

TUTORIAL 0 MA 406
SPRING 2026, IIT BOMBAY

If you have a group of 4 complete problems 4,5(a)(c), 2(a), 2(d)(e) and if there is a 5th person in your group also do 2(b)(c) and submit on Jan 15th 2025, in class.

Each person in the group has to handwrite exactly one problem. Ensure all the group members submit answers to different problems. Marks are allotted individually.

The role of the group is to collaborate on solutions and decide who will write up solution to which problem. If there are any concerns please email me

Part I.

- (1) Let $f : A \rightarrow B$ be a function between sets A and B . Let $A_\alpha \subseteq A$ for $\alpha \in \Lambda$ and $B_\eta \subseteq B$ for all $\eta \in \Omega$.
 - (a) Show that a function between sets $f : A \rightarrow B$ has an inverse if and only if f is one-one and onto.
 - (b) Define arbitrary product and arbitrary union of sets.
 - (c) Prove the following statements if true, else add appropriate condition that f is one-one and/or onto to prove it is true.
 - (i) $f(\cap_\alpha A_\alpha) = \cap_\alpha f(A_\alpha)$
 - (ii) $f(\cup_\alpha A_\alpha) = \cup_\alpha f(A_\alpha)$
 - (iii) $f^{-1}(\cup_\eta B_\eta) = \cup_\eta f^{-1}(B_\eta)$
 - (iv) $f^{-1}(\cap_\eta B_\eta) = \cap_\eta f^{-1}(B_\eta)$
- (2) Let A, B, C be sets then any function $f : A \rightarrow B \times C$ is given by $f = (g, h)$ where $g : A \rightarrow B$ and $h : A \rightarrow C$ are functions. Prove or disprove.
- (3) State and prove the binomial theorem using mathematical induction.
- (4) Give a definition of a finite set, countably infinite set. Show that if A is finite then $\mathcal{P}(A)$ is a finite set and write its cardinality in terms of that of A .
- (5) A relation R on a set X is a subset $R \subseteq X \times X$. We say aRb for $a, b \in X$ if $(a, b) \in R$. A relation R is said to be partial order if R is reflexive, antisymmetric and transitive.
 - (a) Give example of a partial order on a set X in which any set of two elements $\{a, b\}$ has a least upper bound $a \vee b$ and greatest lower bound $a \wedge b$. Such a set is called a lattice.
 - (b) Give an example of a complemented lattice, ie a lattice X with a smallest 0 and largest element 1 such that for every $a \in X$ there is $a' \in X$ such that $a \vee a' = 1$ and greatest lower bound is $a \wedge a' = 0$.
 - (c) Give an example of a distributive lattice L , that is a lattice such that
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$
for all $a, b, c \in L$.
- (6) State the Axiom of Choice, Zorn's lemma and Zermelo's theorem. Read and understand the proof that they are all equivalent.

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Part II.

- (1) Let (X, d) be a metric space.
 - (a) Define open sets in X .
 - (b) Show that ϕ and X are open.
 - (c) Show that finite intersection and arbitrary union of open sets in X is open in X .
 - (d) Define closed set in X in terms of limit points. Show that a set is closed in X if and only if its complement is open in X .
 - (e) Define a continuous function $f : (X, d) \rightarrow (Y, \rho)$ of metric spaces. Show that a f is continuous if and only if for every open set U in Y , $f^{-1}(U)$ is open in X .
- (2) Let X be the set of all **bounded** sequences in \mathbb{R} . Define $\rho : X \times X \rightarrow \mathbb{R}$ as $\rho(x, y) = \sup\{|x_n - y_n| \mid n \in \mathbb{N}\}$.
 - (a) Show that (X, ρ) is a metric space.
 - (b) Describe open neighbourhoods in (X, ρ) .
 - (c) Give 5 different examples of open sets.
 - (d) Give 5 different examples of closed sets.
 - (e) **If we wanted to give a metric structure to all sequences and not just bounded ones what could we do?**
- (3) **Can you use the previous question** to show that the set of sequences in a metric space can be given a metric structure.
- (4) Show that finite union of closed sets in a metric space is closed.
- (5) Show that ϕ and X are both open and closed in a metric space (X, d) .
- (6) Show that arbitrary intersection of closed sets is closed in a metric space.

TUTORIAL 1 MA 406
15 JAN SPRING 2026, IIT BOMBAY

Submission on Jan 22nd 2025, in class: *If you have a group of 4 complete problems Part I -1(b), Part I-5, Part II -7, Part II-9(b) and if there is a 5th person in your group also do PartII-10.*

*Each person in the group has to handwrite exactly one problem. Ensure all the group members submit answers to different problems. Marks are allotted individually. **Please submit solutions only to the assigned problems.***

The role of the group is to collaborate on solutions and decide who will write up solution to which problem. If there are any concerns please email me.

Part I.

- (1) Let (X, d) be a metric space. Define $w \in X$ to be a closure point of a set $A \subset X$ if $\forall \delta > 0, N(w, \delta) \cap A \neq \emptyset$. The set of all closure points of A in X is called the closure of A and denoted by \bar{A} . Prove the following
- (a) $A \subseteq \bar{A}$.
 - (b) \bar{A} is a closed set.
 - (c) If B is closed, and $A \subseteq B$ then $\bar{A} \subseteq B$.
 - (d) Show that

$$\bar{A} = \bigcap_{\{B \subseteq X \mid B \text{ is closed, } A \subseteq B\}} B.$$

- (e) Show that A is closed if and only if $A = \bar{A}$.

