure (in topo. sp.)</u>: Let $(X, \mathcal{T})$ be a topo. sp. Then for any $A \subseteq X$, closure of A denoted by $\bar{A}$

$$\bar{A} = \bigcap_{K \in \Omega} K, \qquad \Omega = \{ K \subseteq X \text{ closed} : A \subseteq K \}$$

It is the smallest closed set containing A.

Ppts :

1. $A \subseteq \overline{A}$
2. $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$
3. $\overline{\emptyset} = \emptyset$
4. $\overline{X} = X$
5. $\overline{(\overline{E})} = \overline{E}$
6. $\overline{A \cup B} = \overline{A} \cup \overline{B}$
7. A is closed iff $A = \overline{A}$

Neighbourhood :

Let X be a top. sp. Let $a \in X$, then N is a neighbourhood of a if $\exists$ an open set $a \in V$ s.t $V \subseteq N$

Interior :

Let $(X, \mathcal{J})$ be a top. sp. & $A \subseteq X$. Interior of A denoted by $\mathring{A}$ is the largest open set in X contained in A.

Rem: Every open set $(\mathbb{R}, |\ |)$ is a union of open intervals.

In general, if given a coll. of sets in X,

$$\mathcal{B} = \{ A_i \subseteq X : i \in \mathcal{I} \}$$

$$\mathcal{T}_{\mathcal{B}} = \{ \bigcup_{j \in J} A_{i_j} : J \text{ is an indexing set} \}$$

Take $J = \emptyset$, to get $\bigcup_{\emptyset} A_{i_j} = \emptyset \in \mathcal{T}_{\mathcal{B}}$

If $\mathcal{B} = \{ \{i\} \subseteq \mathbb{R} : i \in \mathbb{Z}_{>0} \}$, then $\mathcal{T}_{\mathcal{B}}$ cannot be a topology on $\mathbb{R}$.

$\mathcal{T}_{\mathcal{B}}$ is closed under arbitrary unions as it has all possible unions of elems. of $\mathcal{B}$ &

$$\bigcup_{j \in J} \left(\bigcup_{i \in \mathcal{I}} A_i \right) = \bigcup_{(i,j) \in \mathcal{I} \times J} A_i \in \mathcal{T}_{\mathcal{B}}$$

For being closed under finite intersections,

$$\left(\bigcup_{k \in \mathcal{I}_1} A_k \right) \cap \left(\bigcup_{l \in \mathcal{I}_2} A_l \right) = \bigcup_{(k,l) \in \mathcal{I}_1 \times \mathcal{I}_2} A_k \cap A_l \in \mathcal{T}_{\mathcal{B}}$$

So, there should exist J s.t $\bigcup_{(k,l) \in \mathcal{I}_1 \times \mathcal{I}_2} A_k \cap A_l = \bigcup_{j \in J} A_j$

Hence, for $\mathcal{T}_{\mathcal{B}}$ to be a topology on X:

1. $\bigcup_{A_i \in \mathcal{B}} A_i = X$

2. Let $x \in A_k \cap A_l$ then $\exists A_x \in \mathcal{B}$ s.t $x \in A_x \subseteq A_k \cap A_l$

If $\mathcal{B}$ satisfies these two properties, $\mathcal{B}$ is said to be a base for $\mathcal{T}_{\mathcal{B}}$.

Equiv., a coll. $\mathcal{B} \subseteq \mathcal{T}$ is said to be a base for a topology $\mathcal{T}$ if $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$

Ex 8: Find a base for

1. $(\mathbb{R}^2, \| \ \|)$
2. $(X$, discrete metric$)$
3. $(X$, cofinite topology$)$

Ex 9: Let $\mathcal{B} = \{ [x, y) \in \mathbb{R} : x, y \in \mathbb{R} \}$

1. then $\mathcal{B}$ a base for a topology on $\mathbb{R}$.
2. How is it diff. from $(\mathbb{R}, | \ |)$?
3. Find closure $\{ \frac{1}{n} : n \in \mathbb{Z}_{>0} \}$ in $\mathcal{T}_{\mathcal{B}}$

Ex 10: Let $(X, <)$ be a totally ordered set.

Define $\mathcal{D} = \{ (a,b) : a, b \in X \}$

$$(a,b) = \{ x \in X : a < x < b \}$$

Does give a base for a topo. sp. on X.

A base $\mathcal{B}$ of topology $\mathcal{T}$ is a coll. $\mathcal{B} \subseteq \mathcal{T}$ s.t

$$\mathcal{T}_{\mathcal{B}} = \{ \bigcup_{j \in J} B_j : B_j \in \mathcal{B} \text{ for some indexing set } J \}$$
$$= \mathcal{T}$$

A coll. $\mathcal{B} = \{ B_i \subseteq X : i \in \mathcal{I} \}$ generates a topology

$\mathcal{T}_{\mathcal{B}} = \{ \bigcup_{j \in J} B_j : B_j \subseteq \mathcal{B}, J \subseteq \mathcal{I} \}$ on X iff

(1) $\bigcup_{i \in \mathcal{I}} B_i = X$

(2) $\forall B_e, B_k \in \mathcal{B}$ if $x \in B_e \cap B_k$, then

$\exists\, x \in B_l \subseteq B_e \cap B_k$

and in this case $\mathcal{B}$ is a base for the topology $\mathcal{T}_{\mathcal{B}}$.

Ex: Find base for a cofinite topology $\mathcal{C}$ on X.

A. 1. X is finite: The cofinite topology = discrete topology
The set of singleton sets forms a base for discrete topology.

$$\mathcal{B} = \{ \underbrace{\{a\}}_{} : a \in X \} \subseteq \mathcal{C} = P(X)$$

↳ is open since X is finite
$\Rightarrow X - \{a\}$ is finite

Since any $A \in \mathcal{C}$, $A = \bigcup_{a \in A} \{a\} \in \mathcal{T}_{\mathcal{B}} \Rightarrow \mathcal{C} \subseteq \mathcal{T}_{\mathcal{B}}$

Since $\mathcal{C}$ is a topology, $\mathcal{B} \subseteq \mathcal{C} \Rightarrow \mathcal{T}_{\mathcal{B}} \subseteq \mathcal{C}$

Hence, $\mathcal{T}_{\mathcal{B}} = \mathcal{C}$.

2. X is not finite:
First we need to know description of base in terms of closed sets.

A base $\mathcal{B}$ of topology $\mathcal{F}$ of closed sets is a coll. $\mathcal{A} \subseteq \mathcal{F}$ s.t

$$\mathcal{F}_{\mathcal{A}} = \left\{ \bigcap_{j \in J \subseteq I} A_j : A_j \in \mathcal{A}, \ J \text{ is some indexing set} \right\}$$

$$= \mathcal{F}$$

Now, consider $\mathcal{F} = \{ Z \subseteq X : Z \text{ is finite} \}$ which is the topo. of closed sets on X.

A coll. of closed sets $\mathcal{A} = \{A_i \in X : i \in \mathcal{I}\}$ generates a topo. of closed sets on X.

$$\mathcal{F}_{\mathcal{A}} = \{\bigcap_{j \in J} A_i : A_i \in \mathcal{A},\ J \subseteq \mathcal{I}\} \quad \text{iff}$$

(1) $\bigcap_{i \in \mathcal{I}} A_i = \emptyset$

(2) For any $A_k, A_\ell \in \mathcal{A}$, $\exists A_r \in \mathcal{A}$, s.t $A_k \cup A_\ell \subseteq \bigcap_{r \in R \subseteq \mathcal{I}} A_r$

and in this case $\mathcal{A}$ is a base for the topology $\mathcal{F}_{\mathcal{A}}$ of closed sets on X.

Ex: Show that $\{[a,b) : a,b \in \mathbb{R}\}$ forms a base for a topo. on $\mathbb{R}$.

A. 1. $\mathbb{R} = \bigcup_{x \in \mathbb{R}} [x, x+1) \subseteq \bigcup_{a,b \in \mathbb{R}} [a,b) \subseteq \mathbb{R}$

2. Let $x \in [a_1, b_1) \cap [a_2, b_2)$

Then $x = [a,b) \subseteq [a_1, b_1) \cap [a_2, b_2)$, $\quad a = \max\{a_1, a_2\}$

$b = \min\{b_1, b_2\}$

Q. Can you generate a topology from any arbitrary coll.?

eg - $\mathcal{R} = \{(a, \infty) : a \in \mathbb{R}\}$

Claim: $\mathcal{T}_{\mathcal{R}} = \mathcal{R}$

eg: 2. $\mathcal{L} = \{(a,\infty), (-\infty,b) : a,b \in \mathbb{R}\}$

$\mathcal{L}$ is not a base since for $a<b$, $\nexists\, S \in \mathcal{L}$ s.t
$S \subseteq (-\infty,b) \cap (a,\infty) = (a,b)$

However, $\mathcal{L}$ can still generate a topology (just not as a base) if we consider finite intersections of elems. of $\mathcal{L}$ as well.

In general, given any coll. of subsets $\mathcal{D}$ of X s.t their union is X, it can generate a topology by considering $\mathcal{T}_{\mathcal{D}}$ along with finite intersections of elems. of $\mathcal{D}$.

Let $(X, \mathcal{J})$ be a topological sp. Then for $A \subseteq X$,

$$\mathcal{J}_A = \{ A \cap W : W \in \mathcal{J} \}$$

is a topology on A, & $(A, \mathcal{J}_A)$ is called a subspace of X.

eg: 1. $\mathbb{Q} \subseteq \mathbb{R}$

$\downarrow$ $\hookrightarrow$ topo. is a set is open iff it is a union of open intervals.

topo. is

$\mathbb{Q} \cap (0, 1)$

2. $\mathbb{R} \subseteq (\mathbb{R}^2, \|.\|_2)$

Let $(X, \mathcal{J})$ be a topo. sp. & $(A, \mathcal{J}_A)$ be a subsp.

What are closed in $(A, \mathcal{J}_A)$?

Ppn: A set $W \subseteq A$ is closed iff $\exists$ closed set $Z \subseteq X$, s.t

$W = Z \cap A$

Pf: $(\Leftarrow)$ Let $W = Z \cap A$, where Z is closed in X.

$$A - W = A - (Z \cap A) = (X \cap A) - (Z \cap A) = (X - Z) \cap A$$

$A - W$ is open in A, so W is closed in A

($\Rightarrow$) W is closed in A $\Rightarrow$ $A-W$ is open in A

$\Rightarrow \exists\, Y \in \mathcal{T}$ s.t $A-W = Y \cap A$

$$\Rightarrow W = A-(Y\cap A) = (X\cap A)-(Y\cap A)$$

$$= \underbrace{(X-Y)}_{\text{closed}} \cap A$$

Hence, $\exists$ closed $Z = X - Y \subseteq X$ s.t $W = Z \cap A$.

<u>Ex</u>: Let $E \subseteq A \subseteq X$,

Prove <u>1.</u> $Cl_A(E) = Cl_X(E) \cap A$

<u>2.</u> $int_A(E) = int_X(E \cup (X\setminus A)) \cap A$

Comparing topological spaces

$(X, \mathcal{T})$ & $(Y, \mathcal{F})$

$g : X \to Y$ is a bijection
we want that for every $U \in \mathcal{T}$, $g(U) \in \mathcal{F}$ &
$\forall\, V \in \mathcal{F}$, $g^{-1}(V) \in \mathcal{T}$

Continuous fn^n: A fn^n b/w topo. sp. $g : (X, \mathcal{T}) \to (Y, \mathcal{F})$ is called cont. if $\forall\, V \in \mathcal{F}$, $g^{-1}(V) \in \mathcal{T}$

Homeomorphism: A cont. bij. b/w $g : (X, \mathcal{T}) \to (Y, \mathcal{F})$ topo. sp. is said to be a homeomorphism if g^{-1} is also cont.

eg: 1. $Id : (\mathbb{R}^2, \| \|_2) \to (\mathbb{R}^2, \| \|_2)$

2. $Id : (\mathbb{R}^2, \| \|_2) \to (\mathbb{R}^2, \| \|_\infty)$
Open neighbourhoods in $(\mathbb{R}^2, \| \|_\infty)$ are open rectangles.
Their pre-image under Id are open rect. which are also open in $(\mathbb{R}^2, \| \|_2)$.
Checking this suffices as every open set in $(\mathbb{R}^2, \| \|_\infty)$ is a union of open rectangles.

$$V = \bigcup_{i \in \mathcal{I}} R_i \Rightarrow Id^{-1}(V) = \bigcup_{i \in \mathcal{I}} R_i$$

open in $(\mathbb{R}^2, \| \|_\infty)$ open in $(\mathbb{R}^2, \| \|_2)$

<u>Thm</u>: A fxn $f : (X, \mathcal{T}) \to (Y, \mathcal{F})$ is cont. iff given a base $\mathcal{B} \subseteq \mathcal{F}$ $\forall A \in \mathcal{B}$, $f^{-1}(A) \in \mathcal{T}$ (open in X)

<u>Pf</u>: $(\Rightarrow)$ f is cont., so for any $A \in \mathcal{B} \subseteq \mathcal{F}$, $f^{-1}(A) \in \mathcal{T}$ (by defn)

$(\Leftarrow)$ Consider $V \in \mathcal{F}$. Then $V = \bigcup_{i \in \mathcal{I}} A_i$, $A_i \in \mathcal{B}$ $\forall i \in \mathcal{I}$

$$\Rightarrow f^{-1}(V) = f^{-1}\left(\bigcup_{i \in \mathcal{I}} A_i\right) = \bigcup_{i \in \mathcal{I}} f^{-1}(A_i)$$

$\because$ $f^{-1}(A_i) \in \mathcal{T}$ $\forall i \in \mathcal{I}$ $\Rightarrow$ $f^{-1}(V) \in \mathcal{T}$

($\because$ topo. is closed under arbitrary unions)

So, f is cont.

eg: $\underline{0}$. $(X, \mathcal{T}) \xrightarrow{Id} (X, \mathcal{T})$ is always cont.

$\underline{1}$. $(X, \mathcal{T}) \xrightarrow{Id} (X, \mathcal{U})$ is cont. if $\mathcal{U} \subseteq \mathcal{T}$

2. $(X, \mathcal{T}_d) \xrightarrow{f} (Y, \mathcal{U})$ is always cont.

↑ discrete topo.

$\underline{3}$. $(X, \mathcal{T}) \xrightarrow{f} (Y, \mathcal{U})$ is always cont.

$\mathcal{U} = \{\emptyset, Y\}$

<u>Ppts</u>: $(X, \mathcal{T})$, $(Y, \mathcal{U})$ & $(Z, \mathcal{F})$ be topo. sp.

Let $A \subseteq X$ & let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ & $g : (Y, \mathcal{U}) \to (Z, \mathcal{F})$ be fn's

$\underline{1}$. If f is cont., $f|_A$ is also cont.

$\underline{2}$. If f, g are cont., then $g \circ f$ is cont.

$\underline{3}$. f is cont. iff pre-image of closed sets is closed.

$\underline{4}$. f is cont. iff $f(Cl_X A) \subseteq Cl_Y(f(A))$

$\underline{5}$. If $X = A \cup B$ where A, B are both open (closed) sets, then f is cont. iff $f|_A$ & $f|_B$ are cont.

Pf: 1. Consider any $V \in \mathcal{U}$. Since f is cont., $f^{-1}(V) \in \mathcal{T}$

So, $f^{-1}|_A(V) = f^{-1}(V) \cap A \in \mathcal{T}_A$

Hence, $f|_A$ is cont.

2. Consider $W \in \mathcal{F}$. Since g is cont., $g^{-1}(W) = V \in \mathcal{U}$.

Since f is cont., $f^{-1}(V) \in \mathcal{T} \Rightarrow f^{-1}(g^{-1}(V)) \in \mathcal{T}$

$\Rightarrow (g \circ f)^{-1}(V) \in \mathcal{T}$

Hence, $g \circ f$ is cont.

3. Consider closed $V \in \mathcal{Y}$. So, $V^c \in \mathcal{U}$

Since f is cont., $f^{-1}(V^c) \in \mathcal{T} \Rightarrow f^{-1}(V)^c \in \mathcal{T}$

So, $f^{-1}(V)$ is closed in $(X, \mathcal{T})$.

Hence, pre-image of closed sets is closed.

4. $(\Rightarrow)$ Let f be cont.

To show: $Cl_X(A) \subseteq f^{-1}(Cl_Y(f(A)))$

Note, $Cl_Y(f(A))$ is closed $\Rightarrow f^{-1}(Cl_Y(f(A)))$ is closed.

$Cl_X(A)$ is the smallest closed set in X containing A.

$$f(A) \subseteq Cl_Y(f(A)) \Rightarrow f^{-1}(f(A)) \subseteq f^{-1}(Cl_Y(f(A)))$$

$$\Rightarrow A \subseteq f^{-1}(Cl_Y(f(A))) \quad [\because A \subseteq f^{-1}(f(A))]$$

$$\Rightarrow \mathrm{Cl}_X(A) \subseteq f^{-1}(\mathrm{Cl}_Y(f(A)))$$

($\Leftarrow$) $f(\mathrm{Cl}_X(A)) \subseteq \mathrm{Cl}_Y(f(A)) \quad \forall\ A \subseteq X$

<u>To show</u>: If E is closed in Y, then $f^{-1}(E)$ is closed in X.

$f^{-1}(E)$ is closed iff $\mathrm{Cl}_X(f^{-1}(E)) = f^{-1}(E)$

Given $\mathrm{Cl}_X(f^{-1}(E)) \subseteq f^{-1}(\mathrm{Cl}_Y(f(f^{-1}(E))))$

Now, $f(f^{-1}(E)) \subseteq E \Rightarrow \mathrm{Cl}_Y(f(f^{-1}(E))) \subseteq \mathrm{Cl}_Y(E) = E$

$\Rightarrow \mathrm{Cl}_X(f^{-1}(E)) \subseteq f^{-1}(E)$

$\Rightarrow f^{-1}(E)$ is closed.

<u>5</u>. ($\Rightarrow$) Holds by part <u>1</u>.

($\Leftarrow$) Suppose A & B are open and $f|_A$ & $f|_B$ are cont.

<u>To show</u>: f is cont. $\Leftrightarrow$ $\forall$ open set $V \subseteq Y$, $f^{-1}(V)$ is open in X.

$f|_A$ & $f|_B$ are cont. $\Rightarrow f^{-1}(V) \cap A$ & $f^{-1}(V) \cap B$ are open.

$\Rightarrow (f^{-1}(V) \cap A) \cup (f^{-1}(V) \cap B)$ is open

$\Rightarrow f^{-1}(V) \cap \underbrace{(A \cup B)}_{X} = f^{-1}(V)$ is open.

Metrizable space:

Let $(X, \mathcal{J})$ be a topo. sp. We say $(X, \mathcal{J})$ is metrizable if $\exists$ a metric $d: X \times X \to \mathbb{R}$ s.t the topology on (X, d) is the same as $\mathcal{J}$.

eg: 1. On $\mathbb{R}$, $\{(a,b) : a, b \in \mathbb{R}\}$ generates a topo. on $\mathbb{R}$ as a base.

This is metrizable with the Euclidean metric on $\mathbb{R}$.

2. $(X, \mathcal{J}_d)$ ← discrete topo.

$= \mathcal{P}(X)$

If we take discrete metric, it gives the same topo.

3. $(X, \{\emptyset, X\})$

Is this metrizable?

Suppose $\exists\, x, y \in X$ s.t $d(x,y) > 0$

Then $B_x(d(x,y)/2)$ is an open set not containing y.

$$\left.\begin{array}{l} \because\ x \in B_x(d(x,y)/2) \Rightarrow B_x(d(x,y)/2) \neq \emptyset \\ y \notin B_x(d(x,y)/2) \Rightarrow B_x(d(x,y)/2) \neq X \end{array}\right\} \rightarrow \text{Contd}^n$$

Hence, X is metrizable only if it contains only one element.

4. (X, d_1) & (Y, d_2) are metric sp.

Is $X \sqcup Y$ metrizable? Is $X \times Y$ metrizable?

$$d(a,b) = \begin{cases} d_1(a,b), & \text{if } a,b \in X \\ d_2(a,b), & \text{if } a,b \in Y \\ 1, & \text{if } a \in X, b \in Y \\ & \text{or } b \in X, a \in Y \end{cases}$$

Does not satisfy triangle inequality as

for $a, c \in X$, $b \in Y$, $\quad d(a,c) \leq \underbrace{d(a,b)}_{1} + \underbrace{d(b,c)}_{1}$

violated when $d(a,c) > 2$

Instead consider

$$d(a,b) = \begin{cases} \min\{1, d_1(a,b)\}, & \text{if } a,b \in X \\ \min[1, d_2(a,b)], & \text{if } a,b \in Y \\ 1, & \text{if } a \in X, b \in Y \\ & \text{or } b \in X, a \in Y \end{cases}$$

Check whether $X \overset{i_1}{\hookrightarrow} X \sqcup Y$, $\quad i_1$ & i_2 are cont.

$Y \overset{i_2}{\hookrightarrow} X \sqcup Y$

Consider $B_d(a, \delta)$ for $a \in X$, $\delta \leq 1 \Rightarrow i_1^{-1}(B_d(a,\delta)) = B_{d_1}(a, \delta)$

for $a \in X$, $\delta > 1 \Rightarrow i_1^{-1}(B_d(a,\delta)) = X$

$B_d(a,\delta)$ for $a \in Y$, $\delta \leq 1 \Rightarrow i_1^{-1}(B_d(a,\delta)) = \emptyset$

for $a \in Y$, $\delta > 1 \Rightarrow i_1^{-1}(B_d(a,\delta)) = X$

So, i_1 & i_2 are cont.

5. $(a_1,b_1), (a_2,b_2) \in X \times Y$

$$d((a_1,b_1),(a_2,b_2)) = \left(d_1^2(a_1,a_2) + d_2^2(b_1,b_2)\right)^{1/2}$$

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

Consider $B_{d_1}(a,\delta)$ for $a \in X$, $\pi_1^{-1}(B_{d_1}(a,\delta)) = B_{d_1}(a,\delta) \times Y$ which is open

So, π_1 & π_2 are cont.

6.

$$\begin{tikzcd} & X \times Y \arrow[dl, "P_1"'] \arrow[dr, "P_2"] & \\ (X, \mathcal{J}) & & (Y, \mathcal{F}) \end{tikzcd}$$

Let $\mathcal{U}$ be a topo. on $X \times Y$ s.t P_1 & P_2 are cts.

Clearly $\mathcal{S} = \{ \underbrace{P_1^{-1}(U)}_{U \times Y}, \underbrace{P_2^{-1}(V)}_{X \times V} : U \in \mathcal{J}, V \in \mathcal{F} \} \subseteq \mathcal{U}$

$$(U \times Y) \cap (X \times V) = U \times V \notin \mathcal{S}$$

So, this set by itself is not a topo.

Neither is this a base ($\because$ not closed under finite intersections)

Let $\mathcal{B} = \{ \bigcap_{i=1}^{k} S_i : S_i \in \mathcal{S} \}$
$= \{ U \times V : U \in \mathcal{J}, V \in \mathcal{F} \}$

Clearly $\{ U \times V : U \in \mathcal{J}, V \in \mathcal{F} \} \subseteq \{ \bigcap_{i=1}^{k} S_i : S_i \in \mathcal{S} \}$

since $(U \times V) = (U \times Y) \cap (X \times V)$

Consider $\{S_k\}_{k=1}^{m+n} = \{ U_i \times Y \}_{i=1}^{m} \cup \{ X \times V_j \}_{j=1}^{n} = \{ U_k \times V_k \}_{k=1}^{m+n}$

where $V_k = Y \;\; \forall \; 1 \leq k \leq m$ & $U_k = X \;\; \forall \; m+1 \leq k \leq m+n$

$$\bigcap_{k=1}^{m+n} S_k = \bigcap_{k=1}^{m+n} (U_k \times V_k) = \left(\bigcap_{k=1}^{m+n} U_k \right) \times \left(\bigcap_{k=1}^{m+n} V_k \right) = U \times V \text{ for } U \in \mathcal{J}, V \in \mathcal{F}$$

So, $\{\bigcap_{i=1}^{k} S_i : S_i \in \mathcal{S}\} \subseteq \{U \times V : U \in \mathcal{T}, V \in \mathcal{F}\}$

Hence, the smallest topo. on $X \times Y$ s.t P_1 & P_2 are cts.
is the topo. generated by $\mathcal{B}$

7.

$$\begin{array}{ccc} X & \xrightarrow{i_1} & X \sqcup Y \\ Y & \xrightarrow{i_2} & \nearrow \end{array}$$

Find largest topo. on $X \sqcup Y$ s.t i_1 & i_2 are cts.
Every elem. of $X \sqcup Y$ is of the form $U \sqcup V$
where $U \subseteq X$, $V \subseteq Y$.

Consider $\mathcal{U} = \{U \sqcup V : U \in \mathcal{T}, V \in \mathcal{F}\}$
Then $i_1^{-1}(U \sqcup V) = U$
$i_2^{-1}(U \sqcup V) = V$

Clearly $\emptyset, X \sqcup Y \in \mathcal{U}$
Let $U_1 \sqcup V_1$, $U_2 \sqcup V_2 \in \mathcal{U}$
$\Rightarrow (U_1 \sqcup V_1) \cap (U_2 \sqcup V_2) = \underbrace{(U_1 \cap U_2)}_{\in \mathcal{T}} \sqcup \underbrace{(V_1 \cap V_2)}_{\in \mathcal{F}} \in \mathcal{U}$

Let $\{U_i \sqcup V_i\}_{i\in I} \subseteq \mathcal{U}$

$$\Rightarrow \bigcup_{i\in I} U_i \sqcup V_i = \underbrace{\left(\bigcup_{i\in I} U_i\right)}_{\in \mathcal{I}} \sqcup \underbrace{\left(\bigcup_{i\in I} V_i\right)}_{\in \mathcal{J}} \in \mathcal{U}$$

Hence, $\mathcal{U}$ is a topo. on $X \sqcup Y$ & the largest one s.t i_1 & i_2 are cts.

Ex: X is a discrete sp. Is $X \sqcup X$ discrete?