- (2) Let (X, d) be a metric space. Define $w \in X$ to be an interior point of a set $A \subset X$ if $\exists \delta > 0, N(w, \delta) \subseteq A$. The set of all interior points of A in X is denoted by $\overset{\circ}{A}$. Prove the following.
- (a) $\overset{\circ}{A} \subseteq A$.
 - (b) $\overset{\circ}{A}$ is an open set.
 - (c) If B is open, and $B \subseteq A$ then $B \subseteq \overset{\circ}{A}$.
 - (d) Show that

$$\overset{\circ}{A} = \bigcup_{\{B \subseteq X \mid B \text{ is open, } B \subseteq A\}} B.$$

- (e) Show that A is open if and only if $A = \overset{\circ}{A}$.

- (3) Show that every open set in $(\mathbb{R}, | \cdot |)$ can be written as a union of open intervals. Modify the argument to show that in fact every open set $(\mathbb{R}, | \cdot |)$ is a union of open intervals which arise from a countable collection.
- (4) Show that there exists a countable set \mathcal{O} of open sets in \mathbb{R}^2 such that every open set in $(\mathbb{R}^2, \|\cdot\|_2)$ can be written as a union of elements of the \mathcal{O} .
- (5) Find the closure and interior of the set $B = (2, 3) \cup \{5\}$ in $(\mathbb{R}, | \cdot |)$. Justify your answer

Part II.

- (1) Let X be a set. Define $\mathcal{U} = \{A \subseteq X \mid X - A \text{ is finite}\} \cup \{\emptyset, X\}$. Show that this forms a topology.
- (2) Let (X, \mathcal{U}) be a topological space. Then show that for any subset $A \subseteq X$ and $\mathcal{U}_A = \{A \cap U \mid U \in \mathcal{U}\}$ is a topology on A .
- (3) The collection of sets of the form $(a, b]$ in \mathbb{R} give a base for a topology. Prove or disprove.
- (4) Let \mathcal{U}_1 and \mathcal{U}_2 be topologies on a set X . Let $\mathcal{U} = \{S \subseteq X \mid S \in \mathcal{U}_1, S \in \mathcal{U}_2\}$. Is \mathcal{U} a topology on X ? Justify your answer.
- (5) Let X be a topological space and $A, B \subseteq X$
 - (a) Show that $\overset{\circ}{X} = X$.
 - (b) Show that $A \subseteq B \subseteq X$ then $\overset{\circ}{A} \subseteq \overset{\circ}{B} \subseteq X$
 - (c) Show that $\overset{\circ}{\overset{\circ}{A}} = \overset{\circ}{A}$.
 - (d) Show that $(A \overset{\circ}{\cap} B) = \overset{\circ}{A} \overset{\circ}{\cap} \overset{\circ}{B}$.
 - (e) Is $(A \overset{\circ}{\cup} B) \subseteq (\overset{\circ}{A} \overset{\circ}{\cup} \overset{\circ}{B})$? Prove or disprove.
 - (f) Is $(\overset{\circ}{A} \overset{\circ}{\cup} \overset{\circ}{B}) \subseteq (A \overset{\circ}{\cup} B)$? Prove or disprove.
 - (g) The set A is open if and only if $\overset{\circ}{A} = A$. Prove or disprove.
- (6) If a topology on a set X is defined using closed sets. Give a definition of a base of the topology in terms of closed sets.
- (7) Let \mathcal{T} be the topology generated by the collection $\{(a, b) \mid a, b \in \mathbb{R}\} \cup \{\{n\} \mid n \in \mathbb{N}\}$. Describe the interior and closure operation in this collection.
- (8) Let X be a set with cofinite topology. Find a base for X .
- (9) Let $\mathcal{D} = \{[a, b) \mid a, b \in \mathbb{R}\}$.
 - (a) Show that \mathcal{D} is a base for a topology $\mathcal{T}_{\mathcal{D}}$ on \mathbb{R} .
 - (b) How is $(\mathbb{R}, \mathcal{T}_{\mathcal{D}})$ different from $(\mathbb{R}, |\cdot|)$ as a topological space?
 - (c) Find the closure of $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ in $(\mathbb{R}, \mathcal{T}_{\mathcal{D}})$.
- (10) Let X be a totally ordered set, is a partially ordered set where all elements are related. **Assume that there is no maximum or minimum element in this set.** Let $(w, z) = \{x \in X \mid w \leq x \leq z, w \neq x, z \neq x\}$. Then $\mathcal{D} = \{(w, z) \mid w, z \in X\}$ is a base for a topology on X .

TUTORIAL 2 MA 406
22 JAN SPRING 2026, IIT BOMBAY

Submission on Jan 29th 2026, in class: *If you have a group of 4 complete problems Part I -(2), Part I-(3)(b), Part I -(4)(c), Part II-(4) and if there is a 5th person in your group also do Part II- (5) (b).*

*Each person in the group has to handwrite exactly one problem. Ensure all the group members submit answers to different problems. Marks are allotted individually. **Please submit solutions only to the assigned problems.***

The role of the group is to collaborate on solutions and decide who will write up solution to which problem. If there are any concerns please email me.