Is $X \times X$ discrete? (with topo. we described above)

A. Yes

Ex: Let X, Y be discrete sp. Show that $X \times Y$ with base $\{U \times V : U \text{ open in } X,\ V \text{ open in } Y\}$ gives rise to discrete topo.

Ex: $f: X \to Y$ is cts. iff $f^{-1}(S)$ is open in X
$\forall S$ in a sub-base of Y.

Pf: ($\Rightarrow$) Trivial (elems. of sub-base are also elems. of base)

($\Leftarrow$) Suppose $f^{-1}(S_i)$ is open $\forall S_i \in \mathcal{S}$

$$\because \quad \underbrace{\bigcap_{i=1}^{n} f^{-1}(S_i)}_{\text{open}} \;\Rightarrow\; \underbrace{= f^{-1}\left(\bigcap_{i=1}^{n} S_i\right)}_{\text{open}}$$

Since finite intersections of S_i form a base, so
f is cts.

Let X be a set & $\sim$ be an equivalence relation

$X/\sim$: set of all equivalence classes under $\sim$

$$X \xrightarrow{q} X/\sim$$
$$a \mapsto [a]$$

<u>Quotient topology</u> on $X/\sim$ is defined as
$V \subseteq X/\sim$ is open if $q^{-1}(V)$ is open in X.

Let X, Y be topo. sp.
Let $q : X \to Y$ be an onto fn. Then Y is said to have quotient topo. wrt X if $V \subseteq Y$ is open iff $q^{-1}(V)$ is open & q is said to be a quotient map.

Given,

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[dr, "g"'] & Y \arrow[d, dotted, "f"] \\ & Z \end{tikzcd}$$

$f \circ q = g$ & q is a qt. map., then f is cts. iff g is cts.

Pf: ($\Rightarrow$) Trivial since composition of cts. fn's is cts.

($\Leftarrow$) Consider open $W \subseteq Z$.

$\because$ g is cts., $U = g^{-1}(W) \subseteq X$ is open

$= (f \circ q)^{-1}(W)$

$= q^{-1}(f^{-1}(W))$

$= q^{-1}(V)$ is open for $V = f^{-1}(W)$

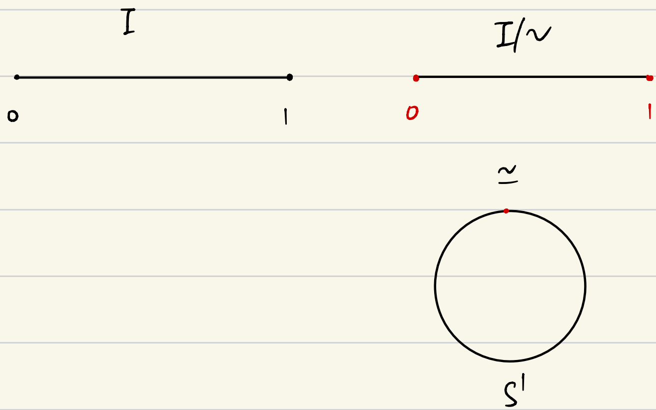
Now, $q^{-1}(V)$ is open iff $V \subseteq Y$ is open ($\because$ q is qt. map.)

$\Rightarrow f^{-1}(W) \subseteq Y$ is open

Hence, f is cts.

eg: Consider $I=[0,1]$ with eq. rel^n $\sim$ s.t

$0\sim 1$, $1\sim 0$ & $t\sim t$ $\forall\, t\in[0,1]$



$$S^n = \{ x\in\mathbb{R}^{n+1} : \|x\|_2 = 1 \}$$

$$D^{n+1} = \{ x\in\mathbb{R}^{n+1} : \|x\|_2 \le 1 \}$$

$$I \xrightarrow{g} S^1$$

$$t \longmapsto (\cos(2\pi t), \sin(2\pi t))$$

$$\begin{tikzcd} I \arrow[r, "g"] \arrow[dr, "q"'] & S^1 \\ & I/\sim \arrow[u, dashed, "f"'] \end{tikzcd}$$

$$f([t]) = g(t)$$

In a metric space, we can always separate pts. using open sets.
That is, if (X,d) is a metric space & $a \neq b$, then $\exists$ open sets $U, V \subseteq X$ s.t. $U \cap V = \emptyset$ & $a \in U$, $b \in V$.

Such a space is said to be <u>Hausdorff</u>.

<u>Pf</u>: Let $a, b \in X$, $a \neq b \Rightarrow \delta = d(a,b) \neq 0$

Let $U = B(a, \delta/3)$, $V = B(b, \delta/3)$. Then $a \in U$, $b \in V$.

For $z \in U \cap V$, $d(z,a) < \delta/3$ & $d(z,b) < \delta/3$

$$\Rightarrow \underbrace{d(a,b)}_{\delta} \leq d(z,a) + d(z,b) < 2\delta/3$$

$\Rightarrow \delta < 2\delta/3 \rightarrow$ Contdn

So, $U \cap V = \emptyset$

Hence, (X,d) is Hausdorff.

Are all topological spaces Hausdorff ?

No.

Consider $X = \{a, b\}$

$\mathcal{T} = \{\emptyset, X, \{a\}\}$ is not Hausdorff

Is $(\mathbb{R}$, cofinite$)$ Hausdorff ?

For $0 \neq 1$, $0 \in U$, $1 \in V$, $U \cap V = \emptyset$

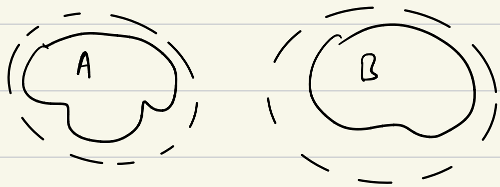
$$\Rightarrow \underbrace{U^c}_{\text{finite}} \cup \underbrace{V^c}_{\text{finite}} = \mathbb{R} \rightarrow \text{Contd}^n$$

In a metric space, we can always separate disjoint closed sets using open sets. That is, if (X, d) is a metric sp. & closed $A, B \subseteq X$, $A \cap B = \emptyset$, then $\exists$ open sets $U, V \subseteq X$ s.t $U \cap V = \emptyset$, $A \subseteq U$, $B \subseteq V$.

Such a space is said to be normal.

Pf : Let (X,d) be a metric sp. & $A, B \subseteq X$ be closed, $A \cap B = \emptyset$

Then $d(x, A) = 0 \iff x \in \bar{A}$



Define $\underset{a \in A}{\delta_a = d(a, B) > 0}$, $\underset{b \in B}{\varepsilon_b = d(b, A) > 0}$

Let $U = \bigcup_{a \in A} B(a, \delta_a)$, $V = \bigcup_{b \in B} B(b, \varepsilon_b)$

Then $A \subseteq U$, $B \subseteq V$

For $z \in U \cap V$, $z \in B(a, \frac{\delta_a}{3})$, $z \in B(b, \frac{\varepsilon_b}{3})$ for some $a \in A$, $b \in B$

$\Rightarrow d(z, a) < \frac{\delta_a}{3}$ $\Rightarrow d(z, b) < \frac{\varepsilon_b}{3}$

$$d(a,b) \leq d(z,a) + d(z,b) < \frac{\delta_a}{3} + \frac{\varepsilon_b}{3} \leq \frac{d(a,b)}{3} + \frac{d(a,b)}{3} = \frac{2}{3} d(a,b)$$

$\Rightarrow \delta < 2\delta/3 \rightarrow$ Contd^n

Let X be a topo sp.

A subset $A \subseteq X$ is said to be <u>disconnected</u> if it can be written as a distinct union of open sets in A.

That is, $A = (U \cap A) \cup (V \cap A)$ s.t $(U \cap A) \cap (V \cap A) = \emptyset$,
U, V open in X.

A set is said to be <u>connected</u> if it is <u>not</u> disconn.

<u>Note</u>: In the defn, we can replace open sets with closed sets.

$\mathbb{R}_\ell$ i.e $\mathbb{R}$ with base $\{[x,z) : x, z \in \mathbb{R}\}$ is <u>not</u> connected

$$\mathbb{R} = (-\infty, n) \cup [n, \infty)$$

$$\bigcup_{k \in \mathbb{Z}_{>0}} [-k, n) \qquad \bigcup_{\substack{k > n \\ k \in \mathbb{Z}_{>0}}} [n, k)$$

open open

Moreover, they're non-empty.

<u>Ex</u>: Show $[0,1]$ is connected

<u>Pf</u>: Suppose $[0,1]$ is disconnected. Then $\exists$ disjoint H, K open (in $[0,1]$) non-empty sets s.t $[0,1] = H \cup K$

If $x \in H$, $\exists \epsilon > 0$ s.t $(x-\epsilon, x+\epsilon) \subseteq H$

Wlog let $0 \in H$.

Consider $R = \{ l : [0, l) \subseteq H \} \neq \emptyset$ ($\because 0 \in H$ & H is open)

If $\sup R = 1$, $K = \{1\}$ which is <u>not</u> open

Else, $\sup R = r \neq 1$.

If $\sup R \in H \Rightarrow [0, r] \subseteq H$

Since $r \in H$, $\exists \epsilon > 0$ s.t $(r-\epsilon, r+\epsilon) \subseteq H$

$$\Rightarrow [0, r+\epsilon) \subseteq H \Rightarrow r \neq \sup R$$

If $\sup R \in K \Rightarrow \exists \epsilon > 0$ s.t $(r-\epsilon, r+\epsilon) \subseteq K$

$\Rightarrow \exists\, r-\epsilon < r' < r'' < r$ s.t $r' \in K$ & $[0, r'') \subseteq H$

by defn of r. $\rightarrow$ Contdn as $H \cap K = \emptyset$

Ppn: If $f: X \to Y$ is cts. & X is conn. then $f(X)$ is conn.

Pf: (Contra+)

Let $f(X)$ be disconnected.

Then $f(X) = H \cup K$, $\quad H, K$ open in $f(X)$

$H \cap K = \emptyset$, $H \neq \emptyset$, $K \neq \emptyset$

Alt., $f(X) \subseteq H \cup K$, $\quad H, K$ open in Y

$H \cap K = \emptyset$, $H \cap f(X) \neq \emptyset$, $K \cap f(X) \neq \emptyset$

Since f is cts., $f^{-1}(H)$, $f^{-1}(K)$ are open in X.

Moreover, $f^{-1}(H) \neq \emptyset$ & $f^{-1}(K) \neq \emptyset$ as $H \cap f(X) \neq \emptyset$, $K \cap f(X) \neq \emptyset$

$$\underbrace{f^{-1}(f(X))}_{X} = f^{-1}(H) \cup f^{-1}(K)$$

Note, $f^{-1}(H) \cap f^{-1}(K) = \emptyset$ as $H \cap K = \emptyset$

$\Rightarrow$ X is disconnected

This proves the contra+.

<u>Ppn</u>: $X_\alpha \subseteq X$ are all conn. $\forall \alpha \in J$, $\bigcap X_\alpha \neq \emptyset \Rightarrow \bigcup_{\alpha \in J} X_\alpha$ is conn.

<u>Pf</u>: Suppose $\bigcup_{\alpha \in J} X_\alpha$ is disconn.

$\bigcup_{\alpha \in J} X_\alpha = A \cup B$, open A, B in X

$A \cap B = \emptyset$, $\bigcup_{\alpha \in J} X_\alpha \cap A \neq \emptyset$

$\bigcup_{\alpha \in J} X_\alpha \cap B \neq \emptyset$

X_α is conn. $\Rightarrow X_\alpha \subseteq A$ or $X_\alpha \subseteq B$

Consider $X_\alpha \subseteq A$ & $X_\beta \subseteq B$ $\rightarrow$ Contd^n since $\bigcap X_\alpha \neq \emptyset$

but $A \cap B = \emptyset$

$\Rightarrow \bigcup_{\alpha \in J} X_\alpha \subseteq A$ or $\bigcup_{\alpha \in J} X_\alpha \subseteq B$

<u>Rem</u>: The pp^n still holds if the sets merely pairwise intersection of X_α s were non-empty.

Hence, $\mathbb{R}$ is connected

Since $\mathbb{R} = \bigcup_{n \in \mathbb{Z}_{>0}} [-n, n]$ & $\bigcap_{n \in \mathbb{Z}_{>0}} [-n, n] \neq \emptyset$

Ex: Show that $\mathbb{R}^2$ is conn.

Find X_α s.t $\mathbb{R}^2 = \bigcup_{\alpha \in \Lambda} X_\alpha$ s.t X_α is conn. & $\bigcap_{\alpha \in \Lambda} X_\alpha \neq \emptyset$

A. Consider $X_k = \{(x, kx) : x \in \mathbb{R}\}$, $X_* = \{(0, y) : y \in \mathbb{R}\}$

Then each X_k is conn. ($\because \simeq \mathbb{R}$ conn.) & $\mathbb{R}^2 = X_* \cup \bigcup_{k \in \mathbb{R}} X_k$

s.t $X_* \cap \bigcap_{k \in \mathbb{R}} X_k = \{(0,0)\} \neq \emptyset$

Hence, $\mathbb{R}^2$ is conn.

Ppn: Let $E \subseteq X$ topo. sp. be conn. s.t $E \subseteq A \subseteq \bar{E}$, then A is conn.

Pf: Show that $\bar{E} = Cl_X(E)$ is conn.

Why is that enough?

Suppose $\bar{E} \subseteq H \cup K$ s.t H, K closed disjoint in X, $H \cap \bar{E} \neq \emptyset$, $K \cap \bar{E} \neq \emptyset$

$\because$ E is conn. & $E \subseteq \bar{E}$ $\Rightarrow$ $E \subseteq H$ or $E \subseteq K$

$\Rightarrow$ $\bar{E} \subseteq H$ or $\bar{E} \subseteq K$ ($\because$ H & K are closed)

$\rightarrow$ Contd^n

Now, $Cl_A E = A \cap Cl_X E = A$

Repeat this argument with ambient sp. being A instead of X.

<u>Ex</u>: Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{Q} \text{ or } x_2 \in \mathbb{Q}\}$.

Then prove or disprove that X is conn.

<u>Pf</u>: $X_\alpha = \{(\alpha, y) : y \in \mathbb{R}\} \cup \{(x, \alpha) : x \in \mathbb{R}\}$, $\alpha \in \mathbb{Q}$

Clearly, X_α is conn. $\forall \alpha \in \mathbb{Q}$

Moreover, $\bigcup_{\alpha \in \mathbb{Q}} X_\alpha = X$.

Show for any $\alpha, \beta \in \mathbb{Q}$, $X_\alpha \cap X_\beta \neq \emptyset$ & we're done.

<u>Ex</u>: Show $\mathbb{R}_\ell$ is normal

<u>Pf</u>: Suppose A, B are closed, $A \cap B = \emptyset$

For $x \in A$, $x \in \underset{\text{open}}{B^c}$ $\Rightarrow \exists\, z_x$ s.t $[x, z_x) \subseteq B^c$

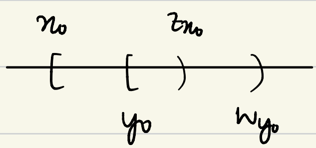
Sim. for $y \in B$, $\exists\, w_y$ s.t $[y, w_y) \subseteq A^c$

$$U = \bigcup_{x \in A} [x, z_x) \quad , \quad V = \bigcup_{y \in B} [y, w_y)$$

<u>Claim</u>: $U \cap V = \emptyset$

Suppose $\exists\, x_0 \in A$, $y_0 \in B$ s.t $[x_0, z_{x_0}) \cap [y_0, w_{y_0}) \neq \emptyset$

$\Rightarrow y_0 \in B$ & $y_0 \in B^c \rightarrow$ Contd^n



<u>Lem</u>: Let X be any space. Let S be a closed subsp. which is discrete.

Let D be a dense set, $\overline{D} = X$ & $|S| \geq \underbrace{|\mathcal{P}(D)|}_{2^{|D|}}$,

then X is <u>not</u> normal.

Recall we were talking about topological invariants (under homeomorphism)

(1) Separating distinct pts. by open sets – Hausdorff

Separating disjoint closed sets by disjoint open sets – Normal

Note, Normal $\not\Rightarrow$ Hausdorff

Normal + singleton closed sets $\Rightarrow$ Hausdorff

Consider $\mathbb{R}_0 = \mathbb{R}$ with base $\{(a,\infty) : a \in \mathbb{R}\}$

All open sets in this topology are of the form (a,∞)

& all closed set are of the form $(-\infty, b]$

Normal: Any two closed sets always intersect.

Hence, $\mathbb{R}_0$ is normal

Hausdorff: Any two open sets always intersect, no two points can be separated.

Hence, $\mathbb{R}_0$ is not Hausdorff.

If X is top. sp. s.t all singleton sets are closed in X, then X is said to be a T_1 space.

Ex: Show that X is T_1 if & only if $\forall\, a, b \in X$, $\exists$ open set U s.t $a \in U$ & $b \notin U$.

($\Leftarrow$) ($\Rightarrow$)

Pf: ($\Rightarrow$) $\because$ a is closed $\Rightarrow$ $X \setminus \{a\}$ is open s.t $a \notin X \setminus \{a\}$ & $b \in X \setminus \{a\}$.

Sim., $X \setminus \{b\}$ is an open set s.t $a \in X \setminus \{b\}$ & $b \notin X \setminus \{b\}$

($\Leftarrow$) For every $a \in X$ s.t $a \neq b$, consider open set U_a s.t $a \in U_a$ & $b \notin U_a$

Then $X \setminus \{b\} = \bigcup_{\substack{a \in X \\ a \neq b}} U_a$ $\rightarrow$ open $\Rightarrow$ b is closed.

Caution: Some textbooks define normal as (normal + T_1).

	Hausdorff	Normal
Subspace	True	False
Product sp.	True	False
Cts. image	False	False
Open image	False	False
Quotient sp.	False	False

Subsp. of Hausdorff : $A \subseteq X$

Let $x, y \in A$. Then $\exists$ disj. open $U, V \subseteq X$ s.t. $x \in U$ & $y \in V$

$$\Rightarrow x \in \underbrace{U \cap A}_{\text{open in } A} \ \& \ y \in \underbrace{V \cap A}_{\text{open in } A}$$

Hence, A is Hausdorff.

Subsp. of normal : $A \subseteq X$

Let D_1, D_2 disj. closed in A. $\Rightarrow$ $D_1 = L_1 \cap A$

$D_2 = L_2 \cap A$

L_1, L_2 closed in A.

However, L_1 & L_2 need not be disjoint.

Note, if A is closed in X, then X is normal $\Rightarrow$ A is normal

($\because$ D_1 & D_2 with be disj. closed in X.)

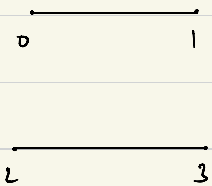
eg: H_M - Moore plane is not normal

$H_M \subseteq \prod_{i \in J} [0,1]$ is normal.

Product sp. of normal : $\mathbb{R}_l$ is normal

but $\mathbb{R}_l \times \mathbb{R}_l$ is not normal.

Quotient sp.:



$x \sim x+2$

except when

$x = \frac{1}{2}$

Since $\frac{1}{2} \not\sim \frac{5}{2}$ yet any open ball around $\frac{1}{2}$ contains pt. around $\frac{5}{2}$ (& vice verse).

Hence, <u>not</u> Hausdorff.

Note, since $\{1/2\}$ & $\{5/2\}$ are closed, the space isn't <u>normal</u> either.

Lem: Let X be any space. Let S be a closed subsp. which is discrete.
Let D be a dense set, $\bar{D} = X$ & $|S| \geq \underbrace{|P(D)|}_{2^{|D|}}$,
then X is not normal.

Pf: Suppose X is normal.

Let $T_1 \subseteq S$ $\leftarrow$ discrete in subsp. top.
s.t T_1 is closed in S.
$\Rightarrow T_1 = S \cap W$
closed subset of X
$\Rightarrow T_1$ is closed in X.

Now, T & $S - T$ are disjoint closed subsets of X.
X is normal $\Rightarrow \exists$ $U(T) \supseteq T$ & $V(T) \supseteq S - T$
both open s.t $U(T) \cap V(T) = \emptyset$

Let $T \subseteq S$, $U(T) \cap D \neq \emptyset$ because D is dense in X.

Consider $T_1, T_2 \subseteq S$ s.t $T_1 \neq T_2$. Wlog $T_1 - T_2 \neq \emptyset$
Then $U(T_1) \cap V(T_2) \neq \emptyset$ since $T_1 - T_2 \neq \emptyset$ & $S - T_2 \subseteq V(T_2)$
$\Rightarrow T_1 - T_2 \subseteq V(T_2)$
& since $T_1 \subseteq U(T_1) \Rightarrow U(T_1) \cap V(T_2) \neq \emptyset$

So, $\underbrace{U(T_1) \cap V(T_2)}_{\text{open}} \cap D \neq \emptyset$

$$\subseteq$$

$U(T_1) \cap D$

$U(T_2) \cap V(T_2) = \emptyset$

$\Rightarrow \underset{\neq \emptyset}{U(T_2) \cap D} \neq \underset{\neq \emptyset}{U(T_1) \cap D}$ $\quad$ ($\because$ $U(T_1) \cap V(T_2) \neq \emptyset$ whereas $U(T_2) \cap V(T_2) = \emptyset$)

$\therefore$ We have a fn^n $P(S) \to P(D)$ which is one-one

$T \mapsto U(T) \cap D$

$\Rightarrow$ $|P(S)| \leq |P(D)|$ $\to$ Contd^n

Thm: Arbitrary product of conn. sp. is conn.

Pf: Idea - Give a dense conn. subset of $\prod_{\alpha \in \Lambda} X_\alpha$,

Let $(a_\alpha) \in \prod X_\alpha$

$$E = \{ (x_\alpha) \in A \; : \; A \text{ is conn. } \& \; (a_\alpha) \in A \subsetneq \prod X_\alpha \}$$

$$E = \bigcup_{\substack{A \subseteq \prod X_\alpha \\ (a_\alpha) \in A \\ A \text{ is conn.}}} A$$ $\rightarrow$ conn. because (a_α) is their intersection.

Claim: E is dense in X.

Enough to show that E intersects every open set in base.

ie E intersects $\pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \pi_{\alpha_2}^{-1}(U_{\alpha_2}) \dots \cap \pi_{\alpha_k}^{-1}(U_{\alpha_k}) \neq \emptyset$

for all α_i & U_{α_i} open in X_{α_i}

Fix $(b_\alpha) \in$

$$L_1 = \{ (x_\alpha) \in \prod X_\alpha : x_{\alpha_i} \in X_{\alpha_i}, \; x_\alpha = a_\alpha \; \forall \alpha \neq \alpha_i \}$$

$\simeq X_{\alpha_1}$ is conn.

$$L_2 = \{ (x_\alpha) \in \prod X_\alpha : x_{\alpha_1} = b_{\alpha_1}, \; x_{\alpha_2} \in X_{\alpha_2}, \; x_\alpha = a_\alpha \; \forall \alpha \neq \alpha_1, \alpha_2 \}$$

$\simeq X_{\alpha_2}$ is conn.

$\vdots$

$$L_k = \{ (x_\alpha) \in \prod X_\alpha : x_{\alpha_i} = b_{\alpha_i}, \; i = 1, \dots, k-1, \; x_{\alpha_k} \in X_{\alpha_k}, \; x_\alpha = a_\alpha \; \forall \alpha \neq \alpha_1, \dots, \alpha_k \}$$

$\simeq X_{\alpha_k}$ is conn.

Thm: If $f : X \to Y$ is a homeomorphism, then $f|_{X-\{a\}} : X \setminus \{a\} \to Y \setminus \{f(a)\}$ is a homeomp.

Pf: Let $g = f|_{X-\{a\}}$

Well-defined: Let $x \in X \setminus \{a\}$. Then $x \neq a$

Since f is injective, $f(x) \neq f(a)$

So, $f(x) \in Y \setminus \{f(a)\}$

Thus, $g(x) \in Y \setminus \{f(a)\}$, so g is well-defined

Bijectivity of g follows from bijectivity of f.

Continuity: Let $U \subseteq Y \setminus \{f(a)\}$ be open

By def^n of subsp. topo., $\exists$ open $V \subseteq Y$ s.t $U = V \cap (Y \setminus \{f(a)\})$

$$
\begin{aligned}
g^{-1}(U) &= \{x \in X \setminus \{a\} : f(x) \in U\} \\
&= \{x \in X \setminus \{a\} : f(x) \in V\} \\
&= f^{-1}(V) \cap (X \setminus \{a\})
\end{aligned}
$$

which is open in $X \setminus \{a\}$

Similarly we can show g^{-1} is cts.

$\underline{\text{Ppn}}$: X is conn. iff the only non-empty subset of X which is both open & closed is X.

$\underline{\text{Pf}}$: $(\Rightarrow)$ Suppose $\emptyset \neq A \subset X$.

Then $X = \underbrace{A}_{\text{open}} \cup \underbrace{A^c}_{\text{open}}$ & both are non-empty

($\because A$ is also closed)

$\Rightarrow X$ is disconn. $\rightarrow$ Contd^n

$(\Leftarrow)$ Suppose X is disconn.

Then $X = A \cup B$ where A, B non-empty open disj.

$\Rightarrow B = A^c \rightarrow$ closed ($\because A$ is open)

$\Rightarrow \emptyset \neq B \subset X$ s.t B is both open & closed $\rightarrow$ Contd^n

Def^n: A connected component in X is a maximal connected subset.

Pp^n: There will always exist a maximal conn. subset.

Pf: $\emptyset \neq \{C : \text{non-empty conn. subset } C \subseteq X\} = \mathcal{A}$

Take a totally ordered subset J of $\mathcal{A}$.

If $C_1, C_2 \in J \Rightarrow C_1 \subseteq C_2$ or $C_2 \subseteq C_1$

Let $K = \bigcup_{C \in J} C \rightarrow$ conn. & pairwise intersecting

$\Rightarrow K$ is conn. & $C \subseteq K \ \forall \ C \in J$

$\Rightarrow$ Zorn's lemma can be applied & $\mathcal{A}$ has maximal element.

eg: $\mathbb{R} \setminus \{5\} = \underbrace{(-\infty, 5)}_{\text{conn. comp.}} \cup \underbrace{(5, \infty)}_{\text{conn. comp.}}$

$\mathbb{R} \setminus \{(0,0)\}$ has only one conn. comp., namely the entire space

Ppⁿ: Every conn. comp. of X is both open & closed in X.