Part I.

- (1) Let (X, d) be a metric space. Let $A \subseteq X$.
 - (a) Show that $d_{A \times A}$, that is the metric restricted to A defines a metric on A .
 - (b) Show that metric topology on A via $d_{A \times A}$ and the subspace topology $A \subset X$ where X has metric topology are equal.

- (2) Recall $\{[a, b] \mid a, b \in \mathbb{R}\}$ is a base for a topology in \mathbb{R} . Is $[a, b]$ closed in this topology? Can you describe a base of closed sets for this topology?

- (3) Consider $H = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ and let $H_+ = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$.
 - (a) Show that $\{N((a, b), \delta) \cap H_+ \mid (a, b) \in H_+\} \cup \{(N((a, 0), \delta) \cap H_+) \cup \{(a, 0)\} \mid a \in \mathbb{R}\}$ is a base for a topology on the upper half plane H .
 - (b) Find the subspace topology on H_+ . Is this topology different from the subspace topology on $H_+ \subseteq \mathbb{R}^2$ where \mathbb{R}^2 has standard topology?
 - (c) Find the subspace topology on \mathbb{R} . Is this topology different from the subspace topology on $\mathbb{R} \subseteq \mathbb{R}^2$ where \mathbb{R}^2 has standard topology.

- (4) Let X be a totally ordered set, is a partially ordered set where all elements are related. **Assume that there is no maximum or minimum element in this set.** Let $(w, z) = \{x \in X \mid w \leq x \leq z, w \neq x, z \neq x\}$. Recall that $\mathcal{D} = \{(w, z) \mid w, z \in X\}$ is a base for a topology on X .
 - (a) Let $I = [0, 1]$. Define lexicographic order on $I \times I$ as $(x_1, x_2) < (y_1, y_2)$ if $x_1 < y_1$ or if $x_1 = y_1$, and $x_2 < y_2$. Describe open sets containing points $(x, 0)$ and $(1, y)$ where $x, y \in I$.
 - (b) Let X be a linearly ordered set and Y be a subset of X . Show that the order topology on Y need not be same as the subspace topology induced by the order topology on X . (Hint: Use $X = \mathbb{R}$.)
 - (c) If X has a maximum or a minimum, how should we define the order topology on it. Explain and prove your answer.

- (5) A topological space (X, \mathcal{T}) is said to be second countable if it has a countable base.
 - (a) Show that \mathbb{R} with standard topology is second countable.

- (b) In fact, show that \mathbb{R}^n with standard metric is second countable space.
 - (c) When is a topological space with cofinite topology second countable?
- (6) Is the subspace of a second countable space second countable? Prove or disprove.
- (7) Let $A \subset X$ and (X, \mathcal{T}) be a topological space. Let A have subspace topology. Let $E \subseteq A$.
- (a) $\text{Cl}_A(E) = \text{Cl}_X(E) \cap A$. Prove or disprove.
 - (b) $\text{int}_A(E) = \text{int}_X(E) \cap A$. Prove or disprove.

Part II.

- (1) Define continuous function between metric spaces. Show that $f : (X, d) \rightarrow (Y, \rho)$ is a continuous if and only if preimage of open sets is open.
- (2) Give an example of a continuous bijective function whose inverse is not continuous.
- (3) Show that all open intervals in \mathbb{R} are homeomorphic with subspace topology.
- (4) Show that all closed intervals in \mathbb{R} are homeomorphic with subspace topology.
- (5) Let $(X, d_1), (Y, d_2)$ and (Z, d_3) be metric spaces.
- (a) Then show that $d((a, b), (c, d)) = \sqrt{(d_1^2(a, c) + d_2^2(b, d))}$ is a metric on $X \times Y$.
 - (b) Show that the projections $\pi_X : (X \times Y, d) \rightarrow (X, d_1)$ and $\pi_Y : (X \times Y, d) \rightarrow (Y, d_2)$ are continuous.
 - (c) Show that a function $f : (Z, d_3) \rightarrow (X \times Y, d)$ is continuous if and only if $\pi_X \circ f$ and $\pi_Y \circ f$ are continuous.
- (6) Show that $D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ and $I \times I$ where $I = [0, 1]$ are homeomorphic with their subspace topology in \mathbb{R}^2 with standard metric topology.

TUTORIAL 3 MA 406
29 JAN SPRING 2026, IIT BOMBAY

Submission on Feb 5th 2026, in class: *Please turn in solutions to problems Part I -(1(b)), Part I-(4(d)), Part I -(5), Part I-(7(b)), Part II- (3(b)).*

*Each person in the new group has to handwrite exactly one problem. Ensure all the new group members submit answers to different problems. Marks are allotted individually. **Please submit solutions only to the assigned problems.***

The role of the group is to collaborate on solutions and decide who will write up solution to which problem. If there are any concerns please email me.

Part I.

- (1) Let \mathbb{R}_l denote the topological space generated by the base $\{[a, b] \mid a, b \in \mathbb{R}\}$.
 - (a) Give two different examples of continuous functions $(\mathbb{R}, | \cdot |) \rightarrow \mathbb{R}_l$ if it exists with justification.
 - (b) Give two different examples of continuous functions $\mathbb{R}_l \rightarrow (\mathbb{R}, | \cdot |)$ if it exists with justification.

- (2) A set D in (X, \mathcal{T}) is said to be dense if $\text{Cl}_X(D) = X$.
 - (a) If D is dense in X then $D \cap A$ is dense in A . Prove or disprove.
 - (b) Give two different examples of proper dense sets of \mathbb{R}^2 . with standard topology.

- (3) Let X be a topological space. Show that if $\{B_\beta\}$ is any collection of open subsets of X whose union is X , then a function on X is continuous if its restriction to each B_β , is continuous.

- (4) Let X be a topological space and $\{A_\alpha\}$ be a collection of subsets of X covering X , that is $X = \cup_\alpha A_\alpha$. Let $f : X \rightarrow Y$ be a function such that $f|_{A_\alpha}$ is continuous.
 - (a) Show that if the collection $\{A_\alpha\}$ is finite and each set A_α is closed, then f is continuous.
 - (b) Find an example where the collection $\{A_\alpha\}$ is countable and each A_α is closed, but f is not continuous. Give justification
 - (c) An indexed family of sets $\{A_\alpha\}$ is said to be locally finite if each point $a \in X$ there exists an open set $a \in V$ which intersects A_α for only finitely many values of α . Show that the union of any subfamily from a locally finite family of closed sets is closed.
 - (d) Show that if the family $\{A_\alpha\}$ is locally finite and each A_α is closed, then f is continuous.

- (5) Show that if $g : \mathbb{R}^2 \rightarrow Y$ is continuous, then $g(-, 0) : \mathbb{R} \rightarrow Y$ and $g(0, -) : \mathbb{R} \rightarrow Y$ are both continuous. Show that the converse to part is not true. (Let $g(x, y) = \frac{xy}{x^2 + y^2}$, when $(x, y) \neq (0, 0)$ with $g(0, 0) = 0$.)

- (6) Let X, Y, Z be topological spaces. Show that if X is homeomorphic to Y and Y is homeomorphic to Z then X is homeomorphic to Z .
- (7) Let Y be an ordered set with order topology. Let $f, g : X \rightarrow Y$ be continuous.
- Show that the set $\{x \in X \mid f(x) \leq g(x)\}$ is closed in X .
 - Let $h : X \rightarrow Y$ be the function $h(x) = \min\{f(x), g(x) \mid x \in X\}$. Show that h is continuous.

Part II.

- (1) Let (X, d_1) and (Y, d_2) be metric spaces. Let $i_1 : X \rightarrow X \sqcup Y$ and $i_2 : Y \rightarrow X \sqcup Y$ be natural inclusions.
- Is there is a canonical choice of metric on $X \sqcup Y$?
 - Are i_1, i_2 are continuous with this choice of metric on $X \sqcup Y$?
- (2) Let (X, d_1) and (Y, d_2) be metric spaces. Let $p_1 : X \times Y \rightarrow X$ and $p_2 : X \times Y \rightarrow Y$ be natural projections.
- Is there is a canonical choice of metric on $X \times Y$?
 - Are p_1, p_2 are continuous with this choice of metric on $X \times Y$?
- (3) Let $f : X \rightarrow Y$ be a function.
- If (Y, \mathcal{F}) is a topology on Y , find the smallest topology on X with which f is continuous if it exists.
 - If (X, \mathcal{T}) is a topology on X , find the largest topology on Y with which f is continuous if it exists.
- (4) Let (X, \mathcal{T}) and (Y, \mathcal{F}) be topological spaces. Find the largest topology on $X \sqcup Y$ such that $i_1 : X \rightarrow X \sqcup Y$ and $i_2 : Y \rightarrow X \sqcup Y$ are continuous.
- (5) Let (X, \mathcal{T}) and (Y, \mathcal{F}) be topological spaces. Find the smallest topology on $X \times Y$ such that $p_1 : X \times Y \rightarrow X$ and $p_2 : X \times Y \rightarrow Y$ are continuous.

TUTORIAL 4 MA 406
5 FEB SPRING 2026, IIT BOMBAY

Submission on Feb 12th 2026, in class: *Please turn in solutions to problems Part I -(2(a)), Part I -(5), Part I-(7(a)), Part II- (2(a)) and Part II- (2(c))*

*Each person in the new group has to handwrite exactly one problem. Ensure all the new group members submit answers to different problems. Marks are allotted individually. **Please submit solutions only to the assigned problems.***

The role of the group is to collaborate on solutions and decide who will write up solution to which problem. If there are any concerns please email me.

Part I.