Ppⁿ: Every connected comp. is closed.

Pf: We know C is conn. $\Rightarrow \bar{C}$ is conn.
If C is the maximal conn. subset of X, then $\bar{C} \subseteq C$
$\Rightarrow C = \bar{C}$
$\Rightarrow C$ is closed

Ex: $\mathbb{R} \setminus \mathbb{Q}$ is not conn.
Write it as a union of its conn. comps.

A. $\mathbb{R} \setminus \mathbb{Q} = (-\infty, 0) \cap \mathbb{R} \setminus \mathbb{Q} \cup (0, \infty) \cap \mathbb{R} \setminus \mathbb{Q}$ $\rightarrow$ hence disconn.

Claim: If $x_1, x_2 \in L \subseteq \mathbb{R} \setminus \mathbb{Q}$, then L is not conn.

Ex: Show that $\mathbb{R} \setminus \mathbb{Q}$ as a subsp. does not have discrete topology.

Path connectedness

A path is a cts. fn^n from $[0,1] \to X$.

We say X is path conn. if $\forall a,b \in X$, $\exists \gamma : [0,1] \to X$

s.t $\gamma(0) = a$, $\gamma(1) = b$

<u>Pp^n</u>: If X is path conn., then X is conn.

<u>Pf</u>: Let X be disconn.

Then $X = A \cup B$, A, B non-empty disjoint open

Consider $a \in A$, $b \in B$. Since X is path conn.,

$\exists$ path $\gamma : I \to X$, $\gamma(0) = a$ & $\gamma(1) = b$.

$\because$ I is conn. & γ is cts. $\Rightarrow$ $\gamma(I)$ is conn. $\to$ Contd^n

$\because \gamma(I) \cap A \neq \emptyset$

& $\gamma(I) \cap B \neq \emptyset$

Note: Connected $\not\Rightarrow$ Path-conn.

Topologist's sine curve

$$(-\infty, 0] \times \{0\} \cup \left\{ \left(x, \sin\left(\tfrac{1}{x}\right) \right) : x \in \mathbb{R}_{>0} \right\} \subseteq \mathbb{R}^2$$

Locally connected: A sp. X is locally conn. if $\forall a \in X$ & open set $a \in U$, $\exists$ an open conn. set W s.t $a \in W \subseteq U$.

eg: $X = (1,2) \cup (3,4)$

Clearly, X is disconn.

Claim: X is locally conn.

Let $a \in X$ & U open in X s.t $a \in U$.

$\exists \delta > 0$ s.t $a \in (a-\delta, a+\delta) \cap X \subseteq U$

Choose δ small enough s.t $a \in \underbrace{(a-\delta, a+\delta)}_{\text{conn.}} \subseteq U$

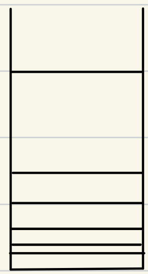
$\Rightarrow X$ is locally conn.

<u>Note</u>: Locally conn. $\not\Rightarrow$ conn.

Conn. $\not\Rightarrow$ Locally conn.

Consider $S = [0,1]\times\{0\} \cup \bigcup_{n\in\mathbb{Z}_{>0}} \left\{ \left(x, \frac{1}{n}\right) : x \in [0,1] \right\} \cup \{0,1\}\times[0,1]$

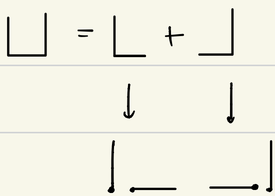
$$S = \bigcup_{n\in\mathbb{Z}_{>0}} (B \cup L_n)$$



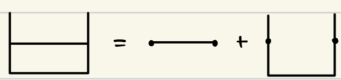
where $L_n = \left\{\left(x,\frac{1}{n}\right) : x\in[0,1]\right\}$

& $B = \{0,1\}\times[0,1] \cup [0,1]\times\{0\}$

$\uparrow$ conn. since



$B \cup L_n$ is conn. since



& pairwise intersecting

Hence, S is conn.

For any pt. <u>not</u> on $[0,1]\times\{0\}$, it is possible to find a conn. neighbourhood.

However for any $(x,y) \in [0,1] \times \{0\}$ say $(\frac{1}{2},0)$,

any nbd. $B((\frac{1}{2},0),\delta) \cap X = \bigcup_{\substack{n \in \mathbb{Z}_{>0} \\ n \geq N}} \{(a_n, b_n) \times \{\frac{1}{n}\}\} \cup (\frac{1}{2}-\delta, \frac{1}{2}+\delta) \times \{0\}$

So, S is not locally conn.

Def^n: A sp. X is called locally path conn. if $\forall\, a \in X$ & $a \in U$ open $\exists$ an open path conn. set W s.t $a \in W \subseteq U$.

Note: The above example also illustrates
Path conn. $\not\Rightarrow$ Locally path conn.

In general, Locally path conn. $\not\Rightarrow$ Path conn.

Pp^n: Conn. & loc. path conn. $\Rightarrow$ Path conn.

Pf: Consider $a \in X$, let $E = \{ b \in X : \exists$ a path from a to $b \}$

E is open

E is locally path conn. $\Rightarrow$ $b \in E$, $\forall$ open sets $U \ni b$ $\exists$ open path conn. $b \in W \subseteq U$

Let $c \in W$.

$\exists$ path γ_2 s.t $\gamma_2(0) = b$, $\gamma_2(1) = c$ & $b \in E$

$\Rightarrow$ $\exists$ a path γ_1 s.t $\gamma_1(0) = a$, $\gamma_1(1) = b$

$\Rightarrow$ $\exists$ path $\gamma_3 : [0,1] \to X$

$\gamma_3[0] = a$, $\gamma_3(1) = c$

$\Rightarrow$ $c \in E$ $\Rightarrow$ $b \in W \subseteq E$

$\Rightarrow$ E is open

<u>E is closed</u>

Consider $\bar{E}$. For $c \in \bar{E}$, let U be an open path conn. subset containing c. Then $U \cap E \neq \emptyset$.

For $b \in U \cap E$, c is path conn. to b ($\because$ $b, c \in U$) which is in turn path conn. to a ($\because$ $b \in E$)

Hence, $c \in E$, so $\bar{E} \subseteq E$ $\Rightarrow$ E is closed

Therefore, $E = X$

Ex: $\gamma_1 : [0,1] \to X$ is a path from a to b

$\gamma_2 : [0,1] \to X$ is a path from b to c

1) Give a path from $c \to b$

2) Give a path from $a \to c$

A. 1) $\gamma(t) = \gamma_2(1-t)$

$\gamma(0) = \gamma_2(1) = c$, $\gamma(1) = \gamma_2(0) = b$

$\gamma(t) = \gamma_2 \circ f(t)$ where $f : [0,1] \to [0,1]$, $f(t) = 1-t$ is cts.

Composition of cts. fx^n's $\Rightarrow$ Cts.

2) $$\gamma(t) = \begin{cases} \gamma_1(2t) & , t \in [0, 1/2] \\ \gamma_2(2t-1) & , t \in [1/2, 1] \end{cases}$$

$\gamma(0) = \gamma_1(2 \cdot 0) = \gamma_1(0) = a$

$\gamma(1) = \gamma_2(2 \cdot 1 - 1) = \gamma_2(1) = c$

Since $\underset{(=\gamma_1(2t))}{\gamma}$ is cts. on $[0, \frac{1}{2}]$ & $\underset{(=\gamma_2(2t-1))}{\gamma}$ is cts. on $[\frac{1}{2}, 1]$

& γ agrees on $[0, \frac{1}{2}] \cap [\frac{1}{2}, 1] = \{\frac{1}{2}\}$ & both intervals are closed, so γ is cts. on $[0,1]$.

Date: 18/03/2026

Ppn: If X is open, conn. & locally path conn.

$\Rightarrow X$ is path-conn.

Cor: Every open conn. subset $(\mathbb{R}^n, \|.\|_2)$ is path-conn.

Why?

Any open subset of a Euclidean sp. is locally path conn!

Because $\forall\, a \in A$, $\exists\, \underbrace{B(a, \epsilon)}_{\text{path-conn.}} \subseteq A$

Since any pt. $b \in B(a, \epsilon)$ is conn. to 0

by the straight line path $\gamma(t) = (1-t)b$, $t \in [0,1]$

Ppn.

1) Loc. conn. $\xrightarrow[\text{img.}]{\text{cts.}}$ Not necessarily loc. conn.

2) Path conn. $\xrightarrow[\text{img.}]{\text{cts.}}$ Path conn.

3) Conn. $\xrightarrow[\text{img.}]{\text{cts.}}$ Conn

Pf: 2) Let $f: V \to W$ f - cts., V - path conn.

For any $f(a), f(b) \in W$. Consider $a, b \in V$

$\because V$ is path conn., $\exists\, \gamma$ conn. a & b

ie $\gamma(0) = a$, $\gamma(1) = b$

Then $f \circ \gamma$ is a path conn $f(a)$ & $f(b)$

since $f \circ \gamma(0) = f(a)$ & $f \circ \gamma(1) = f(b)$ Date:

& $f \circ \gamma$ is cts. (composition of cts. fx^n s)

1) Give counterexample

3) Already proved.

Date: ______________

Quotient maps are cts., open & surjective

Pp^n: 1) Conn. $\xrightarrow{qt}$ Conn.

2) Path conn. $\xrightarrow{qt}$ Path conn.

3) Locally conn. $\xrightarrow{qt}$ Loc. conn.

Pf. 3) Let Y be a quotient of X

$q : X \to Y$, q is quotient map.

Let $W \subseteq Y$ be open in Y

$W = \bigcup_{i \in J} C_i$ ← conn. components

(To show: C_i is open
$y \in C_i \Rightarrow y = f(x), x \in X$)

W is open $\Leftrightarrow f^{-1}(W)$ is open

For any $x \in f^{-1}(W)$, $\exists$ a conn. comp. $C_x \subseteq f^{-1}(W)$
(open)

(since conn. comp of an open set in a loc. conn. set is open)

Now, $f(x) \in C_x \subseteq f(f^{-1}(W)) = W$

Ex: $f^{-1}(f(C_x)) \cap f^{-1}(W) = C_x$ Then we're done

Rem: If all path components of X are open, then X is locally conn.

Date: ______

Def^n: A topo. sp. X is said to be compact if given a coll. of open sets $\mathcal{U} = \{U_i : i \in J\}$ s.t $\bigcup_{i \in J} U_i = X$ i.e given an open covering of X, $\exists$ a finite subset of $\mathcal{U}$, $U_1, U_2, \dots, U_n$ s.t $X = \bigcup_{i=1}^{n} U_i$ (i.e it has a finite subcover)

eg: $\mathbb{R}$ is <u>not</u> compact

Consider the open cover $\mathcal{U} = \{(-n, n) : n \in \mathbb{Z}_{>0}\}$
This has no finite subcover.

Q7. Let $\mathcal{T}, \mathcal{U}$ be topologies on X s.t $\mathcal{T} \subseteq \mathcal{U}$
If X is compact with $\mathcal{T}$, it is compact with $\mathcal{U}$? What about other way around?

A. <u>No</u>. Consider $\mathbb{R}$ with $\mathcal{T} = \{\emptyset, \mathbb{R}\}$ indiscrete
$(\mathbb{R}, \mathcal{T})$ is compact, but $(\mathbb{R}, |\cdot|)$ is <u>not</u> compact.

Other way around is true.
Consider an open cover of X in $\mathcal{T}$.
It is also an open cover of X in $\mathcal{U}$.
Since $(X, \mathcal{U})$ is compact, $\exists$ finite subcover, which is also a finite subcover in $\mathcal{T}$.
Hence $(X, \mathcal{T})$ is compact.

Date: ____

Def^n: A topo. sp. X is said to be compact if given a coll. of open sets $\mathcal{U} = \{U_i : i \in J\}$ s.t $\bigcup_{i \in J} U_i = X$ i.e given an open covering of X, $\exists$ a finite subset of $\mathcal{U}$, $U_1, U_2, \dots, U_n$ s.t $X = \bigcup_{i=1}^{n} U_i$ (i.e it has a finite subcover)

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Date: 20/03/2026

Q8. Is $\mathbb{R}_\ell$ compact? Why or why not?
Identify compact subsets of $\mathbb{R}_\ell$.

A. Since $\mathbb{R}_\ell$ contains the std. topology

By prev. ex. $\underbrace{(X,\mathcal{U})}_{\mathbb{R}_\ell}$ compact $\Rightarrow \underbrace{(X,\mathcal{T})}_{(\mathbb{R},|\cdot|)}$ compact

So, by contrapositive, $(\mathbb{R},|\cdot|)$ not compact

$\Rightarrow \mathbb{R}_\ell$ is not compact □

Show $I=[0,1]$ is compact

Let $\mathcal{U}$ be an open cover.