- (1) Let \mathcal{S} be a collection of subsets of X such that $\cup_{A \in \mathcal{S}} A = X$. Let $\mathcal{B} = \{\cap_{i=1}^n S_i \mid S_1, \dots, S_n \in \mathcal{S}, n \in \mathbb{N}\}$. Show that \mathcal{B} is a base for a topology generated on X . In this case \mathcal{S} is called a **sub base**.
- (2) (a) Show that $(A \overset{\circ}{\times} B) = \overset{\circ}{A} \times \overset{\circ}{B}$ for subsets A and B of topological spaces X and Y respectively.
(b) Show that $\overline{A \times B} = \overline{A} \times \overline{B}$ for subsets A and B of topological spaces X and Y respectively.
- (3) Let $p_\beta : \prod_{\alpha \in I} X_\alpha \rightarrow X_\beta$ be projections for family of topological spaces $\{X_\alpha \mid \alpha \in I\}$. Find the smallest topology on $\prod_{\alpha \in I} X_\alpha$ such that the projections are continuous. This is called the *product topology*.
- (4) Show that a function from $W \xrightarrow{f} \prod_{\alpha \in I} X_\alpha$ is continuous if and only $p_\alpha \circ f$ is continuous for all $\alpha \in I$.
- (5) Let $i_\beta : Y_\beta \rightarrow \coprod_{\alpha \in I} Y_\alpha$ be inclusions for family of topological spaces $\{Y_\alpha \mid \alpha \in I\}$. Find the largest topology on $\coprod_{\alpha \in I} Y_\alpha$ such that the inclusions are continuous. This is called the *weak topology*.
- (6) Show that a function from $\coprod_{\alpha \in I} Y_\alpha \xrightarrow{g} Z$ is continuous if and only $g \circ i_\alpha$ is continuous for all $\alpha \in I$.
- (7) (a) Describe the topology on $\prod_{i \in \mathbb{R}} X_i$ where X_i has discrete topology.
(b) Describe the topology on $\coprod_{i \in \mathbb{R}} X_i$ where X_i has discrete topology.

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5 FEB SPRING 2026, IIT BOMBAY

Part II.

- (1) A function of topological spaces X, Y is said to be open if for every open set $U \subseteq X$, $f(U)$ is open in Y .
 - (a) What condition is needed on f for the statement “ \mathcal{S} is a subbase of the topology on X , then f is open if and only if $f(S)$ is open for all $S \in \mathcal{S}$ ” to be true
 - (b) Assume the required condition and prove the statement in part (a)

- (2) Let $\{f_\alpha : Z \rightarrow X_\alpha \mid \alpha \in I\}$ be a family of functions where X_α are topological spaces.
 - (a) Find the smallest topology on Z such that these functions are continuous.
 - (b) Show that with this topology the map $Z \xrightarrow{\prod f_\alpha} \prod_\alpha X_\alpha$, where $\prod_\alpha X_\alpha$ has product topology, is continuous.
 - (c) Denote $\prod f_\alpha$ by f . If f is one-one show that f is a open map onto its image. (What does this mean exactly?)

- (3) Let $f : X \rightarrow Y$ be a function of sets. Show that $a \sim b$ if $f(a) = f(b)$ is a equivalence relation.

- (4)
 - (a) Describe the smallest equivalence relation generated on $I = [0, 1]$ where $0 \sim 1$.
 - (b) Find the largest topology on I / \sim , that is the set of all equivalence classes such that $I \rightarrow I / \sim$ is continuous.

- (5)
 - (a) Describe the smallest equivalence relation generated on $I \times I$ where $(s, 0) \sim (s, 1)$ for all $s \in I$.
 - (b) Find the largest topology on $I \times I / \sim$, that is the set of all equivalence classes such that $I \times I \rightarrow I \times I / \sim$ continuous.

- (6) Let $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ act on S^2 by antipodal map. (What does this mean?). Find the smallest topology on the orbit spaces S^2/\mathbb{Z}_2 .

TUTORIAL 5 MA 406
12 FEB SPRING 2026, IIT BOMBAY

Submission on Feb 19th 2026, in class: *Please turn in solutions to problems Part I - (6), (8), Part II - (2), (3) and (7). Each person in the new group has to handwrite exactly one problem. Ensure all the new group members submit answers to different problems. Marks are allotted individually. **Please submit solutions only to the assigned problems.***

The role of the group is to collaborate on solutions and decide who will write up solution to which problem. If there are any concerns please email me.

Part I.

- (1) Let $X = I \times I$ with subspace topology. Let \sim_1 be the smallest equivalence relation generated by $(s, 0) \sim_1 (s, 1)$ for all $s \in I$. Then give a base for the quotient topology on $Y = X / \sim_1$.
- (2) Show that Y is homeomorphic to $S^1 \times I$.
- (3) Let $X = I \times I$. Let \sim_2 be the smallest equivalence relation generated by $(s, 0) \sim_2 (1 - s, 1)$ for all $s \in I$. Then give a base for the quotient topology on $Z = X / \sim_2$.
- (4) ¹ Are the two topological spaces Y and Z defined above homeomorphic? Explain your answer.
- (5) Let $X = I \times I$. Let \sim_3 be the smallest equivalence relation generated by $(s, 0) \sim_3 (s, 1), (0, t) \sim_3 (1, t)$ for all $s, t \in I$. Then give a base for the quotient topology on $W = X / \sim_3$.
- (6) Show that W is homeomorphic to $S^1 \times S^1$.
- (7) Let $X = I \times I$. Let \sim_4 be the smallest equivalence relation generated by $(s, 0) \sim_4 (0, 0), (s, 1) \sim_4 (0, 0), (0, t) \sim_4 (0, 0), (1, t) \sim_4 (0, 0)$ for all $s, t \in I$. Then give a base for the quotient topology on $R = X / \sim_4$.
- (8) Show that R is homeomorphic to S^2 .
- (9) Let $X = I \times I$. Let \sim_5 be the smallest equivalence relation generated by $(s, 0) \sim_5 (1 - s, 1), (0, t) \sim_5 (1, 1 - t)$ for all $s, t \in I$. Then give a base for the quotient topology on $Q = X / \sim_5$.
- (10) ² Are R and Q homeomorphic? What about W and Q ? Explain your answer.