Let $K=\{c\in I : [0,c]$ is covered by a finite subcover of $\mathcal{U}\}$

$K\neq\emptyset$: since $\mathcal{U}$ covers I $\Rightarrow \exists$ open set $V\in\mathcal{U}$ s.t. $0\in V$

$\exists\delta>0$ s.t. $[0,\delta/2]\subseteq[0,\delta)\subseteq V$

K is open: Consider $l\in K$ $(l\neq 1)$ $\Rightarrow [0,l]\subseteq\bigcup_{i=1}^{n}V_i$, $V_i\in\mathcal{U}$

$\Rightarrow l\in V_i$ for some i

$\Rightarrow (l-\delta, l+\delta)\subseteq V_i$ for some i (assume $l+\delta<1$)

$\Rightarrow [l-\frac{\delta}{2}, l+\frac{\delta}{2}]\subseteq V_i$

$\Rightarrow [0, l+\frac{\delta}{2}]\subseteq\bigcup_{i=1}^{n}V_i \Rightarrow (l-\frac{\delta}{2}, l+\frac{\delta}{2})\subseteq K$

($\because \Rightarrow \forall x\in(l-\frac{\delta}{2}, l+\frac{\delta}{2})$, $[0,x]\subseteq[0,l+\delta/2]\subseteq\bigcup_{i=1}^{n}V_i$)

K is an interval

Date:

K is closed :

Let $\sup K = s$

Claim : $K = [0, s]$

Clearly, $[0, s) \subseteq K$ $\Big(\because s$ is sup so for any $b < s$,
$\exists\, b < c \leq s$ s.t $c \in K \Rightarrow b \in K$
$\Rightarrow [0, b] \subseteq [0, c] \subseteq I \Big)$

Since $s \in I$, $\exists\, W \in \mathcal{U}$ s.t $s \in W$. open
$\Rightarrow \exists\, \epsilon > 0$ s.t $(s - \epsilon, s + \epsilon) \cap I \subseteq W$

Now, $s = \sup K \Rightarrow \exists\, r \in K$ s.t $s - \epsilon < r \leq s$
$\Rightarrow \exists\, V_1, V_2, \ldots, V_n \subseteq \mathcal{U}$ s.t $[0, r] \subseteq \bigcup_{i=1}^{n} V_i$
$\Rightarrow [0, s] \subseteq [0, r] \cup (r, s + \epsilon) \subseteq \bigcup_{i=1}^{n} V_i \cup W$
$\Rightarrow s \in K$.
$\Rightarrow K$ is closed.

Since K is both open & closed (non-empty)
T'fore $K = [0, 1]$.

(Closed, Bijective, Cts. f^{xn} = Homeomorphism)

Date:

Ppn:

1. Cts. img of a compact set is compact

2. Every closed subset of a compact set is compact

3. Every compact subset of a Hausdorff space is closed.

4. Any cts f^{xn} from a compact space to a Hausdorff sp is closed.

Pf: 1) Let $f: X \to Y$ s.t X compact. & f is cts. onto

Consider open cover $\mathcal{U} = \{U_i\}_{i \in J}$ of Y $\Rightarrow Y = \bigcup_{i \in J} U_i$

Then, $\mathcal{V} = \{f^{-1}(U_i)\}_{i \in J}$ is an open cover of X

Since X is compact, $\exists$ finite subcover $\{f^{-1}(U_{i_j})\}_{j=1}^{n}$ of X

$\Rightarrow \{U_{i_j}\}_{j=1}^{n}$ is a finite subcover of Y ($\because f$ is onto)

2) Let $A \subseteq X$, A closed, X compact.

Let $A \subseteq \bigcup_{\alpha \in A} U_\alpha$ be an open cover.

$X = A \cup X \setminus A \subseteq \bigcup_{\alpha \in A} U_\alpha \cup (X \setminus A)$ is an open cover.

X is compact $\Rightarrow \exists$ finite subcover $\Rightarrow A \subseteq \bigcup_{i=1}^{n} W_i$

If $\exists$ some k s.t $W_k = X \setminus A$, then $\{W_i\}_{i=1}^{n} - \{W_k\}$ is a finite subcover of $\{U_\alpha\}_{\alpha \in A}$ which covers A ($\because A \cap X - A = \emptyset$)

Date: ______

3) Let $A \subseteq X$, A compact ; X Hausdorff.

Consider $b \in X \setminus A$.

To show : $\exists$ an open set U s.t $b \in U \subseteq X \setminus A$

For every $a \in A$, $a \neq b \Rightarrow \exists V_a$ open s.t $a \in V_a \subseteq X$

& $b \in U_a \subseteq X$ s.t $V_a \cap U_a = \emptyset$

$\because$ $A \subseteq \bigcup_{a \in A} V_a$ & A is compact

$\Rightarrow \exists$ finite subcover $A \subseteq \bigcup_{i=1}^{n} V_{a_i}$

Clearly, $b \in \bigcap_{i=1}^{n} U_{a_i} = U_b$ open

Moreover $U_b \cap \bigcup_{i=1}^{n} V_{a_i} = \emptyset$ ($\because$ For any $x \in \bigcup_{i=1}^{n} V_{a_i}$

$\Rightarrow \exists k$ s.t $x \in V_{a_k}$

If $x \in U_b \Rightarrow x \in U_{a_k} \to x \in \underbrace{U_{a_k} \cap V_{a_k}}_{=\emptyset}$

So $U_b \cap \bigcup_{i=1}^{n} V_{a_i} = \emptyset$ $\to$ Contd^n)

So, $U_b \subseteq X \setminus A$

$\Rightarrow X - A$ is open

$\Rightarrow A$ is closed

Date: 25/03/2026

4) Let $f: X \to Y$, f cts., X compact, Y Hausdorff

Consider $A \subseteq X$, A closed.
By 2), A is compact
By 1), $f(A)$ is compact
By 3), $f(A)$ is closed.

$\square$

<u>Tychnoff's Thm</u>: Arbitrary product of compact spaces is a compact space.

<u>Thm</u>: Finite prod of compact spaces is a compact space.

<u>Pf</u>: If we know X, Y compact $\Rightarrow X \times Y$ is compact, then the above thm is true.

1) <u>Base Case</u>: X, Y compact. So, $X \times Y$ is compact

2) <u>Induction Step</u>: Suppose $X_1, \dots, X_{n-1}$ compact so $X_1 \times \dots \times X_{n-1}$ is compact. Consider X_n compact

By ind^n hypothesis $\underbrace{(X_1 \times \dots \times X_{n-1})}_{\text{compact}} \times \underbrace{X_n}_{\text{compact}} = \underbrace{X_1 \times \dots \times X_n}_{\text{compact}}$

Hence, by ind^n, the statement holds for all finite products

To prove it for $n=2$.

Consider X, Y compact. Take a cover of $X \times Y$.

$$\Omega = \{W : W \text{ is open in } X \times Y\}$$

s.t $\displaystyle X \times Y = \bigcup_{W \in \Omega} W$

Claim : $\{x_0\} \times Y$ has a finite subcover

Since $\{x_0\} \times Y \simeq Y$, $\{x_0\} \times Y$ has subsp topo.

& Y is compact $\Rightarrow \{x_0\} \times Y$ has finite subcover.

$W_1, W_2, \ldots, W_\lambda \in \Omega$

$\displaystyle \{x_0\} \times Y = \bigcup_{i=1}^{\lambda} W_i = N$

$(x_0, y_0) \in W_i$ for some i

Now, each $(x_0, y) \in N \quad \forall y \in Y$

$\Rightarrow \exists$ a basic open set $U_y \times V_y$ s.t $(x_0, y) \in U_y \times V_y \subseteq N$

$\displaystyle \Rightarrow \{x_0\} \times Y \subseteq \bigcup_{y \in Y} U_y \times V_y \subseteq N$

$\Rightarrow \bigcup V_y = Y$, Y is compact $\Rightarrow \exists$ finitely many covering Y

$V_{y_1} \cup \ldots \cup V_{y_k} = Y$

$\displaystyle \{x_0\} \times Y \subseteq \bigcup_{i=1}^{k} U_{y_i} \times V_{y_i}$

Date:

$\Rightarrow \quad x_0 \in \bigcap_{i=1}^{k} U_{y_i} = U_{x_0}$

$\Rightarrow \quad \{x_0\} \times Y \subseteq U_{x_0} \times Y \overset{?}{\subseteq} N$

? follows since $U_{x_0} \times Y \subseteq \bigcup_{i=1}^{k} U_{x_0} \times V_{y_i} \subseteq \bigcup_{i=1}^{k} U_{y_i} \times V_{y_i} \subseteq N$ ⌟

So, $\forall x_0 \in X$, $\exists U_{x_0}$ open X s.t $U_{x_0} \times Y$ is covered by finitely many $W_i \in \Omega$

Then $\{U_{x_0}\}_{x_0 \in X}$ is an open cover of $X \times Y$

$$X = \bigcup_{x_0 \in X} U_{x_0}$$

Since X is compact, $\exists U_{x_1}, \dots, U_{x_n}$ s.t $X = \bigcup_{i=1}^{n} U_{x_i}$

$\Rightarrow \quad X \times Y = \bigcup_{i=1}^{n} U_{x_i} \times Y$

Each of these are covered by finite no. of $W \in \Omega$

T'fore $X \times Y$ has a finite subcover
($\because$ finite union of finite sets is finite)

$\square$

Date:

Thm: Every compact set in $\mathbb{R}^n$ is closed & b'nd

Given : $A \subseteq \mathbb{R}^n$ is compact

To show : A is closed & b'nd

Pf : $\mathbb{R}^n$ is Hausdorff & since every compact subset of a Hausdorff sp. is closed $\Rightarrow$ A is closed

Consider the open cover $\{B(0,n)\}_{n=1}^{\infty}$ of $\mathbb{R}^n$.
Since $A \subseteq \mathbb{R}^n$, it is also an open cover of A.

Since A is compact, $\exists$ finite subcover $A \subseteq \bigcup_{i=1}^{r} B(0,N_i)$
$\subseteq B(0,N)$,
where $N = \max_{1 \le i \le r} N_i$

So, A is b'nd.

Converse :
Given: A is closed & b'nd
To show : A is compact

Pf : A is b'nd. $\Rightarrow A \subseteq B(0,N) \subseteq [-N,N]^n$
Since $[-N,N] \subseteq \mathbb{R}$ is compact $\Rightarrow [-N,N]^n \subseteq \mathbb{R}^n$ is compact.

Then $A \subseteq [-N,N]^n$ is compact
(closed subset of a compact set)

Ex Hausdorff $\Rightarrow T_1$

Date: 26/03/2026

Thm: Every compact Hausdorff space is T_4 (T_1+Normal)

Pf: Since Hausdorff $\Rightarrow T_1$, we only need to show compact Hausdorff $\Rightarrow$ Normal

Let X be compact Hausdorff

Given: A, B closed $A \cap B = \emptyset$

To show: X is normal i.e $\exists$ open sets U, V s.t $U \cap V = \emptyset$, $A \subseteq U$, $B \subseteq V$

Fix $a \in A$. $\forall b \in B$, Hausdorff $\Rightarrow \exists U_{a,b}$ & $V_{a,b}$ open s.t $a \in U_{a,b}$, $b \in V_{a,b}$, $U_{a,b} \cap V_{a,b} = \emptyset$

$\underset{\text{closed}}{A, B} \subseteq \underset{\text{compact}}{X} \Rightarrow A, B$ are compact

So, $B \subseteq \bigcup_{b \in B} V_{a,b}$ open $\Rightarrow \exists$ finitely many $B \subseteq \bigcup_{k=1}^{n} V_{a,b_k}$

Since $U_{a,b_i} \cap V_{a,b_i} = \emptyset \Rightarrow \bigcap_{i=1}^{n} U_{a,b_i} \neq \emptyset$ ($\because$ it contains a)

$\bigcap_{i=1}^{n} U_{a,b_i} \subseteq U_{a,b_i} \quad \forall\ 1 \leq i \leq n$

$\Rightarrow \Big(\underbrace{\bigcap_{i=1}^{n} U_{a,b_i}}_{\substack{\text{open} \\ U_a \text{ say}}}\Big) \cap V_{a,b_j} = \emptyset \quad \forall\ 1 \leq j \leq n$

We have open set $a \in U_a$ & $V_a = \underbrace{\bigcup_{i=1}^{n} V_{a,b_i}}_{(\text{open})} \supseteq B$

s.t $U_a \cap V_a = \emptyset$

So, $\forall a \in A$, $\exists$ $a \in U_a$, $B \subseteq V_a$ open s.t $U_a \cap V_a = \emptyset$

Since $A \subseteq \bigcup_{a \in A} U_a$ & A is compact $\Rightarrow$ $\exists$ finitely many $a_1, \dots, a_l$

s.t $A \subseteq \bigcup_{i=1}^{l} U_{a_i}$

Then $B \subseteq \underbrace{\bigcap_{i=1}^{l} V_{a_i}}_{(\text{open})}$.

Also $\bigcap_{i=1}^{l} V_{a_i} \subseteq V_{a_i}$ $\forall\, 1 \leq i \leq n$, $V_{a_i} \cap U_{a_i} = \emptyset$

$$\Rightarrow \underbrace{\left(\bigcap_{i=1}^{l} V_{a_i}\right)}_{\substack{B \subseteq V \\ \text{open}}} \cap \underbrace{\left(\bigcup_{i=1}^{l} U_{a_i}\right)}_{\substack{A \subseteq U \\ \text{open}}} = \emptyset \quad \text{s.t} \quad U \cap V = \emptyset$$

$\square$

Def^n: A topo. sp. X is <u>regular</u> if for every $b \in X$ & closed set A s.t $b \notin A$, then $\exists$ open sets U, V s.t $b \in U$, $A \subseteq V$, $U \cap V = \emptyset$.