¹give heuristic argument

²give a picture argument

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Part II.

- (1) Let X be a space, define $CX = X \times I / \sim$ where \sim is the equivalence relation generated by $(x, 1) \sim (x', 1)$ for all $x, x' \in X$ to be the cone on X .
Give a base for the quotient topology on CS^1 .
- (2) Show that CS^1 is homeomorphic to D^2 .
- (3) Let X be a space, define $\Sigma X = X \times I / \sim$ where \sim is the equivalence relation generated by $(x, 1) \sim (x', 1), (x, 0) \sim (x', 0)$ for all $x, x' \in X$ to be the suspension of X . Give a base for the quotient topology on ΣS^1 .
- (4) Let $x_0 \in X$ and $y_0 \in Y$, then $X \vee Y = X \sqcup Y / \sim$ where \sim is the equivalence relation generated by $x_0 \sim y_0$. Give a base for the topology on $X \vee Y$.
- (5) ³ Draw/ describe the space $S^1 \vee S^1$ and $S^1 \vee S^2$.
- (6) Consider X to be a space and $Y = CX \sqcup CX / \sim$ with $[(x, 0)] \sim [(x, 0)]$ in the two different copies of CX , $x \in X$. Give a base for the topology on Y when $X = S^1$.
- (7) Show that ΣX is homeomorphic to Y .

³Give a picture argument

TUTORIAL 6 MA 406
19 FEB SPRING 2026, IIT BOMBAY

- (1) Prove or disprove.
 - (a) Continuous image of a Hausdorff space is Hausdorff.
 - (b) Image of a Hausdorff space under an open map is Hausdorff.
 - (c) Image of a connected space under an open map is connected.
 - (d) Image of a normal space under an open map is normal.
 - (e) Every normal space is Hausdorff.
- (2) Show that \mathbb{R}^2 is connected.
- (3) Show that $X_\alpha \subseteq X$ are connected spaces, $\alpha \in J$ such that for any pair $\alpha, \beta \in J$, $X_\alpha \cap X_\beta \neq \emptyset$, then $\cup_\alpha X_\alpha$ is connected.
- (4) Let $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in \mathbb{Q} \text{ or } x_2 \in \mathbb{Q}\}$. Then prove or disprove that X is connected.
- (5) Show that the spaces D^n, S^n are connected for all natural numbers $n \geq 1$.
- (6) Product of Hausdorff spaces is Hausdorff. Prove or disprove.
- (7) Consider $H = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ and let $H_+ = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ with topology generated by the base $\{N((a, b), \delta) \cap H_+ \mid (a, b) \in H_+\} \cup \{N((a, 0), \delta) \cap H_+\} \cup \{(a, 0) \mid a \in \mathbb{R}\}$. This is called the Moore topology on Half plane and denote by H_M
 - (a) Show that $\mathbb{R} \times \{0\}$ with subspace topology on H_M is discrete.
 - (b) Show that $H \cap \mathbb{Q} \times \mathbb{Q}$ is a dense subset of H_M .
 - (c) Show that H_M is not normal.
- (8) Show that the set $D = \mathbb{Q} \times \mathbb{Q}$ is dense in \mathbb{R}_l (the real line with base $\{(x, z) \mid x, z \in \mathbb{R}\}$.)
- (9) Show that the subspace topology on $S = \{(x, -x) \mid x \in \mathbb{R}\}$ is discrete with product topology in $\mathbb{R}_l \times \mathbb{R}_l$.
- (10) Is the product of normal spaces also normal? Why or Why not? (Consider $\mathbb{R}_l \times \mathbb{R}_l$)
- (11) Is a subspace of a normal space also normal? Why or Why not?

TUTORIAL 7 MA 406
5 MARCH SPRING 2026, IIT BOMBAY

Submission on March 13th 2026, in class: *Please turn in solutions to problems 1(f) 1(g), (5), (8), (10)(b). Each person in the new group has to handwrite exactly one problem. Ensure all the new group members submit answers to different problems. Marks are allotted individually. **Please submit solutions only to the assigned problems.***

The role of the group is to collaborate on solutions and decide who will write up solution to which problem. If there are any concerns please email me.

- (1) Prove or disprove.
 - (a) Continuous image of a normal space is normal
 - (b) Quotient space of a Hausdorff space is Hausdorff.
 - (c) Quotient space of a normal space is normal.
 - (d) Product of connected spaces is connected.
 - (e) Continuous image of a connected space is connected.
 - (f) Image of a connected space under an open map is connected.
 - (g) Quotient of a connected space is connected.

- (2) Show that if X and Y are connected then $X \times Y$ is connected.

- (3) Which of the following are homeomorphic? Why or why not?
 - (a) $(0, 1)$.
 - (b) $(0, 1]$
 - (c) $[1, \infty)$
 - (d) \mathbb{R} .
 - (e) $[-11, 0]$.

- (4) Are \mathbb{R} and \mathbb{R}^2 homeomorphic? Why or why not?

- (5) Show that the space obtained on removing $[0, 1] \times \{1/2\}$ in the Mobius strip is homeomorphic to a cylinder.

- (6) Consider $H = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ and let $H_+ = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ with topology generated by the base $\{N((a, b), \delta) \cap H_+ \mid (a, b) \in H_+\} \cup \{(N((a, 0), \delta) \cap H_+) \cup \{(a, 0)\} \mid a \in \mathbb{R}\}$. This is called the Moore topology on Half plane and denote by H_M . Show that $\mathbb{R} \times \{0\}$ is a closed subset of H_M .

- (7) Show that $S = \{(x, -x) \mid x \in \mathbb{R}\}$ is a closed subset of $\mathbb{R}_l \times \mathbb{R}_l$ with product topology.

- (8) Show that $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in \mathbb{Q} \text{ or } x_2 \in \mathbb{Q}\}$ is path connected and hence connected.

- (9) A topological space is said to *second countable* if it has a countable base.

- (a) Subspace of a countable space is countable. Prove or disprove.
 - (b) A finite product of second countable spaces is second countable. Prove or disprove.
 - (c) Continuous image of a second countable space is second countable. Prove or disprove.
- (10) A topological space is said to be *separable* if it has a countable dense subset.
- (a) Show that \mathbb{R} with standard topology is separable.
 - (b) Extend your ideas to show that \mathbb{R}^n with standard topology is separable.
 - (c) Show that H_M , the Moore plane is separable but its subspace $\mathbb{R} \times \{0\}$ is not.

TUTORIAL 8 MA 406
12 MARCH SPRING 2026, IIT BOMBAY

Submission on March 20th 2026, in class: *Please turn in solutions to problems 1(a), 2(b), 2(c), (4), (5)(a), . Each person in the new group has to handwrite exactly one problem. Ensure all the new group members submit answers to different problems. Marks are allotted individually. **Please submit solutions only to the assigned problems.***

The role of the group is to collaborate on solutions and decide who will write up solution to which problem. If there are any concerns please email me.

- (1) Let $V = \{(x, 0) \in \mathbb{R}^2 \mid x \leq 0\} \cup \{(x, \sin(1/x)) \mid x > 0, x \in \mathbb{R}\}$ be the topologist's sine curve. Let $T = \{(x, \sin(1/x)) \mid x > 0, x \in \mathbb{R}\}$.
 - (a) Show that V is connected.
 - (b) Show that T is path connected.
 - (c) Show that V is not path connected.
 - (d) Closure of a path connected space is path connected. Prove or disprove.

- (2) Prove or disprove.
 - (a) The continuous image of a path connected space is path connected.
 - (b) The open image of a path connected space is path connected.
 - (c) Product of finitely many path connected spaces is path connected.

- (3) Show that \mathbb{R}_l is not locally connected.

- (4) Show that V is not locally connected.

- (5) Prove or disprove.
 - (a) The continuous image of a locally connected space is locally connected.
 - (b) Product of finitely many locally connected spaces is locally connected.
 - (c) Infinite product of locally connected spaces where all but finitely many are connected is locally connected.

- (6) Any function from a compact space to a Hausdorff space is closed.

- (7) Let \mathcal{T} and \mathcal{U} be topologies on X such that $\mathcal{T} \subseteq \mathcal{U}$. If X is compact with \mathcal{T} topology, is it compact with \mathcal{U} topology? What about the other way around?

- (8) Is \mathbb{R}_l compact? Why or Why not? Identify compact subsets of \mathbb{R}_l .

- (9) Show that finite union of compact sets is compact.

- (10) Is $[0, 1]$ compact in \mathbb{R} with the topology $\mathcal{T} = \{A \mid \mathbb{R} - A \text{ is countable or all of } \mathbb{R}\}$.

- (11) Let X and Y be topological spaces. If Y is compact show that the projection map $p_1 : X \times Y \rightarrow X$ is a closed map.
- (12) Let X and Y be topological spaces and Y be compact Hausdorff. Then f is continuous if and only if the graph of f , $G_f = \{(x, f(x)) | x \in X\}$ is closed in $X \times Y$.

TUTORIAL 9 MA 406
26 MARCH SPRING 2026, IIT BOMBAY

Submission on April 2 2026, in class: *Please turn in solutions to problems (1),(3), (4), (8), (9). Each person in the new group has to handwrite exactly one problem. Ensure all the new group members submit answers to different problems. Marks are allotted individually. **Please submit solutions only to the assigned problems.***

The role of the group is to collaborate on solutions and decide who will write up solution to which problem. If there are any concerns please email me.

- (1) Show that X is normal if and only if for any closed set A and an open set U such that $A \subseteq U$ there exists an open set V such that $A \subseteq V \subseteq \overline{V} \subseteq U$.
- (2) Show that \mathbb{R}_K which is \mathbb{R} with basis $\{(a, b) \mid a, b \in \mathbb{R}\} \cup \{(a, b) - K \mid a, b \in \mathbb{R}\}$ where $K = \{1/n \mid n \in \mathbb{N}\}$ is Hausdorff.
- (3) A topological space is a regular space if a point and closed set can be separated by disjoint open sets and T_3 if it is also T_1 . Show that a space X is regular if for every $a \in X$ and an open set $a \in U$, there exists open set $V \subseteq X$, $a \in V \subseteq \overline{V} \subseteq U$.
- (4) Prove that the subspace of a regular space is regular.
- (5) Prove that product of regular spaces are regular.
- (6) Show that $\mathbb{R}_l \times \mathbb{R}_l$ is regular.
- (7) Show that \mathbb{R}_K is not regular.
- (8) Give an example of a regular space which is not normal.
- (9) Show that if Y is Hausdorff and $f, g : X \rightarrow Y$ are continuous functions on spaces, then $\{x \in X \mid f(x) = g(x)\}$ is a closed set. Prove then that if two continuous functions into Y agree on a dense set then they are equal.
- (10) Show that subspace of a completely regular space is completely regular.
- (11) Show that the product of completely regular space is completely regular.
- (12) Show that $\mathbb{R}_l \times \mathbb{R}_l$ is completely regular.