If X is regular & T_1, it is called T_3.

<u>Note</u>: $T_4 \Rightarrow T_3$

Date: 27/03/2026

Lindeloff = Every open cover has countable subcover

T_3 + Lindeloff $\Rightarrow$ Normal

Urysohn's lemma : Let X be normal iff for any A, B closed disjoint subsets of X, then $\exists$ a cts. fx^n $f: X \to [0,1]$ s.t $f(A) = 0$, $f(B) = 1$

Pf : ($\Leftarrow$) Let A, B be closed disjoint s.t $f: X \to [0,1]$ cts. s.t $f(A) = 0$, $f(B) = 1$.

Consider $\underbrace{[0, 1/4), (3/4, 1]}_{\text{open}} \subseteq [0,1]$

$\because f$ is cts. $\Rightarrow f^{-1}([0,1/4)), f^{-1}((3/4,1]) \subseteq X$ open

Since $f(A) = 0$ & $f(B) = 1 \Rightarrow A \subseteq f^{-1}([0, 1/4))$
& $B \subseteq f^{-1}((3/4, 1])$

Moreover, if $x \in f^{-1}([0, 1/4)) \cap f^{-1}((3/4, 1])$, then $f(x) < 1/4$ & $f(x) > 3/4$

$\Rightarrow f^{-1}([0,1/4)) \cap f^{-1}((3/4,1]) = \varnothing$ $\to$ Contd^n

$\Rightarrow X$ is normal

Date : ____________

($\Rightarrow$) Given X is normal, A, B closed disjoint

To show: $\exists$ cts. fn $f : X \to [0,1]$ s.t $f(A) = 0$ & $f(B) = 1$.

Pf: X is normal $\Rightarrow$ $\exists$ open set $U_{1/2} \subseteq X$ s.t
$A \subseteq U_{1/2} \subseteq \overline{U}_{1/2} \subseteq X \setminus B$

Since $A \subseteq U_{1/2}$, by normality, $\exists$ $U_{1/4}$ open in X s.t
$A \subseteq U_{1/4} \subseteq \overline{U}_{1/4} \subseteq U_{1/2}$

Sim., $\exists$ $U_{3/4} \subseteq X$ open s.t $\overline{U}_{1/2} \subseteq U_{3/4} \subseteq \overline{U}_{3/4} \subseteq X \setminus B$
$\Rightarrow A \subseteq U_{1/4} \subseteq \overline{U}_{1/4} \subseteq U_{1/2} \subseteq \overline{U}_{1/2} \subseteq U_{3/4} \subseteq \overline{U}_{3/4} \subseteq X \setminus B$

Continuing, we get $A \subseteq U_{1/8} \subseteq \overline{U}_{1/8} \subseteq U_{1/4} \subseteq \overline{U}_{1/4} \subseteq U_{3/8} \subseteq \overline{U}_{3/8} \subseteq U_{1/2} \subseteq \overline{U}_{1/2}$
$\dots \subseteq X - B$

We get open sets U_r, $r = k/2^n$, $k = 1, \dots, 2^n - 1$
By induction, $\forall n \in \mathbb{Z}_{>0}$, $A \subseteq U_r \quad \forall r$
$\overline{U}_r \subseteq U_s$ if $r < s$
& $\overline{U}_r \subseteq X \setminus B \quad \forall r$

Def. $f : X \to [0,1]$
$$f(x) = \begin{cases} \inf\{r : x \in U_r\} & \text{when } x \in U_r \text{ for some } r \\ 1, & \text{if } x \notin U_r \text{ for any } r \end{cases}$$

If $x \in B \Rightarrow f(x) = 1$ $\quad (\because U_r \subseteq X \setminus B \ \forall r)$
If $x \in A \Rightarrow A \subseteq U_r \ \forall r \Rightarrow f(x) = 0$

Date :

1) If $x \notin U_r$ for some $r \Rightarrow f(x) \geq r$

2) If $x \in \bar{U}_r$ for some $r \Rightarrow x \in U_s \ \forall s > r$

$\Rightarrow f(x) \leq r$

To show : f is cts.

Let $(a,b) \subseteq [0,1]$ & $w \in f^{-1}((a,b))$

Sufficient to show $\forall w \in f^{-1}((a,b))$, $\exists$ open set V_w s.t

$w \in V_w \subseteq f^{-1}((a,b))$

$\Rightarrow a < f(w) < b$

Ex: Show that elements of the form $k/2^n$ are dense in $(0,1)$

$a < r < f(w) < s < b$ for some r, s of the form $k/2^n$

$f(w) < s \Rightarrow w \in U_s$

$f(w) > r \Rightarrow w \notin \bar{U}_r$

(Contrapositive of 1 & 2)

$\Rightarrow w \in \underbrace{U_s}_{\text{open}} - \bar{U}_r \subseteq f^{-1}((a,b))$

Date :

Let $w \in f^{-1}([0,a))$

$\Rightarrow 0 \le f(w) < r < a$

$\uparrow$ is of the form $k/2^n$

$\Rightarrow w \in U_r$

$\Rightarrow w \in U_r \subseteq f^{-1}([0,a))$

Let $w \in f^{-1}((b,1])$ $\Rightarrow$ $b < s < f(w) \le 1$

$\uparrow$ is of the form $k/2^n$

$\Rightarrow w \notin \overline{U_s}$

$\Rightarrow w \in X \setminus \overline{U_s} \subseteq f^{-1}((b,1])$

Since we have proved continuity of f for all basic open sets of $[0,1]$.

T'fore f is cts & thus the proof is complete. $\square$

<u>Rem</u>: Recall regular meant we could separate pt. $a \notin B$ closed from closed sets $\Rightarrow \exists\, U, V$ open s.t $a \in U$, $B \subseteq V$, $A \cap B = \emptyset$

$\not\Rightarrow$ $\exists$ a cts fx^n $f : X \to [0,1]$ s.t $f(a) = 0$ & $f(B) = 1$.

Such a space is called a <u>completely regular space</u>.

Date : 01/04/2026

Thm: X is Tychnoff (T_1 & completely regular)

$\Downarrow$

X is subspace of a cube $[0,1]^J$

$\Updownarrow$

X is subspace of compact Hausdorff space

$\Downarrow$

X is a subspace of a T_4 space

$\Downarrow$

X is Tychnoff.

Pf 1) Any T_4 space is a Tychnoff space.

Given: X is normal & T_1

To show. X is completely regular & T_1

Since X is normal, by Urysohn's lemma,

$\forall A, B$ closed disj, $\exists$ cts. fx^n s.t $f(A) = 0$, $f(B) = 1$.

In particular, this holds for $A = \{a\}$ where $a \notin B$

$\Rightarrow f(a) = 0$, $f(B) = 1$ ($\because X$ is T_1) (i.e $\{a\} \cap B = \emptyset$)

So, X is completely regular.

Date : ______

Thm Let X be T_1

2) If $\exists$ cts fx^ns $\{f_\alpha : X \to X_\alpha\}_{\alpha\in\Lambda}$ which separate pts. from closed sets (i.e for any closed set $B \subseteq X$ & $a \notin B$, $\exists \beta \in \Lambda$ s.t $f_\beta(a) \notin \overline{f_\beta(B)}$)

Then $e : X \to \prod X_\alpha$ is an embedding
$x \mapsto (f_\alpha(x))_{\alpha\in\Lambda}$
(i.e it is a homeomorphism onto its image)

Pf: e is one-one.
Since for every $a, b \in X$, $a \neq b \Rightarrow a \notin \{b\}$ closed ($\because X$ is T_1)
$\Rightarrow \exists \beta \in \Lambda$ s.t $f_\beta(a) \notin f_\beta(\{b\}) \Rightarrow f_\beta(a) \neq f_\beta(b)$
$\Rightarrow f(a) \neq f(b)$ ($\because$ differ at one coordinate)

$\Rightarrow e$ is injective

$\Rightarrow e$ is cts. since $\pi_\alpha \circ e = f_\alpha$ is cts $\forall \alpha \in \Lambda$
& $\prod X_\alpha$ has product topology
(fx^n is cts. iff component fx^ns are cts.)

Claim: $\{f_\alpha^{-1}(V) : V \subseteq X_\alpha \text{ open}, \alpha \in \Lambda\}$ is a base for topo. on X.

$$\underbrace{e(f_\beta^{-1}(V))}_{\text{image of basic open set}} = \underbrace{e(X) \cap \pi_\beta^{-1}(V)}_{\text{open set in } e(X)}$$

$\Rightarrow e$ maps basic open sets in X to open sets to open sets in $e(X)$
$\Rightarrow e$ is homeomp. onto its image

Date : ____________

Pf of Claim:

For any $a \in X$, consider $f_\alpha(a)$ for any $\alpha \in \Lambda$

Then $f_\alpha(a) \in X_\alpha$ & $\exists$ open set V s.t $f_\alpha(a) \in V$

$\Rightarrow a \in f_\alpha^{-1}(V)$

Hence $X \subseteq \bigcup f_\alpha^{-1}(V)$

To show: This is a base, we show that $\forall a \in X$ & open $W \subseteq X$, $a \in W$ $\exists$ an open set of the form $a \in f_\beta^{-1}(V) \subseteq W$

$a \in W \Rightarrow a \notin X-W$ ← closed set.

$\exists \beta \in \Lambda$ s.t $f_\beta(a) \notin \overline{f_\beta(X-W)}$

$\exists$ some open set $V \subseteq X_\beta$ s.t $f_\beta(a) \subseteq V$ s.t $V \cap f_\beta(X-W) = \emptyset$

$\Rightarrow f_\beta^{-1}(V) \cap X - W = \emptyset$

$\Rightarrow f_\beta^{-1}(V) \subseteq W$

Hence proved

$\square$

Now, for a completely regular sp. X,

$\forall a \in X$, $B \subseteq X$ closed s.t $a \notin B$, then $\exists$ a cts. fxn

$f_{a,B} : X \to [0,1]$ s.t $f(a) = 0$, $f(B) = 1$

Then $\{f_{a,B} : X \to [0,1],\ a \in X,\ B \subseteq X \text{ closed},\ a \notin B\}$ separates pts. from closed sets $\left(\because \text{In } [0,1], \text{ closure of } \{1\} \text{ is } \{1\} \right)$

Date : ______

Pf : (1 ⇒ 2)

Using the thm, $e : X \longrightarrow \prod_{i \in I} [0,1]$ is an embedding

$$x \longmapsto (f_{a,B}(x))$$

i.e a homeomorphism onto its image
& thus a subspace of the cube $\prod_{i \in I} [0,1]$

Pf : (2 ⇒ 3)
Cube is arbitrary product of compact (& Hausdorff) space, hence compact Hausdorff

Pf : (3 ⇒ 4)
Every compact Hausdorff space is T_4

Pf : (4 ⇒ 1)
Any T_4 space is Tychnoff + Any subspace of Tychnoff is Tychnoff.

Date : 08/04/2026

Tietze extension thm :

Let X be a topo. sp, then X is normal iff for any closed subset $A \subseteq X$ & a cts. fn $f: A \to \mathbb{R}$, $\exists$ cts. fn $F: X \to \mathbb{R}$ extending it, i.e, $F|_A = f$

(Tietze $\Rightarrow$ Urysohn's)

Let A, B be disjoint closed sets

$f: A \cup B \to \{0, \pi/2\}$

$f(A) = 0$, $f(B) = \frac{\pi}{2}$ By pasting lemma, f is cts ($\because$ non-empty intersection, nothing to check)

By Tietze extension,

$\exists$ cts. $F: X \to \mathbb{R}$ s.t $F|_{A \cup B} = f$

i.e $F(A) = 0$, $F(B) = \frac{\pi}{2}$

Consider $G = |\sin \circ F| : X \to [0, 1]$

G is cts. as it is composition of cts fn's.

$G(A) = |\sin(0)| = 0$, $G(B) = |\sin(\frac{\pi}{2})| = 1$

Date : ____________

(Urysohn's $\Rightarrow$ Tietze)

Claim : It suffices to show if $\exists f : A \to [-1,1]$ cts. for any closed set $A \subseteq X$, then it extends to $F : X \to [-1,1]$ s.t $F|_A = f$

Since Claim $\Rightarrow$ any cts map $f : A \to \underset{\underset{(-1,1)}{\cong}}{\mathbb{R}}$ can be extended to X

$f : A \to (-1,1) \subseteq [-1,1]$

Then by claim, extends to cts. $G : X \to [-1,1]$ s.t $G|_A = f$

Let $B = \{x \in X : G(x) = \pm 1\}$.

$B \cap A = \emptyset$ ($\because$ Values of G on A is f on $A \neq 1, -1$)

B is closed ($\because B = G^{-1}(\underbrace{\{-1,1\}}_{\text{closed}})$)

By Urysohn's lemma,

$\exists g : X \to [0,1]$ s.t $g(A) = 1$, $g(B) = 0$

$\Rightarrow F = g \cdot G$ is the required fx^n.

(1) F is cts

(2) $\text{Im} F \subseteq (-1,1)$ ($\because$ whenever G is ± 1, g is 0)

(3) $F|_A = f$

Date : ___________

We have proved Claim $\Rightarrow$ Tietze extension.

Only need to prove the claim.

Assume, X is normal, $A \subseteq X$ closed & $f : A \to [-1,1]$ is cts.

To show : $\exists F : X \to [-1,1]$ cts s.t $F|_A = f$

Define $A_1 = \{x \in A : f(x) \leq -1/3\}$

$B_1 = \{x \in A : f(x) \geq 1/3\}$ both closed

By Urysohn's lemma, $\exists$ cts fn^n $f_1 : X \to [\frac{-1}{3}, \frac{1}{3}]$ s.t

$f_1(A_1) = \frac{-1}{3}$, $f_1(B_1) = \frac{1}{3}$

On A, $|f - f_1| \leq \frac{2}{3}$

So, $f \cdot f_1 : A \to [\frac{-2}{3}, \frac{2}{3}]$

Define $A_2 = \{x \in A : f \cdot f_1(x) \leq \frac{-2}{9}\}$

$B_2 = \{x \in A : f \cdot f_1(x) \geq \frac{2}{9}\}$ both closed.

By Urysohn's lemma, $\exists f_2 : X \to [\frac{-2}{9}, \frac{2}{9}]$ cts. s.t

$f_2(A_2) = \frac{-2}{9}$, $f_2(B_2) = \frac{2}{9}$

On A, $|(f - f_1) - f_2| \leq \left(\frac{2}{3}\right)^2$

Repeat by ind^n to get cts. $f_n : X \to [-\frac{2^{n-1}}{3^n}, \frac{2^{n-1}}{3^n}]$

On A, $|f - \sum_{i=1}^{n} f_i| \leq \left(\frac{2}{3}\right)^n$

Date : ____________

$F(x) = \sum_{i=0}^{\infty} f_i(x)$ is well-defined as $\left|\sum_{i=1}^{n} f_i(x)\right| \leq \underbrace{\sum_{i=1}^{n}\left(\frac{2^{i-1}}{3^i}\right)}_{\text{convergent}}$

Let $a \in X$

$\forall \epsilon > 0$, $\exists$ small enough open set U around a s.t $\forall x \in U$,

$|F(x) - F(a)| < \epsilon$

Choose N large s.t $\sum_{n=N+1}^{\infty} \left(\frac{2}{3}\right)^{n} < \frac{\epsilon}{4}$

$\left|\sum_{i=1}^{\infty} f_i(x) - \sum_{i=1}^{\infty} f_i(a)\right| \leq \sum_{i=1}^{N} |f_i(x) - f_i(a)| + \underbrace{\epsilon/4 + \epsilon/4}_{\text{bound on tail of series}}$

$\forall i = 1, \ldots, N$, $\exists$ an open subset U_i s.t $\forall x \in U_i$, $|f_i(x) - f_i(a)| < \epsilon/2N$

($\because f$ is cts.)

Let $U = \bigcap_{i=1}^{N} U_i$ open.

Then $\forall x \in U$, $|F(x) - F(a)| < \epsilon$, so F is cts

Now, $|F(x)| < \sum_{i=1}^{\infty} \frac{2^{i-1}}{3^i} = 1$

Since $f : X \to [-1, 1]$

f is cts & on A $\left|f(x) - \sum_{i=1}^{n} f_i(x)\right| \leq \left(\frac{2}{3}\right)^{n}$

& $F(x) = \lim_{n \to \infty} \sum_{i=1}^{n} f_i(x)$

$\Rightarrow F(x) = f(x) \ \forall x \in A$

Date : 10/04/2026

<u>Thm</u> : X is a locally cmpt. Hausdorff sp. iff $\exists$ a space $Y \supseteq X$ s.t

(1) Y is cmpt. Hausdorff

(2) $Y - X$ is a pt.

(3) $X \subseteq Y$ has subspace topology

Pf : ($\Leftarrow$) Let $X \subseteq Y$ cmpt. Hausdorff s.t $Y - X$ is a pt

<u>To show</u> : $\forall a \in X$, $a \in U$ open, $\exists V$ open s.t $a \in V \subseteq \underbrace{\bar{V}}_{\text{cmpt}} \subseteq U$

Y is T_4 - ($\because$ it is cmpt. Hausdorff)

$\because T_4 \Rightarrow T_3$

So, $\forall a \in U$ open in X $\rightarrow$ also open in Y

$= \underset{\text{open in } Y}{\mathcal{O}} \cap X = \mathcal{O} - \{p\}$

Y is regular $\Rightarrow \exists$ open set $\underset{(\text{open in } X)}{V \subseteq Y}$ s.t $a \in V \subseteq Cl_Y(V) \subseteq U \subseteq X$

$Cl_X(V) = \underbrace{Cl_Y(V)}_{\subseteq X} \cap X$

$\Rightarrow Cl_X(V) = Cl_Y(V) \subseteq Y$

cmpt. since it is closed & Y is cmpt

$\Rightarrow \forall a \in U$ open in X, $\exists$ open set V in X s.t

$a \in V \subseteq Cl_X(V) \subseteq U$ s.t V is open

Second countable : Countable base

Separable : Countable dense set

Date : 15/04/2026

Def^n: A refinement of a cover $\mathcal{U}$ of X is a cover $\mathcal{V}$ of X s.t $\forall V \in \mathcal{V}$, $\exists U \in \mathcal{U}$ s.t $V \subseteq U$.

Def^n: A cover $\mathcal{U}$ of X is locally finite if every pt. $a \in X$ has open set set $a \in W$ s.t W has non-empty intersection with finitely many sets in $\mathcal{U}$.

Paracompact spaces : Let X be a topo. sp. X is said to be paracompact if every open cover has locally finite open refinement

FACTS

Date : ____________

(Jones' lemma)

Lem : Let X be any space. Let S be a closed subsp. which is discrete. Let D be a dense set, $\overline{D} = X$ & $|S| > \underbrace{|P(D)|}_{2^{|D|}}$, then X is not normal

Pf: Suppose X is normal

Let $T_1 \subset S$ (discrete in subsp. topo) s.t T_1 is closed

$T_1 = S \cap W$ ← closed in X

$\Rightarrow T_1$ is closed in X

Now, T & $S \setminus T$ are disjoint closed subsets of X.

X is normal $\Rightarrow \exists\, U(T) \supseteq T$ & $V(T) \supseteq S \setminus T$ both open s.t $U(T) \cap V(T) = \emptyset$

Let $T \subseteq S$, $U(T) \cap D \neq \emptyset$ since D is dense in X.

Consider $T_1, T_2 \subseteq S$ s.t $T_1 \neq T_2$. Wlog $T_1 - T_2 \neq \emptyset$

Then $U(T_1) \cap V(T_2) \neq \emptyset$ since $T_1 - T_2 \neq \emptyset$ & $S \setminus T_2 \subseteq V(T_2)$

$\Rightarrow T_1 - T_2 \subseteq V(T_2)$

& since $T_1 \subseteq U(T_1) \Rightarrow \underbrace{U(T_1) \cap V(T_2)}_{\text{open}} \neq \emptyset$

$\subseteq U(T_1) \cap D$

$U(T_2) \cap V(T_2) = \emptyset$

$\Rightarrow \underbrace{U(T_2) \cap D}_{\neq \emptyset} \neq \underbrace{U(T_1) \cap D}_{\neq \emptyset}$ ($U(T_1) \cap V(T_2) \neq \emptyset$ whereas $U(T_2) \cap V(T_2) = \emptyset$

Date :

$\therefore$ We have a fn $P(S) \to P(D)$ which is one-one

$T \mapsto U(T) \cap D$

$\Rightarrow |P(S)| \le |P(D)| \to$ Contdn

Let X be a space

$CX = X \times I/\sim$ where $(x,1) \sim (x',1)$ $\forall\ x, x' \in X$

$\Sigma X = X \times I/\sim$ where $(x,1) \sim (x',1)$

(suspension of X) $(x,0) \sim (x',0)$ $\forall\ x, x' \in X$

$X \vee Y = X \sqcup Y/\sim$ where $x_0 \sim y_0$

$Y = CX \sqcup CX/\sim$ where $[(x,0)] \sim [(x,0)]$ in two different copies of CX, $x \in X$.

$\Sigma X \simeq Y$

$CS^1 \simeq D^2$

Date : ______________

Sorgenfrey line ($\mathbb{R}_\ell$)

- For any $a, b \in \mathbb{R}$, $[a, b)$ is clopen in $\mathbb{R}_\ell$
  So, $\mathbb{R}_\ell$ is totally disconnected (only singletons as connected s...

- Any compact subset of $\mathbb{R}_\ell$ is at most countable
- $\mathbb{R}_\ell$ is perfectly normal Hausdorff space (T_6)
  $\Rightarrow$ completely normal Hdf (T_5)
  $\Rightarrow$ normal Hdf (T_4)

- $\mathbb{R}_\ell$ is first countable & separable, but not second countable
- $\mathbb{R}_\ell$ is not metrizable ($\because$ separable metric spaces are second countable)

- $\mathbb{R}_\ell$ is Lindelöff & paracompact but not locally compact
- Not connected : $\mathbb{R}_\ell = (-\infty, 0) \cup [0, \infty)$
- $\mathbb{R}_\ell$ is regular & completely regular (Tychonff)

Sorgenfrey plane ($\mathbb{R}_\ell \times \mathbb{R}_\ell$)

- Regular & completely regular (Tychonoff)
  (as product of $\mathbb{R}_\ell$)
- Not normal (even though $\mathbb{R}_\ell$ is normal)

- first countable & separable but not second countable
- Not Lindeloff (even though $\mathbb{R}_\ell$ is Lindeloff)
- Not metrizable (same logic as $\mathbb{R}_\ell$)

Date : ______

Moore plane (or Half plane)

- Completely regular Hausdorff space (Tychonoff)
- Not normal
- Separable
- The subspace $\{(x,0) : x \in \mathbb{R}\}$ is discrete under subspace topology
  - $\Rightarrow$ Shows subspace of a separable space need not be separable
- First countable but not second countable or Lindeloff
  (∵ not second countable)
- Not paracompact (∵ Hdf paracompt spaces are normal)

Rational number ($\mathbb{Q}$ as a subspace of $\mathbb{R}$)

- Hausdorff (subsp. of Hdf)
- Normal (metric subspace)
- Regular & completely regular
- Second countable
- Separable (countable discrete set is itself dense)
- Lindeloff (∵ second countable)
- Not compact
- Not locally compact
- Metrizable

Date : ____________

Separation axioms :

$T_4 \Rightarrow T_{3\frac{1}{2}} \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1$

(Tychonoff)

$T_1 \not\Rightarrow T_2$: Cofinite topology on infinite set

$T_2 \not\Rightarrow T_3$: K - topology on $\mathbb{R}$

$T_{3\frac{1}{2}} \not\Rightarrow T_4$: Sorgenfrey plane

K - topology on $\mathbb{R}$

- Hausdorff
- Not compact
- Not regular (K is a closed set not containing 0 but the set K & 0 have no disjoint nbd)
- Conn. but not path conn.

  Two path components $(-\infty, 0]$, $(0, \infty)$
- Not locally path conn at 0 & not locally conn. at 0

For $A \subseteq Y \subseteq X$,

$$Cl_Y(A) = Cl_X(A) \cap Y$$

$$Int_Y(A) = Int_X(A \cup (X \setminus Y)) \cap Y$$

Date :

Proof that Moore plane is not normal

The countable set $S=\{(p,q)\in\mathbb{Q}\times\mathbb{Q} : q>0\}$ is dense in Γ
Hence, every cts fx^n $f:\Gamma\to\mathbb{R}$ is determined by its restriction to S
So, there can be at most $|\mathbb{R}|^{|S|}=2^{\aleph_0}$ many cts. real valued functions on Γ.

However, the real line $L=\{(p,0):p\in\mathbb{R}\}$ is a closed discrete subspace of Γ with $2^{\aleph_0}$ many pts.
So, there are $2^{2^{\aleph_0}} > 2^{\aleph_0}$ many cts. fx^n's from L to $\mathbb{R}$.
Not all these can be extended to cts. fx^n's on Γ.

Hence Γ is not normal, because by Tietze extension theorem, all fx^n's defined on a closed subsp. of normal sp. can be extended to a cts. fx^n on the whole space.

We can use the same idea to prove $\mathbb{R}_\ell\times\mathbb{R}_\ell$ is not normal
Consider $A=\{(x,-x):x\in\mathbb{R}\}\subset\mathbb{R}_\ell\times\mathbb{R}_\ell$

A is an uncountable, discrete, closed subset of $\mathbb{R}_\ell\times\mathbb{R}_\ell$
↳ ∵ for any $(x,-x)\in A$,
If $(y,-y)$ inside basic open set $[x,x+\epsilon)\times[-x,-x+\epsilon)$
$\Rightarrow y\in[x,x+\epsilon)$ & $y\in(x-\epsilon,x]$ $\Rightarrow y=x$
So, nbd contains only one pt of A.

(Jones' lemma) In a normal space, every closed discrete subsp D satisfies $\underbrace{|D|}_{2^{2^{\aleph_0}}}\le 2^{\aleph_0}$ (in separable normal space, closed discrete subsets must be countable)
since D is uncountable here $\rightarrow$ Contd^n