TUTORIAL 10 MA 406
2 APRIL 2026 SPRING 2026, IIT BOMBAY

Submission on April 15 2026, in class: *Please turn in solutions to problems (1),(3), (6)(a), (6)(b), (7)(c). Each person in the new group has to handwrite exactly one problem. Ensure all the new group members submit answers to different problems. Marks are allotted individually. **Please submit solutions only to the assigned problems.***

The role of the group is to collaborate on solutions and decide who will write up solution to which problem. If there are any concerns please email me.

- (1) Prove that product of Tychonoff spaces is Tychonoff.
- (2) Show that Tietze extension theorem implies Uryzohn's lemma and hence normality.
- (3) Show that connected T_4 space with more than one point is uncountable.
- (4) Show that the if X is Hausdorff the following statements are equal. A Hausdorff space is *locally compact* if it satisfies either of these statements
 - (a) For every $x \in X$, there exists a compact set $W \subseteq X$ such that it contains an open set V with $x \in V \subseteq W$. *This is saying every point has a compact neighbourhood*
 - (b) For every $x \in X$ and an open set $x \in U \subset X$, there exists a open set $V \subseteq X$ such that it $x \in V \subseteq \bar{V} \subseteq U$ and \bar{V} is compact. *This is saying that every open set contains a compact neighbourhood.*
- (5) When is a compact space (not necessarily Hausdorff) locally compact? If (a) is taken as definition or if (b) is or both or neither?
- (6) Prove or disprove.
 - (a) \mathbb{Q} is locally compact.
 - (b) \mathbb{R}_l is locally compact.
 - (c) Let $f : X \rightarrow Y$ be a continuous map of Hausdorff spaces. Then X is locally compact implies $f(X)$ is locally compact.
- (7) Let (X, \mathcal{T}) be a locally compact Hausdorff space. Let $X^* = X \sqcup \{+\}$. Define a $\mathcal{U} = \mathcal{T} \cup \{\{+\} \cup X - K \mid K \text{ compact in } X\}$.
 - (a) Show that this defines a topology on X^* .
 - (b) Show that X^* is a compact Hausdorff space.
 - (c) If $X = \mathbb{R}$, show that X^* is homeomorphic to S^1 .

TUTORIAL 11 MA 406 PART I
10 APRIL 2026 SPRING 2026, IIT BOMBAY

Submission on April 17 2026, in class:

Part I. *Please turn in solutions to problems Part I (2),(3),(4), (5), (6). Each person in the group has to handwrite exactly one problem. Ensure all the new group members submit answers to different problems. Marks are allotted individually. **Please submit solutions only to the assigned problems.***

The role of the group is to collaborate on solutions and decide who will write up solution to which problem. If there are any concerns please email me.

- (1) A locally compact space Hausdorff is Tychonoff. Prove or Disprove.
- (2) Show that if X is second countable, every open cover of X has a countable subcover. (This is same as saying X is *Lindeloff*.)
- (3) Show that if a space is T_3 and Lindeloff then it is T_4 .
- (4) Every separable metric space is second countable.
- (5) Show that $\mathbb{R}^{\mathbb{N}}$ is a separable space.
- (6) Let X be T_4 and second countable. Show that there is an embedding from $X \rightarrow I^{\mathbb{N}}$. (Use Uryzohn functions, to separate the open sets in base.)
- (7) Use the above statements to prove the following are equivalent for a T_1 topological space.
 - (a) X is regular and second countable.
 - (b) X is separable and metrizable.
 - (c) X can be embedded as a subspace of the Hilbert cube $I^{\mathbb{N}}$.

Part II.

- (1) Give at least two different locally finite open covers for \mathbb{R} .
- (2) Give a refinement of the cover $(n, n + 2) \times (n - 1, n + 3)$ of \mathbb{R}^2 which is also a cover if it exists.
- (3) Show that every regular Lindeloff space is paracompact.
- (4) Show that every Hausdorff paracompact space is normal.
- (5) Show that \mathbb{R}_l is paracompact but $\mathbb{R}_l \times \mathbb{R}_l$ is not.
- (6) Show every discrete space is paracompact.
- (7) Continuous image of a paracompact space is paracompact.
- (8) ¹ Let X be a paracompact space. Then every open cover of X has a locally finite partition of unity subordinated to it. (A partition of unity is a collection of functions $\phi_\alpha : X \rightarrow \mathbb{R}_{\geq 0}$ such that at each $a \in X$, $\phi_\alpha(a) \neq 0$ only for finitely many α and $\sum_\alpha \phi_\alpha(a) = 1$ for all $a \in X$. It is locally finite if for $a \in X$ there is a neighbourhood $U \subseteq X$ such that all but finitely many ϕ vanish on it. It is subordinated to a cover if each function is zero outside an open subset in the cover.)

¹Beyond syllabus